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UNIQUENESS OF HYPERSPACES FOR PEANO CONTINUA

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ABSTRACT. For a metric continuum X and a positive integer n, let $C_n(X)$ be the hyperspace of nonempty closed subsets of X with at most n components. We say that X has unique hyperspace $C_n(X)$ provided that, if Y is a continuum and $C_n(X)$ is homeomorphic to $C_n(Y)$, then X is homeomorphic to Y. In this paper we study which Peano continua X have a unique hyperspace $C_n(X)$. We find some sufficient and also some necessary conditions for a Peano continuum X to have unique hyperspace $C_n(X)$. Our results generalize all the previously known results on this subject. We also give some significant examples.

1. Introduction. A *continuum* is a nondegenerate compact connected metric space. A *Peano continuum* is a locally connected continuum. For a continuum X and $n \in \mathbf{N}$, consider the following hyperspaces:

$$\begin{aligned} 2^X &= \{A \subset X : A \text{ is closed and nonempty}\}, \\ C(X) &= \{A \in 2^X : A \text{ is connected}\}, \\ C_n(X) &= \{A \in 2^X : A \text{ has at most } n \text{ components}\}. \end{aligned}$$

All the hyperspaces considered are metrized by the Hausdorff metric H_X. Note that $C(X) = C_1(X)$.

We say that a continuum X has *unique hyperspace* $C_n(X)$ provided that the following implication holds: if Y is a continuum and $C_n(X)$ is homeomorphic to $C_n(Y)$, then X is homeomorphic to Y.

Given a continuum X, let

$$\mathcal{G}(X) = \{p \in X : p \text{ has a neighborhood } M \text{ in } X \text{ such that } M \text{ is a finite graph}\} \text{ and } \mathcal{P}(X) = X - \mathcal{G}(X).$$

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A *free arc* in X is an arc $\alpha \subset X$, with end points p and q such that $\alpha - \{p,q\}$ is open in X. The continuum X is said to be *almost meshed* provided that the set $\mathcal{G}(X)$ is dense in X, and an almost meshed continuum X is *meshed* provided that X has a basis of neighborhoods $\mathcal{B}$ such that, for each element $U \in \mathcal{B}$, $U - \mathcal{P}(X)$ is connected. A *dendrite* is a locally connected continuum without simple closed curves. Let $\mathfrak{D}$ denote the class of dendrites with a closed set of end points.

Using the results of Duda in [**11**, subsection 9.1], Acosta [**1**, Theorem 1] observed that finite graphs different from both an arc and a simple closed curve have unique hyperspace $C(X)$. Illanes proved in [**16, 17**] that finite graphs have unique hyperspaces $C_n(X)$, for each $n \geq 2$.

In [**13**], Herrera-Carrasco showed that if X is in $\mathfrak{D}$ and X is not an arc, then X has unique hyperspace $C(X)$. This result was extended in [**15**], where Herrera-Carrasco and Macías-Romero proved that if $X \in \mathfrak{D}$, then X has a unique hyperspace $C_n(X)$ for every $n \geq 3$. The case $n = 2$ has also been solved. It was more difficult so the two papers [**14, 18**] were needed to complete its solution. Acosta and Herrera-Carrasco [**2**] have shown that if X is a dendrite and $X \notin \mathfrak{D}$, then there are uncountable many non-homeomorphic continua Y such that $C(X)$ is homeomorphic to $C(Y)$. Thus, a dendrite X that is not an arc belongs to $\mathfrak{D}$ if and only if X has unique hyperspace $C(X)$.

Recently [**3**], Acosta, Herrera-Carrasco and Macías-Romero have proved that if X is a locally $\mathfrak{D}$-continuum (that is, X is a continuum such that each point has a basis of neighborhoods $\mathfrak{B}$ such that each element in $\mathfrak{B}$ is an element of $\mathfrak{D}$) that is not an arc, then X has unique hyperspace $C(X)$.

On the other hand, the well known Curtis-Schori theorem (see [**9, 10**]) states that if X is a Peano continuum containing no free arcs, then $C(X)$ is homeomorphic to the Hilbert cube. This is why the problem of determining whether a Peano continuum X has unique hyperspace is open only when X contains free arcs.

In this paper we are interested in studying which Peano continua X have a unique hyperspace $C_n(X)$. The main results are the following.

A. If a Peano continuum has a nonempty open subset without free arcs (that is, X is not almost meshed), then X does not have unique hyperspace $C_n(X)$ for any $n \in \mathbf{N}$ (Theorem 20). Thus, for a Peano continuum X to have unique hyperspace, we at least need X to be almost meshed.

B. If X is meshed, we obtain a completely opposite result (Theorem 37). For $n \neq 1$, X has a unique hyperspace $C_n(X)$. If, further, X is neither an arc nor a simple closed curve, then X has unique hyperspace $C(X)$ (Theorem 37). Recall that if X is either an arc or a simple closed curve, then $C(X)$ is a 2-cell. Thus, the problem of determining if a Peano continuum X has unique hyperspace $C_n(X)$ is open only when X is almost meshed but not meshed.

C. The class of meshed continua contains the following classes: (a) finite graphs, (b) $\mathfrak{D}$, (c) locally $\mathfrak{D}$ continua. Hence, Theorem 37 covers all the known cases of continua X having a unique hyperspace $C_n(X)$.

D. If X is almost meshed and $X - \mathcal{P}(X)$ is disconnected, then X does not have a unique hyperspace $C(X)$ (Corollary 23).

E. Let $Z_0 = ([-1,1] \times \{0\}) \cup (\bigcup\{\{1/m\} \times [0,(1/m)] : m \geq 2\})$. Then Z_0 plays an important role in this topic:

(a) if a dendrite X contains Z_0, then $X \notin \mathfrak{D}$ and X does not have a unique hyperspace $C(X)$ [**2**];

(b) Z_0 is almost meshed, $\mathcal{P}(Z_0) = \{(0,0)\}$, $Z_0 - \mathcal{P}(Z_0)$ is disconnected;

(c) Z_0 is not meshed (Lemma 3);

(d) the dendrite $Z_3 = Z_0 \cup (\bigcup\{\{-1/m\} \times [0,(1/m)] : m \geq 2\})$ *has* a unique hyperspace $C_2(Z_3)$ (Example 39);

(e) if we add the segment $\{0\} \times [0,1]$ to Z_3, that is, if $Z_1 = Z_3 \cup (\{0\} \times [0,1])$, then Z_1 does not have a unique hyperspace $C_2(Z_1)$ (Example 43);

(f) if we add the arc $L = (\{-1,1\} \times [0,1]) \cup ([-1,1] \times \{1\})$, that is, if $Z_2 = Z_0 \cup L$, then $Z_2 - \mathcal{P}(Z_2)$ is connected, Z_2 is not meshed and Z_2 *has* a unique hyperspace $C(Z_2)$ (Example 38).

A discussion about uniqueness of other hyperspaces can be found in the introduction of [**18**].

2. Meshed and almost meshed continua. Given a continuum X and a subset A of X, we denote the interior of A in X by A° or $\text{int}_X(A)$.

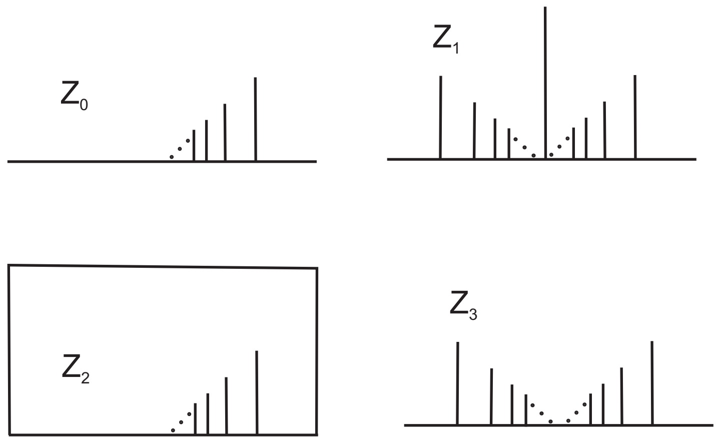

FIGURE 1.

For $\varepsilon > 0$, $p \in X$ and $A \subset X$, let $B(\varepsilon, p)$ denote the ε-ball around p in X, and let $N(\varepsilon, A) = \bigcup\{B(\varepsilon, a) : a \in A\}$. Given $A \in C_n(X)$, we denote by $\dim_A[C_n(X)]$ the dimension of the space $C_n(X)$ at the element A. Let

$$\mathcal{FA}(X) = \bigcup\{J^{\circ} : J \text{ is a free arc } J \text{ in } X\}.$$

Given $n \in \mathbf{N}$ and a continuum X, let

$$\mathfrak{F}_n(X) = \{A \in C_n(X) : \ \dim_A[C_n(X)] \text{ is finite}\}.$$

The set $\mathfrak{F}_1(X)$ is simply denoted by $\mathfrak{F}(X)$.

Given subsets $U_1, \ldots, U_m$ of X, let $\langle U_1, \ldots, U_m\rangle = \{A \in C_n(X) : A \subset U_1 \cup \cdots \cup U_m$ and $A \cap U_i \neq \varnothing$ for each $i \in \{1, \ldots, m\}\}$. It is known (see [**23**, subsection 4.24]) that the family of all sets of the form $\langle U_1, \ldots, U_m\rangle$, where $m \in \mathbf{N}$ and each U_i is open in X, is a basis for the topology in $C_n(X)$.

We describe some examples in the Euclidean plane $\mathbf{R}^2$. Given two different points $p, q \in \mathbf{R}^2$, let pq denote the convex segment joining them.

Let $Z_0 = ([-1,1] \times \{0\}) \cup (\bigcup\{\{1/m\} \times [0,(1/m)] : m \geq 2\})$. Then Z_0 is a dendrite, $Z_0 \notin \mathfrak{D}$, $\mathcal{P}(Z_0) = \{(0,0)\}$, Z_0 is almost meshed but Z_0 is not meshed.

Let $F_\omega = \bigcup\{(0,0)((1/m),(1/m^2)) : m \in \mathbf{N}\}$. Then F_ω is a dendrite, $F_\omega \notin \mathfrak{D}$, $\mathcal{P}(F_\omega) = \{(0,0)\}$, F_ω is almost meshed but F_ω is not meshed.

In [**5**] it was proved that a dendrite X is in $\mathfrak{D}$ if and only if X does not contain a topological copy of neither Z_0 nor F_ω.

Note that meshed continua do not need to be local dendrites. For example, the continuum X described in [**23**, Example 10.38, Figure 10.38 (a)] is meshed and $\mathcal{P}(X)$ is the segment $A_0 = [0,1] \times \{0\}$.

The following lemma is easy to prove.

Lemma 1. *Let X be a continuum. Then* $\mathrm{cl}_X(\mathcal{G}(X)) = \mathrm{cl}_X(\mathcal{FA}(X))$. *Therefore, X is almost meshed if and only if $\mathcal{FA}(X)$ is dense in X.*

Lemma 2. *If X is a meshed continuum, then X is a Peano continuum.*

Proof. Let $\mathcal{B}$ be a basis of neighborhoods of X such that, for each element $U \in \mathcal{B}$, $U - \mathcal{P}(X)$ is connected. Since X is almost meshed, $(\mathcal{P}(X))^\circ = \varnothing$. Thus, for each $U \in \mathcal{B}$, $\mathrm{int}_X(U) \subset \mathrm{cl}_X(U - \mathcal{P}(X))$. Therefore, the family $\{\mathrm{cl}_X(U - \mathcal{P}(X)) : U \in \mathcal{B}\}$ is a basis of connected neighborhoods for X. Hence, X is connected almost certainly and then X is locally connected. $\square$

Lemma 3. *Let X be a continuum. Then X is meshed if and only if X is almost meshed, and X has a basis $\mathcal{D}$ of open connected subsets of X such that, for each element $U \in \mathcal{D}$, $U - \mathcal{P}(X)$ is connected.*

Proof. The sufficiency is immediate from the definition of meshed continuum. Now, suppose that X is meshed. Let $\mathcal{B}$ be a basis of neighborhoods of X such that, for each element $U \in \mathcal{B}$, $U - \mathcal{P}(X)$ is connected. Let $p \in X$ and W be an open subset of X such that $p \in W$. Let $U \in \mathcal{B}$ be such that $p \in \mathrm{int}_X(U) \subset U \subset W$. By Lemma 2, there exists an open connected subset Z of X such that $p \in Z \subset \mathrm{int}_X(U)$. Since $\mathcal{P}(X)$ is a closed subset of X, for each $x \in U - \mathcal{P}(X)$, there exists an open and connected subset of V_x of X such that $x \in V_x \subset W - \mathcal{P}(X)$.

Let $V = Z \cup (\bigcup\{V_x : x \in U - \mathcal{P}(X)\})$. Clearly, V is an open subset of X such that $p \in V \subset W$. Since $(U-\mathcal{P}(X))\cup(\bigcup\{V_x : x \in U-\mathcal{P}(X)\})$ is a connected subset of $V - \mathcal{P}(X)$ and $Z - \mathcal{P}(X) \subset U - \mathcal{P}(X)$, we obtain that $V - \mathcal{P}(X) = (U - \mathcal{P}(X)) \cup (\bigcup\{V_x : x \in U - \mathcal{P}(X)\})$ is an open connected subset of X. Since $V - \mathcal{P}(X) \subset V \subset \mathrm{cl}_X(V - \mathcal{P}(X))$, we conclude that V is connected. This completes the proof of the lemma. □

Theorem 4. *Let X be a Peano continuum, $n \in \mathbf{N}$ and $A \in C_n(X)$. Then the following are equivalent.*

(a) $\dim_A[C_n(X)]$ *is finite,*

(b) *there exists a finite graph D contained in X such that $A \subset D^{\circ}$,*

(c) $A \cap \mathcal{P}(X) = \varnothing$.

Proof. (a) $\Rightarrow$ (b). Let k be the number of components of A. In the case that $k = 1$, since $\dim_A[C(X)] \le \dim_A[C_n(X)]$, we obtain that $\dim_A[C(X)]$ is finite. Thus, [**18**, Lemma 2.2, Claim 1] guarantees the existence of D. Suppose then that $k > 1$. Let $A_1, \dots, A_k$ be the components of A. Let $Z_1, \dots, Z_k$ be pairwise disjoint subcontinua of X such that $A_i \subset Z_i^{\circ}$ for each $i \in \{1, \dots, k\}$.

Let $\varphi : C(Z_1) \times \cdots \times C(Z_k) \to \langle Z_1, \dots, Z_k\rangle \cap C_k(X)$ be given by $\varphi(B_1, \dots, B_k) = B_1 \cup \ldots \cup B_k$. Notice that φ is a homeomorphism. Given $i \in \{1, \dots, k\}$, $\dim_{A_i}[C(Z_i)] \le \dim_{(A_1,\dots,A_k)}[C(Z_1) \times \cdots \times C(Z_k)] = \dim_A[\langle Z_1, \dots, Z_k\rangle \cap C_k(X)] \le \dim_A[C_n(X)] < \infty$. Since $C(Z_i)$ is a neighborhood of A_i in $C(X)$, $\dim_{A_i}[C(X)] = \dim_{A_i}[C(Z_i)]$. Since A_i is connected, by the first case we considered ($k = 1$), there exists a finite graph D_i, contained in X, such that $A_i \subset D_i^{\circ}$. We may assume that $D_i \subset Z_i$. Since the finite graphs $D_1, \dots, D_k$ are pairwise disjoint and X is arcwise connected [**23**, subsection 8.23], it is possible to construct a finite number of arcs $\alpha_1, \dots, \alpha_r$ in X such that $D = D_1 \cup \cdots \cup D_k \cup \alpha_1 \cup \cdots \cup \alpha_r$ is a finite graph. Since $A \subset D^{\circ}$, the proof of (a) $\Rightarrow$ (b) is finished.

(b) $\Rightarrow$ (a). Suppose that $A \subset D^{\circ}$ for some finite graph D in X. Then $C_n(D)$ is a neighborhood of A in $C_n(X)$. Thus, $\dim_A[C_n(X)] = \dim_A[C_n(D)]$. By the main result in [**21**], $\dim_A[C_n(D)]$ is finite (in fact, in [**21**, Theorem 2.4] there is an explicit formula for computing $\dim_A[C_n(D)]$).

(b) $\Rightarrow$ (c) is immediate from the definition of $\mathcal{P}(X)$.

(c) $\Rightarrow$ (b). Suppose that $A \cap \mathcal{P}(X) = \varnothing$. For each point $a \in A$, let D_a be a finite graph in X such that $a \in \mathrm{int}_X(D_a)$. Then there exists a finite graph F_a in X such that $a \in \mathrm{int}_X(F_a) \subset F_a \subset \mathrm{int}_X(D_a) - \mathcal{P}(X)$. By the compactness of A, there exist $m \in \mathbf{N}$ and $a_1, \dots, a_m \in A$ such that $A \subset \mathrm{int}_X(F_{a_1}) \cup \dots \cup \mathrm{int}_X(F_{a_m})$. Let $F = F_{a_1} \cup \dots \cup F_{a_m}$. Notice that F has a finite number of components and $A \subset F^{\circ}$. Since each point $p \in F$ belongs to the interior in X of a finite graph contained in X, it is easy to check that each component of F satisfies conditions (1) and (2) of [**23**, Theorem 9.10]. Thus, each component of F is a finite graph. Joining the components of F by appropriate arcs in X, we obtain the required graph D. This completes the proof of the theorem. □

Theorem 5. *For a Peano continuum X, the following are equivalent.*

(a) *X is meshed*,

(b) *for each $n \in \mathbf{N}$, $\mathfrak{F}_n(X)$ is dense in $C_n(X)$*,

(c) *there exists an $n \in \mathbf{N}$ such that $\mathfrak{F}_n(X)$ is dense in $C_n(X)$.*

Proof. (a) $\Rightarrow$ (b). Suppose that X is meshed. Let $n \in \mathbf{N}$, $A \in C_n(X)$ and $\varepsilon > 0$. Let $A_1, \dots, A_k$ be the components of A. We assume that $N(\varepsilon, A_1), \dots, N(\varepsilon, A_k)$ are pairwise disjoint. For each $a \in A$, by Lemma 3, there exists an open connected subset U_a of X such that $a \subset U_a \subset B(\varepsilon, a)$ and the open set $V_a = U_a - \mathcal{P}(X)$ is connected. Notice that V_a is nonempty. Fix a point $b(a)$ in V_a. Given $i \in \{1, \dots, k\}$, by the compactness of A_i, there exist $m \in \mathbf{N}$ and $a_1, \dots, a_m \in A_i$ such that $A_i \subset U_{a_1} \cup \dots \cup U_{a_m} \subset N(\varepsilon, A_i)$. Let $U = U_{a_1} \cup \dots \cup U_{a_m}$ and $V = V_{a_1} \cup \dots \cup V_{a_m}$. Notice that U is connected. We see that V is connected. Suppose to the contrary that V is disconnected. Then, we may assume that there exists an $r \in \{1, \dots, m-1\}$ such that $(V_{a_1} \cup \dots \cup V_{a_r}) \cap (V_{a_{r+1}} \cup \dots \cup V_{a_m}) = \varnothing$. Since U is connected, the open set $W = (U_{a_1} \cup \dots \cup U_{a_r}) \cap (U_{a_{r+1}} \cup \dots \cup U_{a_m})$ is nonempty. Since $\mathrm{int}_X(\mathcal{P}(X)) = \varnothing$, $(V_{a_1} \cup \dots \cup V_{a_r}) \cap (V_{a_{r+1}} \cup \dots \cup V_{a_m}) = W - \mathcal{P}(X)$ is nonempty, a contradiction. Therefore, V is connected. By [**23**, Theorem 8.26], V is arcwise connected. Hence, there exists a tree $T_i \subset V$ such that $\{b(a_1), \dots, b(a_m)\} \subset T_i$. Clearly, $H_X(A_i, T_i) < 2\varepsilon$ and $T_i \cap \mathcal{P}(X) = \varnothing$. Let $T = T_1 \cup \dots \cup T_k \in C_n(X)$. Then $H_X(A, T) < 2\varepsilon$ and $T \cap \mathcal{P}(X) = \varnothing$. By Theorem 4, $\dim_T[C_n(X)]$ is finite, so $T \in \mathfrak{F}_n(X)$.

(b) $\Rightarrow$ (c) is immediate.

(c) $\Rightarrow$ (a). Suppose that $\mathfrak{F}_n(X)$ is dense in $C_n(X)$. First, we see that $\mathcal{G}(X)$ is dense in X. Let $p \in X$ and $\varepsilon > 0$. Then there exists an $A \in \mathfrak{F}_n(X)$ such that $H_X(\{p\}, A) < \varepsilon$. By Theorem 4, there exists a finite graph D contained in X such that $A \subset D^{\circ}$. Fix a point $a \in A$. Then $a \in B(\varepsilon, p)$ and D is a neighborhood of a. Thus, $a \in B(\varepsilon, p) \cap \mathcal{G}(X)$. Therefore, $\mathcal{G}(X)$ is dense in X.

Now suppose that X is not meshed. Then there exist $p \in X$ and a neighborhood W of p such that, for each open subset U of X such that $p \in U \subset W$, $U - \mathcal{P}(X)$ is not connected. Since X is a Peano continuum, there exists an open connected subset V of X such that $p \in V \subset W$. Then $V - \mathcal{P}(X) = S \cup T$, where S and T are disjoint open nonempty subsets of X. Fix $x \in T$ and pairwise different points $p_1, \ldots, p_n \in S$. Since V is arcwise connected, there exists an arc $\alpha \subset V$ such that α joins x to a point p_i and $\alpha \cap \{p_1, \ldots, p_n\} = \{p_i\}$. We may suppose that $i = n$. Let $A = \{p_1, \ldots, p_{n-1}\} \cup \alpha \in C_n(X)$. Let $\varepsilon > 0$ be such that $B(\varepsilon, p_1), \ldots, B(\varepsilon, p_{n-1}), N(\varepsilon, \alpha)$ are pairwise disjoint, $B(\varepsilon, p_1) \cup \cdots \cup B(\varepsilon, p_n) \subset S$, $B(\varepsilon, x) \subset T$ and $N(\varepsilon, \alpha) \subset V$. By the density of $\mathfrak{F}_n(X)$, there exists a $B \in \mathfrak{F}_n(X)$ such that $H_X(B, A) < \varepsilon$. Notice that B is contained in the union of the sets $B(\varepsilon, p_1), \ldots, B(\varepsilon, p_{n-1}), N(\varepsilon, \alpha)$ and intersects each one of them. Thus, the components of B are the sets $B_1 = B \cap B(\varepsilon, p_1), \ldots, B_{n-1} = B \cap B(\varepsilon, p_{n-1})$ and $B_n = B \cap N(\varepsilon, \alpha)$. Notice that $B_n \cap B(\varepsilon, p_n) \neq \varnothing$ and $B_n \cap B(\varepsilon, x) \neq \varnothing$. Thus, B_n is connected, $B_n \subset V$ and B_n intersects S and T. This implies that $B_n \cap \mathcal{P}(X) \neq \varnothing$ and, by Theorem 4, $B \notin \mathfrak{F}_n(X)$, a contradiction. This proves that X is meshed and completes the proof of the theorem. □

Theorem 6. *The class of meshed continua contains the following classes.*

(a) *Finite graphs,*

(b) $\mathfrak{D}$,

(c) *locally* $\mathfrak{D}$ *continua.*

Proof. Since the class of locally $\mathfrak{D}$ continua contains class $\mathfrak{D}$ and all the finite graphs, we only need to check that locally $\mathfrak{D}$ continua are meshed. Let X be a locally $\mathfrak{D}$ continuum. Clearly, X is a Peano continuum. By [**3**, Theorem 3.9], $\mathfrak{F}(X)$ is dense in $C(X)$, so Theorem 5 implies that X is meshed. □

3. Free arcs. A *free circle* S, in a continuum X, is a simple closed curve S in X such that there exists a $p \in S$ such that $S - \{p\}$ is open in X. A *maximal free arc* is a free arc in X which is maximal with respect to inclusion. Let

$$\mathfrak{A}(X) = \{J \subset X : J \text{ is a maximal free arc in } X\}$$

and

$$\mathfrak{A}_S(X) = \mathfrak{A}(X) \cup \{S \subset X : S \text{ is a free circle in } X\}.$$

A *simple triod* is a continuum T homeomorphic to the cone over the discrete space $\{1, 2, 3\}$. The point of T corresponding to the vertex of the cone is called the *vertex* of T.

Given an arc J in a continuum X and points x, y in J, let $[x, y]_J$ be the subarc of J joining x and y, if $x \neq y$, and $[x, y]_J = \{x\}$, if $x = y$. We also define $[x, y)_J = [x, y]_J - \{y\}$ and $(x, y)_J = [x, y]_J - \{x, y\}$.

The following lemma is easy to prove.

Lemma 7. *Let X be a continuum, and let J be a free arc in X. Then*:

(a) *no point of J° can be the vertex of a simple triod in X,*

(b) *if J and K are free arcs in X and $J^\circ \cap K^\circ \neq \varnothing$, then $J \cup K$ is a free arc or a free circle in X.*

Lemma 8. *For a Peano continuum X, let $\{J_m\}_{m=1}^{\infty}$ be a sequence of pairwise different elements of $\mathfrak{A}_S(X)$ and $x_m \in J_m$, for each $m \in \mathbf{N}$. If* $\lim x_m = x$ *for some $x \in X$, then* $\lim J_m = \{x\}$ *(in $C(X)$).*

Proof. Note that X is neither an arc nor a simple closed curve. For each $m \in \mathbf{N}$, $x_m \in \operatorname{cl}_X(J_m^\circ)$, so we may assume that $x_m \in J_m^\circ$. For each $m \in \mathbf{N}$, $\operatorname{Fr}_X(J_m)$ is a nonempty subset of X with at most two elements. Thus, we can put $\operatorname{Fr}_X(J_m) = \{p_m, q_m\}$. Suppose that the sequence $\{J_m\}_{m=1}^{\infty}$ does not converge to $\{x\}$ in $C(X)$. Since $C(X)$ is compact, there exists a subsequence of $\{J_m\}_{m=1}^{\infty}$ that converges to some $A \in C(X)$, where $A \neq \{x\}$. We may assume that $\lim J_m = A$,

$\lim p_m = p$ and $\lim q_m = q$, for some $p, q \in X$. Note that $p, q, x \in A$. Since $A \neq \{x\}$, we can choose an element $y \in A - \{p, q\}$. Then there exists a sequence $\{y_m\}_{m=1}^{\infty}$ in X such that $y_m \in J_m$, for each $m \in \mathbf{N}$ and $\lim y_m = y$. By [**14**, Lemma 3], $J_m^{\circ} \cap J_k^{\circ} = \varnothing$, if $m \neq k$. Thus, $y \notin J_m^{\circ}$ for every $m \in \mathbf{N}$. Let U be an open connected (then arcwise connected) set in X such that $y \in U$ and $p, q \notin \mathrm{cl}_X(U)$. Let $m_0 \in \mathbf{N}$ be such that, for each $m \geq m_0$, $y_m \in U$. For each $m \geq m_0$, let α_m be an arc in U with end points y_m and y. Since $y \notin J_m^{\circ}$, α_m contains one of the points p_m or q_m. This implies that $p \in \mathrm{cl}_X(U)$ or $q \in \mathrm{cl}_X(U)$, a contradiction. This completes the proof of the lemma. □

Lemma 9. *Let X be a Peano continuum and J a free arc with an end point e such that $e \in J^{\circ}$. Then there exists a free arc K such that $J \subset K$, e is an end point of K, $e \in K^{\circ}$ and K contains every free arc in X containing J.*

Proof. We may assume that X is not an arc. Let $\mathcal{F} = \{L \subset X : L$ be a free arc in X such that $J \subset L\}$. Given $L \in \mathcal{F}$, let p_L and q_L be the end points of L. We claim that $e \in \{p_L, q_L\}$. Suppose to the contrary that $e \notin \{p_L, q_L\}$. Since $e \in J^{\circ}$, there exist points $x, y \in L$ such that $e \in (x, y)_L \subset J$. This is a contradiction since e is an end point of J. Hence, $e \in \{p_L, q_L\}$, and we may assume that the end points of L are p_L and e. Since $e \in J^{\circ}$, we have that $e \in L^{\circ}$. Thus, $L - \{p_L\}$ is open in X.

By Lemma 7 (a), it follows that if $L, M \in \mathcal{F}$, then $L \subset M$ or $M \subset L$.

Let $U = \cup\{L - \{p_L\} : L \in \mathcal{F}\}$ and $K = \mathrm{cl}_X(U)$. We claim that $K \neq U$. Suppose to the contrary that $K = U$. Since K is compact and $L - \{p_L\}$ is open for each $L \in \mathcal{F}$, by the previous paragraph, there exists an $L \in \mathcal{F}$ such that $K = L - \{p_L\}$. This is impossible since $L - \{p_L\}$ is not compact. Hence, $K \neq U$. Fix a point $p \in K - U$. Since X is arcwise connected, there exists an arc M in X joining p and e.

We see that $K = M$. Let $L \in \mathcal{F}$ and $z \in L - \{e, p_L\}$. Then $X - \{z\} = (X - [z, e]_L) \cup (z, e]_L$ is a separation of $X - \{z\}$. Thus, z separates p and e in X. Hence, $z \in M$. We have shown that $L - \{e, p_L\} \subset M$. Therefore, $U \subset M$ and $K \subset M$. Since $p, e \in K$, we conclude that $K = M$. Thus, U is a connected subset of the arc M, $e \in U$ and $p \in \mathrm{cl}_X(U)$. This implies that $U = M - \{p\} = K - \{p\}$. Since U is open in X, we have that K is a free arc. Thus, $K \in \mathcal{F}$.

Given $L \in \mathcal{F}$, since K is closed in X and $L - \{p_L\} \subset K$, we have $L \subset K$. This completes the proof of the lemma. $\square$

Lemma 10. *Let X be a Peano continuum, and let J be a free arc. Then there exists a $K \in \mathfrak{A}_S(X)$ such that $J \subset K$.*

Proof. We may assume that X is not a simple closed curve and J is not contained in a free circle in X. Let x, y be the end points of J. Fix points $p, q \in (x, y)_J$ such that $[x, p]_J \cap [q, y]_J = \varnothing$. Let $Y = X - (p, q)_J$. Then Y is a compact subset of X. Let X_p and X_q be the components of Y containing p and q, respectively. Notice that $\mathrm{Fr}_X(Y) = \{p, q\}$, $[x, p]_J \subset X_p$ and $[q, y]_J \subset X_q$. By the boundary bumping theorem ([**23**, Theorem 5.4]), each component of Y contains either p or q. This implies that $Y = X_p \cup X_q$, and we have that either $X_p = X_q = Y$ or $X_p \cap X_q = \varnothing$. Clearly, Y is locally connected and each X_p and X_q are Peano continua. Notice that $[x, p]_J$ is a free arc of X_p and $p \in \mathrm{int}_{X_p}([x, p]_J)$. By Lemma 9, there exists a free arc K_p of X_p such that $[x, p]_J \subset K_p$, p is an end point of K_p, $p \in \mathrm{int}_{X_p}(K_p)$ and K_p contains every free arc in X_p containing $[x, p]_J$. Similarly, $[q, y]_J$ is a free arc of X_q, $q \in \mathrm{int}_{X_q}([q, y]_J)$, and there exists a free arc K_q of X_q such that $[q, y]_J \subset K_q$, q is an end point of K_q, $q \in \mathrm{int}_{X_q}(K_q)$ and K_q contains every free arc in X_q containing $[q, y]_J$. Let p_0 (respectively, q_0) be the other end point of K_p (respectively, K_q).

Since $[x, p]_J$ is a free arc of X_p and $p \in \mathrm{int}_{X_p}([x, p]_J)$, p is an end point of each arc in X_p containing p. If $p \in (q, q_0)_{K_q}$, then $p \in X_p \cap X_q$ and $X_p = X_q$. This implies that p is not an end point of the arc $[q, q_0]_{K_q} \subset X_p$, a contradiction. Hence, $p \notin (q, q_0)_{K_q}$. Since $\mathrm{Fr}_X(X_q) \subset \{p, q\}$, we have that $(q, q_0)_{K_q}$ is an open set in X_q such that $(q, q_0)_{K_q} \subset \mathrm{Int}_X(X_q)$. Hence, $(q, q_0)_{K_q}$ is open in X. Similarly, $(p, p_0)_{K_p}$ is open in X. Thus, K_p and K_q are free arcs in X. Since $\varnothing \neq (x, p)_J \subset K_p \cap [x, q]_J$ and J is not contained in a free circle in X, by Lemma 7 (b), $K_p \cup [x, q]_J = K_p \cup [p, q]_J$ is a free arc in X. Similarly, $K_q \cup [p, q]_J$ is a free arc in X. Applying again Lemma 7 (b), $K_p \cup [p, q]_J \cup K_q = K_p \cup J \cup K_q$ is a free arc in X with end points p_0 and q_0.

Suppose that L is a free arc in X such that $K_p \cup J \cup K_q \subset L$. Suppose that the end points of L are u and v and $[u, p_0]_L \cap [q_0, v]_L = \varnothing$. Then

$[u,p]_L \subset X - (p,q)_J$ and $[u,p]_L \subset X_p$. By the maximality of K_p, $[u,p]_L = K_p = [p_0,p]_L$. This implies that $u = p_0$. Similarly, $v = q_0$. Hence, $L = K_p \cup J \cup K_q$. We have shown that $K_p \cup J \cup K_q$ is maximal. This ends the proof of the lemma. □

Lemma 11. *Let X be a Peano continuum and $A \in C_n(X)$. Then* $\dim_A[C_n(X)] \geq 2n$ *and, if* $\dim_A[C_n(X)] = 2n$*, then there exist $k \in \mathbf{N}$ and elements $J_1, \dots, J_k \in \mathfrak{A}_S(X)$ such that $A \in \langle J_1^{\circ}, \dots, J_k^{\circ} \rangle$, where each component of A is contained in some J_i°.*

Proof. We may assume that $\dim_A[C_n(X)]$ is finite. Let $A_1, \dots, A_k$ be the components of A. By Theorem 4, there exists a finite graph D contained in X such that $A \subset D^{\circ}$. Then $C_n(D)$ is a neighborhood of A in $C_n(X)$. Thus, $\dim_A[C_n(X)] = \dim_A[C_n(D)]$. By [**21**, Theorem 2.4],

$$\dim_A[C_n(D)] = 2n + \sum_{x \in R(D) \cap A} (\operatorname{ord}_D(x) - 2),$$

where $R(D)$ is the set of ramification points of the graph D and $\operatorname{ord}_D(x)$ is the order of the point x in D. Since $\operatorname{ord}_D(x) \geq 3$ for each $x \in R(D)$, $\dim_A[C_n(X)] \geq 2n$ and, if $\dim_A[C_n(X)] = 2n$, then $R(D) \cap A = \varnothing$. Now, assume that $\dim_A[C_n(X)] = 2n$. Then, for each $i \in \{1, \dots, k\}$, there exists a free arc L_i in D such that $A_i \subset \operatorname{int}_D(L_i)$. Since $A \subset D^{\circ}$, $A_i \subset \operatorname{int}_X(L_i)$ so we may assume that $L_i \subset D^{\circ}$. This implies that L_i is a free arc in X. By Lemma 10, there exists a $J_i \in \mathfrak{A}_S(X)$ such that $L_i \subset J_i$. Therefore, $A \in \langle J_1^{\circ}, \dots, J_k^{\circ} \rangle$. □

4. Continua that are not almost meshed. Given a continuum X and a nonempty closed subset K of X, let

$$C_n^K(X) = \{A \in C_n(X) : K \subset A\},$$

and

$$C_n(X,K) = \{A \in C_n(X) : A \cap K \neq \varnothing\}.$$

Given A, $B \in 2^X$ such that $A \subsetneq B$, an *order arc from A to B* is a continuous function $\alpha : [0,1] \to 2^X$ such that $\alpha(0) = A$, $\alpha(1) = B$

and, if $0 \leq s < t \leq 1$, then $\alpha(s) \subsetneq \alpha(t)$. It is known (see [**19**, Lemma 15.2]) that if $A \subsetneq B$, then there exists an order arc from A to B if and only if each component of B intersects A. Given a closed subset $\mathfrak{G}$ of 2^X, we call $\mathfrak{G}$ a *growth hyperspace* provided that, for every $A \in \mathfrak{G}$ and $B \in 2^X$ such that $A \subsetneq B$ and each component of B intersects A, we have $B \in \mathfrak{G}$ (equivalently, there is an order arc from A to B). Note that the sets $C_n(X)$, $C_n^K(X) = \{A \in C_n(X) : K \subset A\}$ and $C_n(X, K) = \{A \in C_n(X) : A \cap K \neq \varnothing\}$ are growth hyperspaces. By the comments at the end of Section 2 of [**8**, Section 2], if X is a Peano continuum and $\mathfrak{G} \subset 2^X$ is a growth hyperspace, then $\mathfrak{G}$ is an AR.

A *compactum* is a compact metric space. A *map* is a continuous function. Given a compactum Y with metric d, a closed subset A of Y is said to be a *Z-set* in Y provided that, for each $\varepsilon > 0$, there is a continuous function $f_\varepsilon : Y \to Y - A$ such that $d(f_\varepsilon(y), y) < \varepsilon$ for all $y \in Y$. A continuous function between compacta $f : Y_1 \to Y_2$ is called a *Z-map* provided that $f(Y_1)$ is a Z-set in Y_2.

Given two disjoint continua X and Y, and points $p \in X$ and $y \in Y$, let $X \cup_p Y$ be the continuum obtained by attaching X to Y (identifying p to y).

Given a continuum X, a metric d for X is said to be *convex* provided that, for each of two points $p, q \in X$, there exists an isometry $\gamma : [0, d(p, q)] \to X$ such that $\gamma(0) = p$ and $\gamma(d(p, q)) = q$. It is known that X is a Peano continuum if and only if X admits a convex metric (see [**6, 22**]).

Given a continuum X, $\varepsilon > 0$ and $A \in 2^X$, define $C_d(\varepsilon, A)$, the *generalized closed d-ball in X of radius ε about A*, by $C_d(\varepsilon, A) = \{x \in X : d(x, A) \leq r\}$. If X is a Peano continuum with a convex metric d, then for every $A \in C_n(X)$ and $\varepsilon > 0$, $C_d(\varepsilon, A) \in C_n(X)$.

Definition 12. Given a Peano continuum X with convex metric d and $\varepsilon > 0$, define $\Phi_\varepsilon : 2^X \to 2^X$ by $\Phi_\varepsilon(A) = C_d(\varepsilon, A)$.

Remark 13. By [**19**, Proposition 10.5], Φ_ε is a map within ε of the identity map. Also notice that, if $\mathfrak{G}$ is a growth hyperspace, $A \in \mathfrak{G}$ and $\varepsilon > 0$, then $\Phi_\varepsilon(A) \in \mathfrak{G}$.

We will use the following characterization by Toruńczyk of the Hilbert cube ([**24**], see also [**19**, Theorem 9.3]).

Theorem 14 (Toruńczyk's theorem). *Let Y be an AR. If the identity map on Y is a uniform limit of Z-maps, then Y is a Hilbert cube.*

Lemma 15. *Let X be a Peano continuum, R a closed subset of $\mathcal{P}(X)$ and $K \in C(X)$ such that $\mathrm{int}_X(K) \cap R \neq \varnothing$. Then $C_n^K(X)$ is a Z-set of $C_n(X,R)$.*

Proof. Notice that $C_n^K(X)$ is a closed subset of $C_n(X,R)$. We show that, for each $\varepsilon > 0$, there is a map, $g_\varepsilon : C_n(X,R) \to C_n(X,R) - C_n^K(X)$ such that $H_X(g_\varepsilon(A), A) < \varepsilon$ for all $A \in C_n(X,R)$.

Let $\varepsilon > 0$, and fix a point $p \in \mathrm{int}_X(K) \cap R$. We may assume that $X \neq B(\varepsilon, p) \subset \mathrm{int}_X(K)$. By [**23**, Theorem 8.10], there exist an $m \in \mathbf{N}$ and Peano subcontinua $X_1, \ldots, X_m$ of X such that, for each $i \in \{1, \ldots, m\}$, $\mathrm{diameter}\,(X_i) < \varepsilon/4$ and $X = X_1 \cup \cdots \cup X_m$. We may assume that $\{i \in \{1, \ldots, m\} : p \in X_i\} = \{1, \ldots, r\}$ where $r < m$. Define the star of p by $\mathrm{St}(p) = X_1 \cup \cdots \cup X_r$. Notice that $\mathrm{St}\,(p) \subset \mathrm{int}_X(K)$.

Let $F = \{j \in \{1, \ldots, m\} : p \notin X_j \text{ and } X_j \cap \mathrm{St}\,(p) \neq \varnothing\}$. Since $\mathrm{St}\,(p) \neq X$ and $X = X_1 \cup \cdots \cup X_m$ is connected, it follows that $F \neq \varnothing$. For each $j \in F$, fix a point $p_j \in X_j \cap \mathrm{St}\,(p)$. Note that, by [**19**, Proposition 10.7], $\mathrm{St}\,(p)$ is a locally connected continuum, and therefore it is arcwise connected. Thus, it is possible to construct a tree $T \subset \mathrm{St}\,(p)$ such that $\{p_j : j \in F\} \subset T$ and $p \in T$. Hence, $T \cap X_j \neq \varnothing$ for each $j \in F$.

Let $Y = T \cup (\bigcup \{X_j : j \in F\})$. By [**19**, Proposition 10.7], Y is a Peano continuum, since $C(Y)$ is a growth hyperspace, $C(Y)$ is an AR. Notice that $Y \subset \mathrm{int}_X(K)$.

Let $Z = Y \cap R$. Notice that $p \in Z$ and $C(Y,Z)$ is an AR ($C(Y,Z)$ is a growth hyperspace).

Define $\alpha : Y \to C(Y)$ by $\alpha(y) = \{y\}$, and let $\beta : Z \to C(Y,Z)$ be given by $\beta(z) = \{z\}$. By [**19**, Theorem 9.1], β can be extended to a map $\overline{\beta} : (\mathrm{St}\,(p) \cup Y) \cap R \to C(Y,Z)$. Notice that $\overline{\beta}|_Z = \alpha|_Z$. Thus, the function $\alpha \cup \overline{\beta} : ((\mathrm{St}\,(p) \cup Y) \cap R) \cup Y \to C(Y)$ defined by

$$(\alpha \cup \overline{\beta})(x) = \begin{cases} \alpha(x) & \text{if } x \in Y, \\ \overline{\beta}(x) & \text{if } x \in (\mathrm{St}\,(p) \cup Y) \cap R, \end{cases}$$

is a well-defined map.

By [**19**, Theorem 9.1], we can extend $\alpha \cup \overline{\beta}$ to a map $\overline{\alpha} : \mathrm{St}\,(p) \cup Y \to C(Y)$.

Now extend $\overline{\alpha}$ to a function $\gamma : X \to C(X)$ by the formula

$$\gamma(x) = \begin{cases} \overline{\alpha}(x) & \text{if } x \in \mathrm{St}\,(p) \cup Y, \\ \{x\} & \text{if } x \in X - (\mathrm{St}\,(p) \cup Y). \end{cases}$$

Since $\mathrm{cl}_X(X - (\mathrm{St}\,(p) \cup Y)) \cap (\mathrm{St}\,(p) \cup Y) \subset \bigcup\{X_j : j \in F\} \subset Y$, γ is a well-defined map.

Notice that, if $x \in R \cap (\mathrm{St}\,(p) \cup Y)$, then $\gamma(x) = \overline{\alpha}(x) = (\alpha \cup \overline{\beta})(x) = \overline{\beta}(x) \in C(Y, Z)$. Therefore, γ has the following property:

$$(*) \qquad \text{For every } x \in R \cap (\mathrm{St}\,(p) \cup Y), \gamma(x) \cap R \neq \varnothing.$$

Define $g_\varepsilon : C_n(X) \to C_n(X)$ as $g_\varepsilon(A) = \bigcup\{\gamma(x) : x \in A\}$. Using [**7**, Lemma 2.2], it is easy to see that g_ε is a well-defined map.

Given $x \in \mathrm{St}\,(p) \cup Y$, since diameter $(\mathrm{St}\,(p) \cup Y) < \varepsilon$ and $\gamma(x) \subset Y$, we have that $H_X(\{x\}, \gamma(x)) < \varepsilon$. This implies that $H_X(A, g_\varepsilon(A)) < \varepsilon$ for each $A \in C_n(X)$.

Now we prove that g_ε maps $C_n(X, R)$ into $C_n(X, R) - C_n^K(X)$. Let $A \in C_n(X, R)$, and fix a point $a \in A \cap R$. If $a \in X - (\mathrm{St}(p) \cup Y)$, then $\gamma(a) = \{a\} \subset R$, so $g_\varepsilon(A) \in C_n(X, R)$. If $a \in \mathrm{St}\,(p) \cup Y$, then $a \in R \cap (\mathrm{St}\,(p) \cup Y)$. By property $(*)$, $\gamma(a) \cap R \neq \varnothing$, so $g_\varepsilon(A) \in C_n(X, R)$.

Notice that, by definition of $\mathcal{P}(X)$, p does not have a neighborhood homeomorphic to a finite graph. Since $\mathrm{St}(p) - (\bigcup\{X_j : j \in F\})$ is an open subset of X that contains p and is contained in $\mathrm{int}_X(K)$, we conclude that there exists a point $s \in (\mathrm{St}\,(p) - (\bigcup\{X_j : j \in F\})) - T \subset (\mathrm{St}\,(p) - Y) \cap K$. Thus, for every $x \in X$, we have that $s \notin \gamma(x)$. Therefore, $K \not\subset g_\varepsilon(B)$ for any $B \in C_n(X)$. Hence, $g_\varepsilon|_{C_n(X,R)} : C_n(X, R) \to C_n(X, R) - C_n^K(X)$ is the desired map, and the lemma is proved. $\square$

Theorem 16. *Let X be a Peano continuum and R a nonempty closed subset of $\mathcal{P}(X)$. Then $C_n(X, R)$ is a Hilbert cube.*

Proof. The proof is based on Toruńczyk's theorem (Theorem 14). Since $C_n(X, R)$ is a growth hyperspace, $C_n(X, R)$ is an AR. We verify the second assumption of Theorem 14 for $C_n(X, R)$. For this purpose, we assume that the metric for X is convex.

Let $\varepsilon > 0$. By Remark 13, $\Phi_\varepsilon|_{C_n(X,R)} : C_n(X, R) \to C_n(X, R)$ is a map within ε of the identity on $C_n(X, R)$. We only need to show that $\Phi_\varepsilon|_{C_n(X,R)}$ is a Z-map.

Since R is compact, there are finitely many points $p_1, \dots, p_s$ of R such that $R \subset C_d((\varepsilon/2), \{p_1\}) \cup \dots \cup C_d((\varepsilon/2), \{p_s\})$. For each $i \in \{1, \dots, s\}$, let $K_i = C_d((\varepsilon/2), \{p_i\})$. Since d is convex, K_i is a continuum and $p_i \in \mathrm{int}_X(K_i) \cap R$. Applying Lemma 15, we obtain that $C_n^{K_i}(X)$ is a Z-set in $C_n(X, R)$. By [**19**, Exercise 9.4], the set $\mathcal{G} = C_n^{K_1}(X) \cup \dots \cup C_n^{K_s}(X)$ is a Z-set in $C_n(X, R)$. By the choice of K_i, it is easy to see that, for each $A \in C_n(X, R)$, there exists a $j \in \{1, \dots, s\}$ such that $\Phi_\varepsilon(A) \in C_n^{K_j}(X)$. Therefore, $\Phi_\varepsilon(C_n(X, R)) \subset \mathcal{G}$.

Since a closed subset of a Z-set is a Z-set, we conclude that $\Phi_\varepsilon|_{C_n(X,R)}$ is a Z-map within ε of the identity map. Therefore, the second assumption of Theorem 14 has been verified, and we obtain that $C_n(X, R)$ is a Hilbert cube. □

Theorem 17 (Anderson's homogenetity theorem). *If $h : A \to B$ is a homeomorphism between Z-sets in a Hilbert cube $\mathcal{Q}$, then h extends to a homeomorphism of $\mathcal{Q}$ onto $\mathcal{Q}$.*

The proof of the following lemma is similar to the proof of Theorem 5.1 of [**2**].

Theorem 18. *Let X be a Peano continuum and $p \in X$. Then there exists an uncountable family $\mathcal{D}$ of pairwise non homeomorphic dendrites such that*:

(a) *for each $D \in \mathcal{D}$, D does not contain free arcs,*

(b) *the Peano continuum* $X \cup_p D$ *is not homeomorphic to* X, *and*

(c) *if* $B \neq D$ *are elements of* $\mathcal{D}$, *then* $X \cup_p B$ *and* $X \cup_p D$ *are not homeomorphic.*

Lemma 19. *Let* X, Y *and* D *be continua and* p *a point of* Y *such that* $Y = X \cup D$ *and* $X \cap D = \{p\}$. *Suppose that* E *is a closed subset of* X *that contains* p. *Then* $\mathrm{Fr}_{C_n(X)}(C_n(X,E)) = \mathrm{Fr}_{C_n(Y)}(C_n(Y, E \cup D))$.

Proof. It follows from the easy-to-prove following facts: $C_n(Y) - C_n(Y, E \cup D) = C_n(X) - C_n(X,E) \subset C_n(X)$ and $C_n(X) \cap C_n(Y, E \cup D) = C_n(X,E)$. □

Now, we are ready to prove the main results of this section.

Theorem 20. *Let* X *be a Peano continuum that is not almost meshed. Then, for every* $n \in \mathbf{N}$, X *does not have unique hyperspace* $C_n(X)$.

Proof. We assume that the metric for X is convex. Since X is not almost meshed, there exist a point $p \in \mathcal{P}(X)$ and an $\varepsilon > 0$ such that $B_{2\varepsilon}(p) \subset \mathcal{P}(X)$. Let $E = C_d(\varepsilon, \{p\})$. Notice that E is a continuum with the properties that $E = \mathrm{cl}_X(\mathrm{int}_X(E))$ and $E \subset \mathcal{P}(X)$. By Theorem 16, $C_n(X,E)$ is a Hilbert cube.

Let $Y = X \cup_p D$, where D is a locally connected continuum without free arcs. By Theorem 18 we can choose D in such a way that X and Y are not homeomorphic.

We show that $C_n(X)$ is homeomorphic to $C_n(Y)$. First notice that $E \cup D$ and Y satisfy the hypothesis of Lemma 16, and therefore $C_n(Y, E \cup D)$ is a Hilbert cube. Assume also that the metric for Y is convex.

Claim 1. $\mathrm{Fr}_{C_n(X)}(C_n(X,E))$ *is a* Z*-set of* $C_n(X,E)$ *and* $\mathrm{Fr}_{C_n(Y)}(C_n(Y, E \cup D))$ *is a* Z*-set of* $C_n(Y, E \cup D)$.

Let $\delta > 0$, and consider $\Phi_\delta|_{C_n(X,E)} : C_n(X,E) \to C_n(X,E)$ as in Definition 12. By Remark 13, $\Phi_\delta|_{C_n(X,E)}$ is within δ of the identity map. Since $E = \mathrm{cl}_X(\mathrm{int}_X E)$, if $A \in C_n(X,E)$, then $\Phi_\delta(A) \cap \mathrm{int}_X(E) \neq \varnothing$. Therefore, $\Phi_\delta(A) \notin \mathrm{Fr}_{C_n(X)}(C_n(X,E))$ and $\Phi_\delta|_{C_n(X,E)} : C_n(X,E) \to C_n(X,E) - (\mathrm{Fr}_{C_n(X)}(C_n(X,E)))$. We have proved that $\mathrm{Fr}_{C_n(X)}(C_n(X,E))$ is a Z-set in $C_n(X,E)$. The proof that $\mathrm{Fr}_{C_n(Y)}(C_n(Y,E\cup D))$ is a Z-set of $C_n(Y,E\cup D)$ is analogous, so the claim is proved.

By Lemma 19, the identity map id : $\mathrm{Fr}_{C_n(X)}(C_n(X,E)) \to \mathrm{Fr}_{C_n(Y)}$ $(C_n(Y,E\cup D))$ is a well-defined homeomorphism. By Claim 1 and Theorem 17, the identity map id can be extended to a homeomorphism $h_1 : C_n(X,E) \to C_n(Y,E\cup D)$. We define a homeomorphism $h : C_n(X) \to C_n(Y)$ as follows.

$$h(A) = \begin{cases} h_1(A) & \text{if } A \in C_n(X,E), \\ A & \text{if } A \in C_n(X) - C_n(X,E). \end{cases}$$

Hence, $C_n(X)$ is homeomorphic to $C_n(Y)$, and the theorem is proved. □

Corollary 21. *Let X be a Peano continuum that is not almost meshed. Then there exists an uncountable family $\mathcal{Y}$ of pairwise non-homeomorphic Peano continua such that*:

(a) *for each $Y \in \mathcal{Y}$, X is not homeomorphic to Y,*

(b) *for each $n \in \mathbf{N}$ and each $Y \in \mathcal{Y}$, $C_n(X)$ is homeomorphic to $C_n(Y)$.*

Proof. Let $\mathcal{D}$ be as in Theorem 18. Fix a point $p \in \mathrm{int}_X(\mathcal{P}(X))$. Let $\mathcal{Y} = \{X \cup_p D : D \in \mathcal{D}\}$. □

5. Almost meshed continua without unique hyperspace. In this section we show a class of almost meshed Peano continua that do not have unique hyperspace $C_n(X)$.

Theorem 22. *Let X be an almost meshed Peano continuum and $n \in \mathbf{N}$. Suppose that there exist a closed subset R of $\mathcal{P}(X)$ and pairwise disjoint nonempty open sets $U_1, \dots, U_{n+1}$ such that*:

(a) *$X - R = U_1 \cup \cdots \cup U_{n+1}$ and*

(b) *for each $i \in \{1, \dots, n+1\}$, $R \subset cl_X(U_i)$. Then X does not have a unique hyperspace $C_m(X)$ for every $m \leq n$.*

Proof. Let $m \leq n$. By Theorem 16, $C_m(X,R)$ is a Hilbert cube.

Fix a point $p \in R$, and let $Y = X \cup_p D$, where D is a locally connected continuum without free arcs. By Theorem 18, we can choose D in such a way that X and Y are not homeomorphic. We show that $C_m(X)$ is homeomorphic to $C_m(Y)$. Notice that $R \cup D$ is a closed subset of $\mathcal{P}(Y)$. By Theorem 16, $C_m(Y, R \cup D)$ is a Hilbert cube. Assume that the metrics for X and Y are convex.

Claim 2. $\mathrm{Fr}_{C_m(Y)}(C_m(Y, R \cup D))$ is a Z-set in $C_m(Y, R \cup D)$.

Let $\varepsilon > 0$, and consider the map $\Phi_\varepsilon|_{C_m(Y,R\cup D)} : C_m(Y, R \cup D) \to C_m(Y, R \cup D)$ of Definition 12. By Remark 13, $\Phi_\varepsilon|_{C_m(Y,R\cup D)}$ is within ε of the identity map, so we only have to prove that $\Phi_\varepsilon(C_m(Y, R\cup D))\cap \mathrm{Fr}_{C_m(Y)}(C_m(Y, R \cup D)) = \varnothing$.

Let $A \in C_m(Y, R \cup D)$.

Case 1. $A \cap R \neq \varnothing$. By (b), $\Phi_\varepsilon(A) \cap U_i \neq \varnothing$, for every $i \in \{1, \ldots, n + 1\}$. Consider a sequence $\{A_j\}_{j=1}^{\infty}$ of elements of $C_m(Y)$ such that $\lim A_j = \Phi_\varepsilon(A)$. Then there exists an $M \in \mathbf{N}$ such that, for each $j \geq M$ and every $i \in \{1, \ldots, n + 1\}$, $A_j \cap U_i \neq \varnothing$. Given $j \geq M$, since A_j has at most m components and $m < n + 1$, we have $A_j \cap (R \cup D) \neq \varnothing$. Thus, $A_j \in C_m(Y, R \cup D)$ and $\Phi_\varepsilon(A)$ cannot be approximated by continua that do not intersect $R \cup D$. Hence, $\Phi_\varepsilon(A) \notin \mathrm{Fr}_{C_m(Y)}(C_m(Y, R \cup D))$.

Case 2. $A \cap R = \varnothing$. In this case $p \notin A$ and $\Phi_\varepsilon(A) \cap (D - \{p\}) \neq \varnothing$. Since $D - \{p\}$ is open in Y, we have that $\Phi_\varepsilon(A) \notin \mathrm{Fr}_{C_m(Y)}(C_m(Y, R \cup D))$.

By Cases 1 and 2, we obtain that $\Phi_\varepsilon|_{C_m(Y,R\cup D)} : C_m(Y, R \cup D) \to C_m(Y, R \cup D) - (\mathrm{Fr}_{C_m(Y)}(C_m(Y, R \cup D)))$. This proves Claim 2. ◻

Claim 3. $\mathrm{Fr}_{C_m(X)}(C_m(X, R))$ is a Z-set in $C_m(X, R)$.

The proof is similar and easier to the one in Claim 2 since we only need to consider Case 1.

By Lemma 19, the identity map $\mathrm{id} : \mathrm{Fr}_{C_m(X)}(C_m(X, R)) \to \mathrm{Fr}_{C_m(Y)} (C_m(Y, R \cup D))$ is a homeomorphism. By Claims 2, 3 and Theorem 17, the identity map id can be extended to a homeomorphism $h_1 : C_m(X, R) \to C_m(Y, R \cup D)$. We define a homeomorphism

$h : C_m(X) \to C_m(Y)$ as follows.

$$h(A) = \begin{cases} h_1(A) & \text{if } A \in C_m(X, R), \\ A & \text{if } A \in C_m(X) - C_m(X, R). \end{cases}$$

Hence, $C_m(X)$ is homeomorphic to $C_m(Y)$, and the theorem is proved. □

Corollary 23. *Let X be an almost meshed Peano continuum such that $X - \mathcal{P}(X)$ is disconnected. Then X does not have a unique hyperspace $C(X)$.*

Proof. Suppose that $X - \mathcal{P}(X) = U \cup V$, where U and V are nonempty open disjoint subsets of X. Since X is almost meshed, $\text{int}_X(\mathcal{P}(X)) = \varnothing$. Thus, $X = \text{cl}_X(U) \cup \text{cl}_X(V)$ and $R = \text{cl}_X(U) \cap \text{cl}_X(V)$ is a nonempty closed subset of $\mathcal{P}(X)$. Let $W = X - \text{cl}_X(U)$ and $Z = X - \text{cl}_X(V)$. Hence, W and Z are nonempty open disjoint subsets of X such that $V \subset W$, $U \subset Z$ and $R \subset \text{cl}_X(W) \cap \text{cl}_X(Z)$. By Theorem 22, the corollary follows. □

Corollary 24. *Let X be an almost meshed Peano continuum satisfying the conditions of Theorem* 22. *Then there exists an uncountable family $\mathcal{Y}$ of pairwise non-homeomorphic Peano continua such that*:

(a) *for each $Y \in \mathcal{Y}$, X is not homeomorphic to Y*,

(b) *for each $Y \in \mathcal{Y}$ and each $m \leq n$, $C_m(X)$ is homeomorphic to $C_m(Y)$.*

Corollary 25. *Let X be a dendrite that is not a tree and $k = \sup\{\text{ord}_X(p) : p \in \mathcal{P}(X)\}$, notice $k \in \mathbf{N} \cup \{\omega\}$. Then for every $m < k$, X does not have a unique hyperspace $C_m(X)$.*

Proof. If X is not almost meshed, then by Theorem 20, X does not have unique hyperspace $C_m(X)$ for every $m \in \mathbf{N}$. If X is almost meshed and $m < k$, there exists a point $q \in \mathcal{P}(X)$ such that $\text{ord}_X(q) \geq m + 1$. Hence, X and the closed subset $\{q\}$ satisfy the conditions of Theorem 22 for m, and the corollary follows. □

6. Meshed continua have unique hyperspaces. Given a continuum X and $n \in \mathbf{N}$, let

$$\mathfrak{P}_n(X) = \{A \in C_n(X) : A \text{ has a neighborhood in } C_n(X) \text{ that is a } 2n\text{-cell}\},$$

$$\mathfrak{P}_n^{\partial}(X) = \{A \in C_n(X) : A \text{ has a neighborhood } \mathcal{M} \text{ in } C_n(X) \text{ that is a } 2n\text{-cell and } A \text{ belongs to the manifold boundary of } \mathcal{M}\},$$

and

$$\Gamma_n(X) = \{A \in C_n(X) - \mathfrak{P}_n(X) : A \text{ has a basis of open neighborhoods } \mathfrak{H} \text{ in } C_n(X) \text{ such that, for each } \mathcal{U} \in \mathfrak{H}, \dim \mathcal{U} = 2n \text{ and } \mathcal{U} \cap \mathfrak{P}_n(X) \text{ is arcwise connected}\}.$$

As usual, we denote $\mathfrak{P}(X) = \mathfrak{P}_1(X)$ and $\mathfrak{P}^{\partial}(X) = \mathfrak{P}_1^{\partial}(X)$.

Define

$$\mathfrak{A}_E(X) = \{J \in \mathfrak{A}(X) : \text{ there exists an end point } p \text{ of } J \text{ such that } p \in J^{\circ}\}.$$

In the case that $J \in \mathfrak{A}_E(X)$ and p is an end point of J such that $p \in J^{\circ}$, p is said to be an *extreme* of X.

Lemma 26. *Let X be a Peano continuum and $A \in C(X)$. Then the following are equivalent*:

(a) $A \in \mathfrak{P}^{\partial}(X)$,

(b) *there is a $J \in \mathfrak{A}_S(X)$ such that one of the following two conditions hold*: (1) $A = \{p\}$, *for some $p \in J^{\circ}$*, (2) $J \in \mathfrak{A}_E(X)$ *and there exists an extreme p of X such that $p \in A \subset J^{\circ}$.*

Proof. (a) $\Rightarrow$ (b). Suppose that $A \in \mathfrak{P}^{\partial}(X)$. Then $\dim_A[C(X)] = 2$. Lemma 11 implies that there exists a $J \in \mathfrak{A}_S(X)$ such that $A \subset J^{\circ}$. Let $\mathcal{M}$ be a 2-cell in $C(X)$ such that $A \in \operatorname{int}_{C(X)}(\mathcal{M}) \subset \operatorname{int}_{C(X)}(C(J))$ and A belongs to the boundary, as manifold, of $\mathcal{M}$. Thus, $\mathcal{M}$ is a neighborhood of A in $C(J)$. Since J is either an arc or a simple closed curve, by the geometric models of $C(J)$ constructed in [**19**, Examples 5.1 and 5.2], we obtain that one of the conditions (1) or (2) holds.

(b) $\Rightarrow$ (a). Let $J \in \mathfrak{A}_S(X)$ be such that $A \subset J^{\circ}$. Then $C(J)$ is a neighborhood of A in $C(X)$. By the models in [**19**, Examples 5.1 and 5.2], in both cases, (1) and (2), there exists a neighborhood $\mathcal{M}$ of A in $C(J)$ such that $\mathcal{M}$ is a 2-cell, A belongs to the boundary, as a manifold, of $\mathcal{M}$ and $\mathcal{M} \subset \mathrm{int}_{C(X)}(C(J))$. Then $\mathcal{M}$ is a neighborhood of A in $C(X)$. Therefore, $A \in \mathfrak{P}^{\partial}(X)$. $\square$

Theorem 27. *Let X be a Peano continuum that is not an arc. Then there exists a homeomorphism $h : \mathrm{cl}_X(\mathcal{FA}(X)) \to \mathrm{cl}_{C(X)}(\mathfrak{P}^{\partial}(X))$ such that $h(p) = \{p\}$ for each $p \in \mathrm{cl}_X(\mathcal{FA}(X)) - \bigcup\{J^{\circ} : J \in \mathfrak{A}_E(X)\}$ and, if $h(p) \cap \mathcal{P}(X) \neq \varnothing$, then $p \in \mathcal{P}(X)$ or p is an end point of J, for some $J \in \mathfrak{A}_E(X)$, where $J \cap \mathcal{P}(X) \neq \varnothing$ and $p \in J^{\circ}$.*

Proof. By [**19**, Example 5.2], we can assume that X is not a simple closed curve.

Given $J \in \mathfrak{A}_E(X)$, let p_J and q_J be the end points of J, where $p_J \in J^{\circ}$. Since X is not an arc, $q_J \notin J^{\circ}$. Fix a homeomorphism $h_J : [0,1] \to J$ such that $h_J(0) = q_J$ and $h_J(1) = p_J$.

Let

$$W = \bigcup\{J - \{q_J\} : J \in \mathfrak{A}_E(X)\}.$$

Then W is an open subset of X and $W \subset \mathcal{FA}(X)$.

Define $h : \mathrm{cl}_X(\mathcal{FA}(X)) \to \mathrm{cl}_{C(X)}(\mathfrak{P}^{\partial}(X))$ as follows:

$$h(p) = \begin{cases} \{p\} & \text{if } p \in \mathrm{cl}_X(\mathcal{FA}(X)) - W, \\ \{h_J(2s)\} & \text{if } p \in J \in \mathfrak{A}_E(X),\ p = h_J(s) \\ & \text{and } s \in [0, 1/2], \\ h_J([-2s+2, 1]) & \text{if } p \in J \in \mathfrak{A}_E(X),\ p = h_J(s) \\ & \text{and } s \in [1/2, 1]. \end{cases}$$

Using Lemma 26 it can be shown that h is a well-defined function. Clearly, h is continuous at each point of W. Thus, in order to conclude that h is continuous, take a sequence $\{x_m\}_{m=1}^{\infty}$ of points of W such that $\lim x_m = x$ for some $x \notin W$. We need to show that $\lim h(x_m) = \{x\}$.

For each $m \in \mathbf{N}$, let $J_m \in \mathfrak{A}_E(X)$ be such that $x_m \in J_m$. We may assume that $J_m \neq J_k$, if $m \neq k$, and that $\lim p_{J_m} = q$, for some $q \in X$. By Lemma 8, $\lim J_m = \{q\}$. Since $h(x_m) \subset J_m$ and $x_m \in J_m$ for each $m \in \mathbf{N}$, we have that $\lim h(x_m) = \{q\}$ and $\lim x_m = q$. Therefore, $q = x$ and $\lim h(x_m) = \{x\}$. This completes the proof that h is continuous.

It is easy to see that h is one-to-one. In order to show that h is onto, note that, by Lemma 26, $\mathfrak{P}^{\partial}(X) \subset h(\mathrm{cl}_X(\mathcal{FA}(X)))$. Hence, $\mathrm{cl}_{C(X)}(\mathfrak{P}^{\partial}(X)) \subset h(\mathrm{cl}_X(\mathcal{FA}(X)))$. Thus, h is onto.

Finally, take $p \in \mathrm{cl}_X(\mathcal{FA}(X))$ such that $h(p) \cap \mathcal{P}(X) \neq \varnothing$. In the case that $h(p) = \{p\}$, we obtain that $p \in \mathcal{P}(X)$. In the case that $h(p) \neq \{p\}$, then $p \in J - \{q_J\} = J^{\circ}$ for some $J \in \mathfrak{A}_E(X)$. Since $h(p) \cap \mathcal{P}(X) \neq \varnothing$, $h(p) \not\subseteq J^{\circ}$. Hence, $h(p) = J = h_J([0,1])$ and we are done. ◻

Lemma 28. *Let X be a Peano continuum and $n \geq 3$. Then $\Gamma_n(X) = \{A \in C_n(X) : A$ is connected and there exists a $J \in \mathfrak{A}_S(X)$ such that $A \subset J^{\circ}\} = \mathfrak{P}(X)$.*

Proof. Let $A \in \Gamma_n(X)$. By Lemma 11 and Theorem 4, $\dim_A[C_n(X)] = 2n$, there exist a $k \in \mathbf{N}$, elements $J_1, \dots, J_k \in \mathfrak{A}_S(X)$ such that $A \in \langle J_1^{\circ}, \dots, J_k^{\circ} \rangle$ and a finite graph D in X such that $A \subset D^{\circ}$. Then $C_n(D)$ is a neighborhood of A in $C_n(X)$. Thus, we may assume that the basis of open neighborhoods $\mathfrak{H}$ in the definition of $\Gamma_n(X)$ satisfies that, for each $\mathcal{U} \in \mathfrak{H}$, $\mathcal{U} \subset C_n(D)$. Hence, $\mathfrak{H}$ is a basis of neighborhoods of A in $C_n(D)$ such that, for each $\mathcal{U} \in \mathfrak{H}$, $\dim \mathcal{U} = 2n$ and $\mathcal{U} \cap \mathfrak{P}_n(X)$ is arcwise connected. Given $\mathcal{U} \in \mathfrak{H}$ and $B \in \mathcal{U} \cap \mathfrak{P}_n(X)$, B has a neighborhood $\mathcal{M}$ in $C_n(X)$ that is a $2n$-cell. Then there exists an $2n$-cell $\mathcal{N} \subset \mathcal{M}$ such that $B \in \mathrm{int}_{C_n(X)}(\mathcal{N}) \subset \mathcal{N} \subset \mathcal{U} \cap \mathcal{M} \subset C_n(D)$. Thus, $\mathcal{N}$ is a $2n$-cell that is a neighborhood of B in $C_n(D)$. Hence, $B \in \mathcal{U} \cap \mathfrak{P}_n(D)$. We have shown that $\mathcal{U} \cap \mathfrak{P}_n(X) \subset \mathcal{U} \cap \mathfrak{P}_n(D)$. The other inclusion is easy to prove. Hence, $\mathcal{U} \cap \mathfrak{P}_n(X) = \mathcal{U} \cap \mathfrak{P}_n(D)$ and $\mathcal{U} \cap \mathfrak{P}_n(D)$ is arcwise connected. Since $A \in \mathcal{U} - \mathfrak{P}_n(X) = \mathcal{U} - \mathfrak{P}_n(D)$, we have proved that $A \in \Gamma_n(D)$. By [**17**, Lemma 3.6], A is connected, and we may assume that $A \subset J_1^{\circ}$.

Now suppose that $A \in C_n(X)$ is such that A is connected and there exists a $J \in \mathfrak{A}_S(X)$ such that $A \subset J^{\circ}$. By [**17**, Lemma 3.6], $A \in C_n(J) - \mathfrak{P}_n(J)$ and A has a basis of open neighborhoods $\mathfrak{H}$ in $C_n(J)$ such that, for each $\mathcal{U} \in \mathfrak{H}$, $\dim \mathcal{U} \leq 2n$ (then $\dim \mathcal{U} = 2n$, by Lemma 11) and $\mathcal{U}\cap\mathfrak{P}_n(J)$ is arcwise connected. Since $A\in \text{int}_{C_n(X)}(C_n(J))$, we can take $\mathcal{U}\subset \text{int}_{C_n(X)}(C_n(J))$ so that $\mathcal{U}$ is open in $C_n(X)$ for each $\mathcal{U}\in\mathfrak{H}$. Proceeding as in the previous paragraph, $\mathcal{U}\cap\mathfrak{P}_n(X)=\mathcal{U}\cap\mathfrak{P}_n(J)$ for each $\mathcal{U}\in\mathfrak{H}$. This implies that $A\in\Gamma_n(X)$.

The equality $\mathfrak{P}(X)=\{A\in C_n(X): A$ is connected, and there exists a $J\in\mathfrak{A}_S(X)$ such that $A\subset J^{\text{o}}\}$ follows from [**19**, Examples 5.1 and 5.2] and Lemma 11. □

Theorem 29. *If X and Y are almost meshed Peano continua, $n\geq 3$ and $C_n(X)$ is homeomorphic to $C_n(Y)$, then X is homeomorphic to Y.*

Proof. By [**17**, Theorem 3.8], we may assume that X and Y are not arcs. Let $h: C_n(X)\to C_n(Y)$ be a homeomorphism. Notice that the definition of $\Gamma_n(X)$ is given in terms of topological concepts that are preserved under homeomorphisms. Thus, $h(\Gamma_n(X))=\Gamma_n(Y)$ and $h(\mathfrak{P}(X))=\mathfrak{P}(Y)$. Note that $\mathfrak{P}(X)$ is an open subset of $C(X)$ and $\mathfrak{P}^{\partial}(X)\subset\mathfrak{P}(X)$. Thus, $\mathfrak{P}^{\partial}(X)=\{A\in\mathfrak{P}(X): A$ has a neighborhood $\mathcal{M}$ in $\mathfrak{P}(X)$ that is a 2-cell and A belongs to the manifold boundary of $\mathcal{M}\}$. It follows that $h(\mathfrak{P}^{\partial}(X))=\mathfrak{P}^{\partial}(Y)$. Hence, $h|\text{cl}_{C(X)}(\mathfrak{P}^{\partial}(X)):\text{cl}_{C(X)}(\mathfrak{P}^{\partial}(X))\to\text{cl}_{C(Y)}(\mathfrak{P}^{\partial}(Y))$ is a homeomorphism. Theorem 27 implies that $\text{cl}_X(\mathcal{FA}(X))$ is homeomorphic to $\text{cl}_Y(\mathcal{FA}(Y))$. By Lemma 1, $\text{cl}_X(\mathcal{G}(X))$ is homeomorphic to $\text{cl}_Y(\mathcal{G}(Y))$. Since X and Y are almost meshed, we conclude that X is homeomorphic to Y. □

Theorem 30. *If X and Y are almost meshed Peano continua which are not arcs and $C(X)$ is homeomorphic to $C(Y)$, then X is homeomorphic to Y.*

Proof. Let $h: C(X)\to C(Y)$ be a homeomorphism. Notice that $h(\mathfrak{P}(X))=\mathfrak{P}(Y)$. Proceeding as in the proof of Theorem 29, we conclude that X is homeomorphic to Y. □

In Theorem 35 we will extend the conclusions of Theorems 29 and 30 to the case $n=2$. As in the previous results on finite graphs and class $\mathfrak{D}$, this case is more difficult and requires a different technique. We will use the following conventions.

Given a continuum X that is not a simple closed curve and $J,K\in\mathfrak{A}_S(X)$, let

$$\mathcal{D}(J,K)=\text{cl}_{C_2(X)}(\mathfrak{P}_2^{\partial}(X)\cap\langle J^{\text{o}},K^{\text{o}}\rangle)\cap\text{cl}_{C_2(X)}(\mathfrak{P}_2^{\partial}(X)-\langle J^{\text{o}},K^{\text{o}}\rangle).$$

In the case that J is an arc, let p_J and q_J be its end points, where $q_J \in \mathrm{Fr}_X(J)$. If J is a simple closed curve, let q_J be the unique point in J such that $J - \{q_J\}$ is open. Since X is not a simple closed curve, $q_J \notin J^{\circ}$. Given $J \in \mathfrak{A}_S(X)$, define $\mathcal{E}(J)$ in the following way: If J is an arc, let $\mathcal{E}(J) = C(J)$. In the case that J is a simple closed curve, let $\mathcal{E}(J) = \{A \in C(J) : A = J$ or $A = \{p\}$ for some $p \in J$ or A is a subarc of J such that $q_J \notin A$ or A is a subarc of J such that q_J is one of its end points$\}$. Note that, in both cases, $\mathcal{E}(J) = \mathrm{cl}_{C(X)}(\langle J^{\circ}\rangle \cap C(X))$. Let W_0 be the continuum obtained as $W_0 = D - \mathrm{int}_{\mathbf{R}^2}(E)$, where D and E are discs in the plane $\mathbf{R}^2$, $E \subsetneq D$, and E and D are tangents. The following lemma can be easily proved from [**19**, Examples 5.1 and 5.2].

Lemma 31. *Let X be a continuum that is not a simple closed curve and $J \in \mathfrak{A}_S(X)$. Then*:

(a) *if J is an arc, then $\mathcal{E}(J)$ is a 2-cell*,

(b) *if J is a simple closed curve, then $\mathcal{E}(J)$ is homeomorphic to W_0* (*where the point of tangency corresponds to $\{q_J\}$*).

Lemma 32. *Let X be a Peano continuum. Then $\mathfrak{P}_2^{\partial}(X) = \{A \in \mathfrak{P}_2(X) : A$ is connected or A has a degenerate component or A contains an extreme of $X\}$.*

Proof. By Lemma 11, $\mathfrak{P}_2(X) \subset \bigcup\{\langle J^{\circ}, K^{\circ}\rangle : J, K \in \mathfrak{A}_S(X)\}$, and by [**18**, Lemma 2.1], for every $J, K \in \mathfrak{A}_S(Y)$, $\langle J^{\circ}, K^{\circ}\rangle$ is a component of $\mathfrak{P}_2(X)$. Using Lemma 7, it can be shown that if $J, K, L, M \in \mathfrak{A}_S(X)$ and $\{J, K\} \neq \{L, M\}$, then $\langle J^{\circ}, K^{\circ}\rangle \cap \langle L^{\circ}, M^{\circ}\rangle = \varnothing$. Thus, the components of $\mathfrak{P}_2(X)$ are sets of the form $\langle J^{\circ}, K^{\circ}\rangle$, where $J, K \in \mathfrak{A}_S(X)$.

Given $J \in \mathfrak{A}_S(X)$, let $C(J^{\circ}) = C(X) \cap \langle J^{\circ}\rangle$ and $\mathfrak{P}^{\partial}(J^{\circ}) = \{A \in C(J^{\circ}) : A$ has a neighborhood $\mathcal{M}$ in $C(J^{\circ})$ such that $\mathcal{M}$ is a 2-cell and A is in the manifold boundary of $\mathcal{M}\}$. Notice that J° is homeomorphic to $(0,1)$ when $J \notin \mathfrak{A}_E(X)$ and J° is homeomorphic to $[0,1)$ when $J \in \mathfrak{A}_E(X)$. By [**19**, Example 5.1], $C(J^{\circ})$ is homeomorphic to $[0,1) \times [0,1)$. In the case that $J \notin \mathfrak{A}_E(X)$, $\mathfrak{P}^{\partial}(J^{\circ}) = \{\{p\} : p \in J^{\circ}\}$ and, in the case that $J \in \mathfrak{A}_E(X)$ and p_J is the extreme of X contained in J, $\mathfrak{P}^{\partial}(J^{\circ}) = \{\{p\} : p \in J^{\circ}\} \cup \{A \in C(J^{\circ}) : p_J \in A\}$.

If $J \neq K$, then $J^o \cap K^o = \varnothing$. Let $\varphi : C(J^o) \times C(K^o) \to \langle J^o, K^o \rangle$ be given by $\varphi(B, C) = B \cup C$. It is easy to show that φ is a homeomorphism and $\mathfrak{P}_2^\partial(X) \cap \langle J^o, K^o \rangle = \varphi((\mathfrak{P}^\partial(J^o) \times C(K^o)) \cup (C(J^o) \times \mathfrak{P}^\partial(K^o))) = \{A \in \langle J^o, K^o \rangle : A \cap J^o \in \mathfrak{P}^\partial(J^o)$ or $A \cap K^o \in \mathfrak{P}^\partial(K^o)\} = \{A \in \langle J^o, K^o \rangle$: A has a degenerate component or A contains an extreme of $X\}$.

If $J = K$, $\langle J^o, K^o \rangle = \langle J^o \rangle = \{A \in C_2(J) : A \subset J^o\}$. In [**16**, Lemma 2.2], the following model (due to R.M. Schori) for $C_2([0,1])$ was constructed. Let $\mathcal{C}_0 = \{A \in C_2([0,1]) : 0 \in A\}$ and $\mathcal{C}_0^1 = \{A \in C_2([0,1]) : \{0,1\} \subset A\} = \{[0,a] \cup [b,1] : 0 \leq a \leq b \leq 1\}$. Then $\mathcal{C}_0^1$ is homeomorphic to the space obtained by identifying the diagonal of the triangle $\{(a,b) \in \mathbf{R}^2 : 0 \leq a \leq b \leq 1\}$ to a point. Thus, $\mathcal{C}_0^1$ is a 2-cell, and the manifold boundary of $\mathcal{C}_0^1$ is the set $\partial(\mathcal{C}_0^1) = \{\{0\} \cup [b,1] : 0 \leq b \leq 1\} \cup \{[0,a] \cup \{1\} : 0 \leq a \leq 1\} \cup \{[0,1]\}$. The function $\eta :$ cone$(\mathcal{C}_0^1) \to \mathcal{C}_0$ given by $\eta((A,t)) = (1-t)A$ is a homeomorphism. Thus, $\mathcal{C}_0$ is a 3-cell, and its manifold boundary is the set $\partial(\mathcal{C}_0) = \mathcal{C}_0^1 \cup \{(1-t)A : A \in \partial(\mathcal{C}_0^1)$ and $t \in [0,1]\}$. Finally, the function λ: cone$(\mathcal{C}_0) \to C_2([0,1])$ given by $\lambda((A,t)) = \{t\} + (1-t)A$ is a homeomorphism. Thus, $C_2([0,1])$ is a 4-cell and its manifold boundary is the set $\partial(C_2([0,1])) = \mathcal{C}_0 \cup \{\{t\} + (1-t)A : A \in \partial(\mathcal{C}_0)$ and $t \in [0,1]\}$. Therefore, $\partial(C_2([0,1])) = \{A \in C_2([0,1]) : A$ is connected or A has a degenerate component or $A \cap \{0,1\} \neq \varnothing\}$.

In the case that $J \notin \mathfrak{A}_E(X)$, J^o is homeomorphic to $(0,1)$, so $\mathfrak{P}_2^\partial(X) \cap \langle J^o \rangle = \{A \in C_2(J^o) : A$ is connected or A has a degenerate component$\}$, and in the case that $J \in \mathfrak{A}_E(X)$, J^o is homeomorphic to $[0,1)$, so $\mathfrak{P}_2^\partial(X) \cap \langle J^o \rangle = \{A \in C_2(J^o) : A$ is connected or A has a degenerate component or the extreme of X contained in J belongs to $A\}$. Therefore, for all $J \in \mathfrak{A}_S(Y)$, $\mathfrak{P}_2^\partial(X) \cap \langle J^o \rangle = \{A \in \langle J^o \rangle : A$ is connected or A has a degenerate component or A contains an extreme of $X\}$. This completes the proof of the lemma. ◻

Lemma 33. *Let X be a Peano continuum. Let $J, K \in \mathfrak{A}_S(X)$ be such that* $\mathrm{Fr}_X(J) \subset \mathrm{cl}_X(\mathcal{FA}(X) - J)$ *and* $\mathrm{Fr}_X(K) \subset \mathrm{cl}_X(\mathcal{FA}(X) - K)$. *Then $\mathcal{D}(J,K) = \{\{p\} \cup A : p \in \mathrm{Fr}_X(J)$ and $A \in \mathcal{E}(K)$ or $p \in \mathrm{Fr}_X(K)$ and $A \in \mathcal{E}(J)\}$.*

Proof. ($\subset$). Let $B \in \mathcal{D}(J,K)$. Since $\mathfrak{P}_2^\partial(X) \cap \langle J^o, K^o \rangle \subset \langle J, K \rangle$ and $\langle J, K \rangle$ is closed in $C_2(X)$, $B \in \langle J, K \rangle$.

The first case we consider is when B is disconnected. Let B_1 and B_2 be the components of B. Given a sequence $\{E_m\}_{m=1}^{\infty}$ of elements of $C_2(X)$ such that $\lim E_m = B$, we may assume that each E_m has two components $E_m^{(1)}$ and $E_m^{(2)}$, $\lim E_m^{(1)} = B_1$ and $\lim E_m^{(2)} = B_2$. Since $B \in \mathrm{cl}_{C_2(X)}(\langle J^{\circ}, K^{\circ}\rangle)$, there exists a sequence $E_m = E_m^{(1)} \cup E_m^{(2)}$ of elements of $\langle J^{\circ}, K^{\circ}\rangle$ such that $\lim E_m^{(1)} = B_1$ and $\lim E_m^{(2)} = B_2$. In the case that $J = K$, we have that $E_m \subset J$, for each $m \in \mathbf{N}$ and $B \subset J = K$. In the case that $J \neq K$, $J^{\circ} \cap K^{\circ} = \varnothing$, so we can assume that $E_m^{(1)} \subset J^{\circ}$ and $E_m^{(2)} \subset K^{\circ}$ for each $m \in \mathbf{N}$. This implies that $B_1 \subset J$ and $B_2 \subset K$. So, in both cases ($J = K$ or $J \neq K$), we may assume that $B_1 \subset J$ and $B_2 \subset K$. Since $B \in \mathrm{cl}_{C_2(X)}(\mathfrak{P}_2^{\partial}(X) - \langle J^{\circ}, K^{\circ}\rangle)$, there is also a sequence $F_m = F_m^{(1)} \cup F_m^{(2)}$ of elements of $\mathfrak{P}_2^{\partial}(X) - \langle J^{\circ}, K^{\circ}\rangle$ such that $\lim F_m^{(1)} = B_1$ and $\lim F_m^{(2)} = B_2$. Since $\mathfrak{P}_2^{\partial}(X) \subset \mathfrak{P}_2(X)$, for each $m \in \mathbf{N}$, there exist $L_m, M_m \in \mathfrak{A}_S(X)$ such that $\{L_m, M_m\} \neq \{J, K\}$ and $F_m \in \langle L_m^{\circ}, M_m^{\circ}\rangle$. We may assume that $F_m^{(1)} \subset L_m^{\circ}$, $F_m^{(2)} \subset M_m^{\circ}$ and $K \neq M_m$. Then $B_2 \subset \mathrm{Fr}_X(K)$. Since $\mathrm{Fr}_X(K)$ has at most two elements, we conclude that B_2 is degenerate. If J is an arc, then B is of the form $B = \{p\} \cup B_1$, where $B_1 \in \mathcal{E}(J)$ and $p \in \mathrm{Fr}_X(K)$. If J is a simple closed curve, since $E_m^{(1)} \subset J^{\circ} = J - \{q_J\}$ for each $m \in \mathbf{N}$, $B_1 = \lim E_m^{(1)}$ is either equal to J or $B_1 = \{p\}$ for some $p \in J$ or B_1 is a subarc of J that has q_J as one of its end points or B_1 is a subarc of J such that $q_J \notin J$. Thus, $B_1 \in \mathcal{E}(J)$.

Now, we consider the case when B is connected. If $J \neq K$, we claim that $B \cap J^{\circ} = \varnothing$ or $B \cap K^{\circ} = \varnothing$. Suppose, to the contrary, that $B \cap J^{\circ} \neq \varnothing$ and $B \cap K^{\circ} \neq \varnothing$. Since $B \in \mathrm{cl}_{C_2(X)}(\mathfrak{P}_2^{\partial}(X) - \langle J^{\circ}, K^{\circ}\rangle)$, there is a sequence $\{E_m\}_{m=1}^{\infty}$ of elements of $\mathfrak{P}_2^{\partial}(X) - \langle J^{\circ}, K^{\circ}\rangle$ such that $\lim E_m = B$. For each $m \in \mathbf{N}$, there exist $L_m, M_m \in \mathfrak{A}_S(X)$ such that $\{L_m, M_m\} \neq \{J, K\}$ and $E_m \in \langle L_m^{\circ}, M_m^{\circ}\rangle$. Since J° and K° are open in X, there exists an $m_0 \in \mathbf{N}$ such that, for each $m \geq m_0$, E_m intersects J° and K°. Then $L_m \cup M_m$ intersects J° and K°. If L_m intersects J°, then $L_m = J$. Thus, for each $m \geq m_0$, we may suppose that $L_m = J$ and $M_m = K$. Hence, $\{L_m, M_m\} = \{J, K\}$, a contradiction. We have shown that $B \cap J^{\circ} = \varnothing$ or $B \cap K^{\circ} = \varnothing$. Suppose, for example, that $B \cap J^{\circ} = \varnothing$. Since $B \in \langle J, K\rangle$, $B = (B \cap J) \cup (B \cap K)$ and $\varnothing \neq B \cap J$. This implies that $B \cap J$ is a nonempty subset of $J - J^{\circ}$ which consists of at most two elements. Since $B \cap J$ and $B \cap K$ are closed in B and B is connected, we have that $B \cap J \subset B \cap K$. Hence, $B \subset K$. Fix a point

$p \in B \cap J$. If K is an arc, then B is of the form $B = \{p\} \cup B$, where $B \in \mathcal{E}(K)$ and $p \in \mathrm{Fr}_X(J)$. Now suppose that K is a simple closed curve. Since $B \in \mathrm{cl}_{C_2(X)}(\langle J^{\mathrm{o}}, K^{\mathrm{o}} \rangle)$, there exists a sequence $\{B_m\}_{m=1}^{\infty}$ in $\langle J^{\mathrm{o}}, K^{\mathrm{o}} \rangle$ such that $\lim B_m = B$. Thus, the components of B_m are $B_m \cap J^{\mathrm{o}}$, $B_m \cap K^{\mathrm{o}}$ and $B = \lim((B_m \cap J^{\mathrm{o}}) \cup (B_m \cap K^{\mathrm{o}}))$. We may suppose that the sequences $\{B_m \cap J^{\mathrm{o}}\}_{m=1}^{\infty}$ and $\{B_m \cap K^{\mathrm{o}}\}_{m=1}^{\infty}$ are convergent in $C(X)$. Recall that $B \cap J$ has at most two elements. If $q \in B$ and $q = \lim q_m$, where $q_m \in B_m \cap J^{\mathrm{o}}$, for each $m \in \mathbf{N}$, then $q \in \mathrm{Fr}_X(J)$. Thus, there are at most two points q of B of this form. So $\lim(B_m \cap J^{\mathrm{o}})$ is a one-point set. This implies that $B = \lim(B_m \cap K^{\mathrm{o}})$. Given $m \in \mathbf{N}$, since $B_m \cap K^{\mathrm{o}}$ is a connected subset of $K^{\mathrm{o}} = K - \{q_K\}$, we have that $B_m \cap K^{\mathrm{o}}$ is an arc such that $q_K \notin B_m \cap K^{\mathrm{o}}$. Hence, $B = \lim(B_m \cap K^{\mathrm{o}}) \in \mathcal{E}(K)$. Therefore, $B = \{p\} \cup B$, where $p \in \mathrm{Fr}_X(J)$ and $B \in \mathcal{E}(K)$.

Finally, we consider the case when B is connected and $J = K$. Since $B \in \mathrm{cl}_{C_2(X)}(\mathfrak{P}_2^{\partial}(X) - \langle J^{\mathrm{o}} \rangle)$, B is limit of elements in $\mathfrak{P}_2^{\partial}(X) - \langle J^{\mathrm{o}} \rangle$ and $B \subset J$. Thus, $B \not\subseteq J^{\mathrm{o}}$. Hence, we can fix a point $p \in B \cap \mathrm{Fr}_X(J)$. If J is an arc, $B = \{p\} \cup B$ and $B \in \mathcal{E}(J)$. If J is a simple closed curve, let $B = \lim E_m$, where $E_m \in \langle J^{\mathrm{o}} \rangle \cap \mathfrak{P}_2^{\partial}(X)$ for each $m \in \mathbf{N}$. For each $m \in \mathbf{N}$, by Lemma 32, E_m is connected or E_m has a degenerate component. In both cases, we can write $E_m = \{p_m\} \cup F_m$, where $F_m \in C(J^{\mathrm{o}})$. Note that $\lim F_m = B$. Since F_m is a connected subset of $J^{\mathrm{o}} = J - \{q_J\}$, we have that F_m is an arc such that $q_J \notin F_m$. Hence, $B = \lim F_m \in \mathcal{E}(J)$. Therefore, $B = \{p\} \cup B$, where $p \in \mathrm{Fr}_X(J)$ and $B \in \mathcal{E}(J)$.

($\supset$). Let $B = \{p\} \cup A$, where $p \in \mathrm{Fr}_X(J) \subset \mathrm{cl}_X(\mathcal{FA}(X) - J)$ and $A \in \mathcal{E}(K)$. Notice that, in both cases: K being an arc and K being a simple closed curve, $A = \lim A_m$, where $A_m \in K^{\mathrm{o}}$ for each $m \in \mathbf{N}$. Given $m \in \mathbf{N}$, there exists a point $p_m \in B(1/m, p) \cap \mathcal{FA}(X) - J$. Note that $\{p_m\} \cup A_m \notin \langle J^{\mathrm{o}}, K^{\mathrm{o}} \rangle$. By Lemma 32, $\{p_m\} \cup A_m \in \mathfrak{P}_2^{\partial}(X) - \langle J^{\mathrm{o}}, K^{\mathrm{o}} \rangle$. Then $B = \lim(\{p_m\} \cup A_m) \in \mathrm{cl}_{C_2(X)}(\mathfrak{P}_2^{\partial}(X) - \langle J^{\mathrm{o}}, K^{\mathrm{o}} \rangle)$. On the other hand, since $p \in \mathrm{Fr}_X(J)$, there exists a sequence $\{x_m\}_{m=1}^{\infty}$ in J^{o} such that $\lim x_m = p$. Then, for each $m \in \mathbf{N}$, $\{x_m\} \cup A_m \in \langle J^{\mathrm{o}}, K^{\mathrm{o}} \rangle$ and, by Lemma 32, $\{x_m\} \cup A_m \in \mathfrak{P}_2^{\partial}(X) \cap \langle J^{\mathrm{o}}, K^{\mathrm{o}} \rangle$. Hence, $B \in \mathrm{cl}_{C_2(X)}(\mathfrak{P}_2^{\partial}(X) \cap \langle J^{\mathrm{o}}, K^{\mathrm{o}} \rangle)$. Therefore, $B \in \mathcal{D}(J, K)$. This completes the proof of the lemma. □

Theorem 34. *Let X and Y be Peano continua. Let $J, K \in \mathfrak{A}_S(X)$ and $L, M \in \mathfrak{A}_S(Y)$ be such that* $\mathrm{Fr}_X(J) \subset \mathrm{cl}_X(\mathcal{FA}(X) -$

J), $\mathrm{Fr}_X(K) \subset \mathrm{cl}_X(\mathcal{FA}(X) - K)$, $\mathrm{Fr}_Y(L) \subset \mathrm{cl}_Y(\mathcal{FA}(Y) - L)$ and $\mathrm{Fr}_Y(M) \subset \mathrm{cl}_Y(\mathcal{FA}(Y) - M)$. Suppose that $h : C_2(X) \to C_2(Y)$ is a homeomorphism and $h(\langle J^{\circ}, K^{\circ}\rangle) = \langle L^{\circ}, M^{\circ}\rangle$. Then:

(1) *if $J = K$ and J is a simple closed curve, then $L = M$ and L is a simple closed curve,*

(2) *if $J = K$, J is an arc and $J \notin \mathfrak{A}_E(X)$, then $L = M$, L is an arc and $L \notin \mathfrak{A}_E(Y)$,*

(3) *if $J = K$ and $J \in \mathfrak{A}_E(X)$, then $L = M$ and $L \in \mathfrak{A}_E(Y)$,*

(4) *if $J \neq L$, then $M \neq N$,*

(5) *if $J = K$ and $p \in J - J^{\circ}$, then $h(\{p\})$ is a one-point set and $h(p) \subset L - L^{\circ}$.*

Proof. We describe models for the set $\mathcal{D}(J, K)$ considering all possibilities for the sets J and K in $\mathfrak{A}_S(X)$. These models are illustrated in Figure 2.

(a) $J = K$, J is an arc and $J \notin \mathfrak{A}_E(X)$. According to Lemma 33, $\mathcal{D}(J, J) = \{\{p_J\} \cup A : A \in C(J)\} \cup \{\{q_J\} \cup A : A \in C(J)\}$. By [**19**, Example 5.1], $C(J)$ is a 2-cell. Thus, $D(J, J)$ is the union of two 2-cells intersecting in the elements $\{p_J, q_J\}$ and J.

(b) $J = K$, $J \in \mathfrak{A}_E(X)$. Here, $\mathcal{D}(J, J) = \{\{q_J\} \cup A : A \in C(J)\}$ is a 2-cell.

(c) $J = K$ and J is a simple closed curve. Here, $\mathcal{D}(J, J) = \{\{q_J\} \cup A : A \in \mathcal{E}(J)\}$ is homeomorphic to the continuum W_0 described in the paragraph prior to Lemma 31.

From now on, we suppose that $J \neq K$.

(d) Both J and K are arcs and $J, K \notin \mathfrak{A}_E(X)$. Let $\mathcal{D}_1 = \{\{p_J\} \cup A : A \in C(K)\}$, $\mathcal{D}_2 = \{\{q_J\} \cup A : A \in C(K)\}$, $\mathcal{D}_3 = \{\{p_K\} \cup A : A \in C(J)\}$ and $\mathcal{D}_4 = \{\{q_K\} \cup A : A \in C(J)\}$. Note that $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$ are 2-cells and $\mathcal{D}(J, K) = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$. Here, we consider three subcases.

(d.1) $J \cap K = \varnothing$. In this subcase, $\mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing = \mathcal{D}_3 \cap \mathcal{D}_4$, $\mathcal{D}_1 \cap \mathcal{D}_3 = \{\{p_J, p_K\}\}$, $\mathcal{D}_1 \cap \mathcal{D}_4 = \{\{p_J, q_K\}\}$, $\mathcal{D}_2 \cap \mathcal{D}_3 = \{\{q_J, p_K\}\}$ and $\mathcal{D}_2 \cap \mathcal{D}_4 = \{\{q_J, q_K\}\}$.

(d.2) $J \cap K$ is a one-point set. In this subcase we may assume that $J \cap K = \{q_J\} = \{q_K\}$. Then we have the same equalities as in case (d.1), that is: $\mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing = \mathcal{D}_3 \cap \mathcal{D}_4$, $\mathcal{D}_1 \cap \mathcal{D}_3 = \{\{p_J, p_K\}\}$, $\mathcal{D}_1 \cap \mathcal{D}_4 = \{\{p_J, q_K\}\}$, $\mathcal{D}_2 \cap \mathcal{D}_3 = \{\{q_J, p_K\}\}$ and $\mathcal{D}_2 \cap \mathcal{D}_4 = \{\{q_J, q_K\}\}$.

(d.3) $J \cap K$ is a set with exactly two points. We may assume that $p_J = p_K$ and $q_J = q_K$. Then $\mathcal{D}_1 \cap \mathcal{D}_2 = \{\{p_J, q_J\}, K\}$, $\mathcal{D}_1 \cap \mathcal{D}_3 = \{\{p_J\}, \{p_J, q_J\}\}$, $\mathcal{D}_1 \cap \mathcal{D}_4 = \{\{p_J, q_K\}\}$, $\mathcal{D}_2 \cap \mathcal{D}_3 = \{\{q_J, p_K\}\}$, $\mathcal{D}_2 \cap \mathcal{D}_4 = \{\{q_J\}, \{p_J, q_K\}\}$ and $\mathcal{D}_3 \cap \mathcal{D}_4 = \{\{p_K, q_K\}, J\}$.

(e) Both J and K are arcs and $J \notin \mathfrak{A}_E(X)$ and $K \in \mathfrak{A}_E(X)$. Let $\mathcal{D}_1 = \{\{p_J\} \cup A : A \in C(K)\}$, $\mathcal{D}_2 = \{\{q_J\} \cup A : A \in C(K)\}$ and $\mathcal{D}_3 = \{\{q_K\} \cup A : A \in C(J)\}$. Note that $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are 2-cells and $\mathcal{D}(J, K) = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$. Here, we consider two subcases.

(e.1) $J \cap K = \varnothing$. In this subcase, $\mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing$, $\mathcal{D}_1 \cap \mathcal{D}_3 = \{\{p_J, q_K\}\}$ and $\mathcal{D}_2 \cap \mathcal{D}_3 = \{\{q_J, q_K\}\}$.

(e.2) $J \cap K$ is a one-point set. In this subcase we may assume that $J \cap K = \{q_J\} = \{q_K\}$. Then we have the same equalities as in case (e.1), that is, $\mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing$, $\mathcal{D}_1 \cap \mathcal{D}_3 = \{\{p_J, q_K\}\}$ and $\mathcal{D}_2 \cap \mathcal{D}_3 = \{\{q_J, q_K\}\}$.

(f) J is an arc, $J \notin \mathfrak{A}_E(X)$ and K is a simple closed curve. Let $\mathcal{D}_1 = \{\{p_J\} \cup A : A \in \mathcal{E}(K)\}$, $\mathcal{D}_2 = \{\{q_J\} \cup A : A \in \mathcal{E}(K)\}$ and $\mathcal{D}_3 = \{\{q_K\} \cup A : A \in C(J)\}$. Note that $\mathcal{D}(J, K) = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$, $\mathcal{D}_3$ is a 2-cell while $\mathcal{D}_1$ and $\mathcal{D}_2$ are homeomorphic to the continuum W_0. In both cases, when $J \cap K = \varnothing$ or when $J \cap K$ is a one-point set, we have that $\mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing$, $\mathcal{D}_1 \cap \mathcal{D}_3 = \{\{p_J, q_K\}\}$ and $\mathcal{D}_2 \cap \mathcal{D}_3 = \{\{q_J, q_K\}\}$.

(g) J and K are arcs and $J, K \in \mathfrak{A}_E(X)$. Let $\mathcal{D}_1 = \{\{q_J\} \cup A : A \in C(K)\}$ and $\mathcal{D}_2 = \{\{q_K\} \cup A : A \in C(J)\}$. Then $\mathcal{D}(J, K) = \mathcal{D}_1 \cup \mathcal{D}_2$ and $\mathcal{D}_1$ and $\mathcal{D}_2$ are 2-cells. Note that $\mathcal{D}_1 \cap \mathcal{D}_2 = \{\{q_J, q_K\}\}$.

(h) $J \in \mathfrak{A}_E(X)$ and K is a simple closed curve. Let $\mathcal{D}_1 = \{\{q_J\} \cup A : A \in \mathcal{E}(K)\}$ and $\mathcal{D}_2 = \{\{q_K\} \cup A : A \in C(J)\}$. Then $\mathcal{D}(J, K) = \mathcal{D}_1 \cup \mathcal{D}_2$, $\mathcal{D}_1$ is a 2-cell and $\mathcal{D}_2$ is homeomorphic to W_0. Note that $\mathcal{D}_1 \cap \mathcal{D}_2 = \{\{q_J, q_K\}\}$.

(i) J and K are simple closed curves. Let $\mathcal{D}_1 = \{\{q_J\} \cup A : A \in \mathcal{E}(K)\}$ and $\mathcal{D}_2 = \{\{q_K\} \cup A : A \in \mathcal{E}(J)\}$. Then $\mathcal{D}(J, K) = \mathcal{D}_1 \cup \mathcal{D}_2$, $\mathcal{D}_1$ and $\mathcal{D}_2$ are homeomorphic to W_0. Note that $\mathcal{D}_1 \cap \mathcal{D}_2 = \{\{q_J, q_K\}\}$.

We can observe, in Figure 2, that for different cases we obtain different models.

If $J = L$ and J is a simple closed curve, then $\mathcal{D}(J, J)$ is as in case (c). Hence, $\mathcal{D}(L, M)$ is as in case (c). This implies that $L = M$ and L is

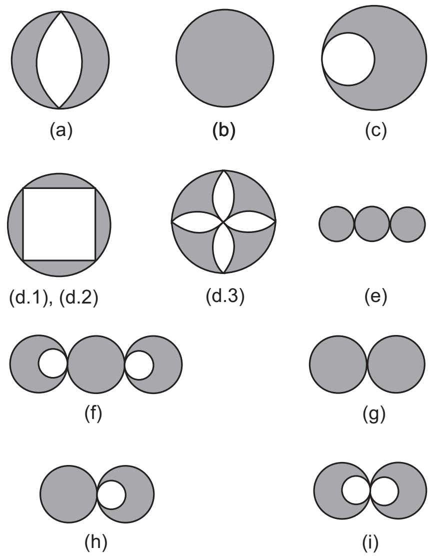

FIGURE 2.

a simple closed curve. This proves (1). The proofs for (2), (3) and (4) are similar.

In order to prove (5), let $B = h(\{p\})$. Since $p \in \mathrm{Fr}_X(J)$, there exists a sequence $\{p_m\}_{m=1}^{\infty}$ of points in J° such that $\lim p_m = p$. Then $\lim h(\{p_m\}) = B$ and $h(\{p_m\}) \subset L^{\circ}$ for each $m \in \mathbf{N}$. Thus, $B \subset L$. Take an open subset U of X such that $p \in U$. Since $\mathrm{Fr}_X(J) \subset \mathrm{cl}_X(\mathcal{FA}(X) - J)$, $U \cap \mathcal{FA}(X) - J \neq \varnothing$. This implies that there exists a sequence $\{x_m\}_{m=1}^{\infty}$ of points of $\mathcal{FA}(X) - J$ such that $\lim x_m = p$. For each $m \in \mathbf{N}$, let $J_m \in \mathfrak{A}_S(X)$ be such that $x_m \in J_m^{\circ}$. Let $L_m \in \mathfrak{A}_S(Y)$ be such that $h(\langle J_m^{\circ} \rangle) = \langle L_m^{\circ} \rangle$. Then $J_m \neq J$, so $L_m \neq L$. Since $h(\{x_m\}) \subset \langle L_m^{\circ} \rangle$, $h(\{x_m\}) \cap L^{\circ} = \varnothing$. Thus, $B = \lim h(\{x_m\}) \subset Y - L^{\circ}$. We have shown that $B \subset \mathrm{Fr}_Y(L)$.

By (1) and (3), if J is a simple closed curve or $J \in \mathfrak{A}_E(X)$, then L is a simple closed curve or $L \in \mathfrak{A}_E(Y)$. In these cases, $\mathrm{Fr}_X(J)$ and $\mathrm{Fr}_Y(L)$ are one-point sets. Then B is a one-point set contained in $\mathrm{Fr}_Y(L)$.

Suppose now that J is an arc and $J \notin \mathfrak{A}_E(X)$. Then L is an arc and $L \notin \mathfrak{A}_E(Y)$. Let u, v be the end points of L. Then $u \neq v$ and

$\mathrm{Fr}_Y(L) = \{u,v\}$. If $B = \{u\}$ or $B = \{v\}$, we are done. Suppose then that $B = \{u,v\}$. Since $h(\mathcal{D}(J,J)) = \mathcal{D}(L,L)$, by the model described in (a), we obtain that $\{p\}$ is not a local cut point of $\mathcal{D}(J,J)$. However, $B = h(p) = \{u,v\}$ is a local cut point of $\mathcal{D}(L,L)$, a contradiction. This completes the proof of (5) and ends the proof of the theorem. ◻

Theorem 35. *Let X and Y be almost meshed Peano continua. If $C_2(X)$ and $C_2(Y)$ are homeomorphic, then X and Y are homeomorphic.*

Proof. By [**16**, Theorem 4.1], we may assume that X and Y are neither an arc nor a simple closed curve. Let $h : C_2(X) \to C_2(Y)$ be a homeomorphism. Proceeding as in the beginning of Lemma 32, we have that the components of $\mathfrak{P}_2(X)$ are the sets of the form $\langle J^{\circ}, K^{\circ}\rangle$ where $J, K \in \mathfrak{A}_S(X)$. Thus, for every $J, K \in \mathfrak{A}_S(X)$, there exist $L, M \in \mathfrak{A}_S(Y)$ such that $h(\langle J^{\circ}, K^{\circ}\rangle) = \langle L^{\circ}, M^{\circ}\rangle$. Since X is almost meshed, for each $J \in \mathfrak{A}_S(X)$, $\mathrm{Fr}_X(J) \subset \mathrm{cl}_X(\mathcal{FA}(X) - J)$ and something similar happens for the elements in $\mathfrak{A}_S(Y)$. Hence, we can apply Theorem 34.

Now, take $p \in X - \bigcup\{L^{\circ} : L \in \mathfrak{A}_S(X)\}$. We claim that $h(\{p\}) = \{y\}$ for some $y \in Y - \bigcup\{K^{\circ} : K \in \mathfrak{A}_S(Y)\}$. Since $X = \mathrm{cl}_X(\mathcal{FA}(X))$, there exists a sequence $\{p_m\}_{m=1}^{\infty}$ in $\mathcal{FA}(X)$ such that $\lim p_m = p$. For each $m \in \mathbf{N}$, let $J_m \in \mathfrak{A}_S(X)$ be such that $p_m \in J_m^{\circ}$ and choose a point $q_m \in \mathrm{Fr}_X(J_m)$. By Lemma 3, $\lim J_m = \{p\}$. This implies that $\lim q_m = p$. By Theorem 34 (5), for each $m \in \mathbf{N}$, $h(\{q_m\}) = \{w_m\}$, for some w_m in the closed set $Y - \bigcup\{K^{\circ} : K \in \mathfrak{A}_S(Y)\}$. Hence, $h(\{p\}) = \{y\}$, for some $y \in Y - \bigcup\{K^{\circ} : K \in \mathfrak{A}_S(Y)\}$.

We define a map $g : X \to Y$. Let $F = X - \bigcup\{L^{\circ} : L \in \mathfrak{A}_S(X)\}$. Given $p \in F$, let $g(p) \in Y$ be such that $h(\{p\}) = \{g(p)\}$. Given $J \in \mathfrak{A}_S(X)$, let $K_J \in \mathfrak{A}_S(Y)$ be such that $h(\langle J^{\circ}\rangle) = \langle K_J^{\circ}\rangle$.

If J is a simple closed curve, by Theorem 34 (5), $g(q_J) \in K_J - K_J^{\circ}$. Hence, $g(q_J)$ is the only point in K_J such that $K_J - \{g(q_J)\}$ is open in Y. Fix a homeomorphism $g_J : J \to K_J$ such that $g_J(q_J) = g(q_J)$. If $J \in \mathfrak{A}_E(X)$, by Theorem 34, $K_J \in \mathfrak{A}_E(Y)$ and $g(q_J)$ is the only point in the arc K_J such that $K_J - \{g(q_J)\}$ is open in Y. Fix a homeomorphism $g_J : J \to K_J$ such that $g_J(q_J) = g(q_J)$. Finally, if J is an arc and $J \notin \mathfrak{A}_E(X)$, then K_J is an arc in $\mathfrak{A}_S(Y) - \mathfrak{A}_E(Y)$

and $g(p_J)$ and $g(q_J)$ are the end points of K_J. Fix a homeomorphism $g_J : J \to K_J$ such that $g_J(p_J) = g(p_J)$ and $g_J(q_J) = g(q_J)$.

Now, define $g : X \to Y$ as the common extension of g (defined in F) and the maps g_J for $J \in \mathfrak{A}_S(X)$. Note that g is well defined and continuous in the open set $X - F$. In fact, $g \mid J$ is continuous for each $J \in \mathfrak{A}_S(X)$. In order to complete the proof that g is continuous, take a sequence $\{p_m\}_{m=1}^{\infty}$ in $X - F$ such that $\lim p_m = p$ for some $p \in F$. For each $m \in \mathbf{N}$, let $J_m \in \mathfrak{A}_S(X)$ be such that $p_m \in J_m^{\circ}$. Then $q_{J_m} \in \mathrm{Fr}_X(J_m)$. We may assume that $J_m \neq J_k$ for $m \neq k$. By Lemma 8, $\lim J_m = \{p\}$. Then $\lim q_{J_m} = p$. Since $q_{J_m} \in F$ for each $m \in \mathbf{N}$, $\{g(p)\} = h(\{p\}) = \lim h(\{q_{J_m}\}) = \lim\{g(q_{J_m})\}$. Hence, $\lim g(q_{J_m}) = g(p)$. Given $m \in \mathbf{N}$, $g(p_m) = g_{J_m}(p_m) \in K_{J_m}$ and $g(q_{J_m}) \in K_{J_m}$. By Lemma 8, $\lim K_{J_m} = \{g(p)\}$. Hence, $\lim g(p_m) = g(p)$. This completes the proof that g is continuous.

It is easy to check that g is one-to-one. In order to see that g is onto, let $K \in \mathfrak{A}_S(Y)$. Applying Theorem 34 to h^{-1}, there exists a $J \in \mathfrak{A}_S(X)$ such that $\langle J^{\circ}\rangle = h^{-1}(\langle K^{\circ}\rangle)$. This implies that $K = K_J$, so $K \subset g(X)$. Since $\bigcup\{K : K \in \mathfrak{A}_S(Y)\}$ is dense in Y, we conclude that g is onto. Therefore, g is a homeomorphism. This ends the proof of the theorem. □

By Theorems 29, 30 and 35, we obtain the following.

Theorem 36. *Suppose that X and Y are almost meshed Peano continua and $C_n(X)$ is homeomorphic to $C_n(Y)$ for some $n \in \mathbf{N}$. Then*:

(a) *if $n = 1$ and X and Y are neither arcs nor simple closed curves, then X is homeomorphic to Y*,

(b) *if $n \neq 1$, then X is homeomorphic to Y.*

Theorem 37. *Suppose that X is a meshed continuum. If $n \neq 1$, then X has a unique hyperspace $C_n(X)$. If X is neither an arc nor a simple closed curve, then X has unique hyperspace $C(X)$.*

Proof. Suppose that $C_n(X)$ and $C_n(Y)$ are homeomorphic. Let $h : C_n(X) \to C_n(Y)$ be a homeomorphism. Since X is meshed, by Lemma 2, X is a Peano continuum. Then (see [**20**, Theorem

3.2]), Y is a Peano continuum. Note that $h(\mathfrak{F}_n(X)) = \mathfrak{F}_n(Y)$. By Theorem 5, $\mathfrak{F}_n(X)$ is dense in $C_n(X)$. Thus, $\mathfrak{F}_n(Y)$ is dense in $C_n(Y)$. By Theorem 5, Y is meshed. Applying Theorem 36, we conclude the proof of the theorem. $\square$

7. An almost meshed continuum with unique hyperspace. Consider the example $Z_0 = ([-1,1] \times \{0\}) \cup (\bigcup\{\{1/m\} \times [0,1/m] : m \geq 2\})$ mentioned at the end of the introduction and illustrated in Figure 1. If a dendrite Z contains a topological copy of Z_0, then the hyperspace $C(Z)$ is not unique [**2**]. Roughly speaking, this happens because there is a Hilbert cube $\mathfrak{C}$ near the element $\{(0,0)\}$ of $C(Z)$: consider the continuum W that is obtained by attaching a Peano continuum D without free arcs at $(0,0)$ to Z, that is, $W = Z \cup D$. Then $C(D)$ and the set $\{A \in C(W) : (0,0) \in A\}$ are Hilbert cubes whose union with $\mathfrak{C}$ is again a Hilbert cube and, moreover, the homeomorphism obtained can be extended to the homeomorphism between $C(Z)$ and $C(W)$. One may think local dendrites behave in the same way.

The next example shows that this does not happen. The "simplest" local dendrite X which is not a dendrite and contains a topological copy of Z_0 *does have* unique hyperspace $C(X)$.

Example 38. There exists a local dendrite X such that X contains a topological copy of Z_0, $\mathcal{P}(X)$ is a one-point set, $X - \mathcal{P}(X)$ is connected and X has unique hyperspace $C(X)$.

Let $S = (\{-1,1\} \times [0,1]) \cup ([-1,1] \times \{0,1\})$. Then S is a simple closed curve. Let $X = Z_0 \cup S$ and $\theta = (0,0)$ (X is the continuum Z_2 illustrated in Figure 1). Then X is an almost meshed Peano continuum that contains a simple closed curve S, $\mathcal{P}(X) = \{\theta\}$, $X - \mathcal{P}(X)$ is connected and X is not meshed. Observe that X is a local dendrite.

For each $m \geq 2$, let $B_m = \{1/m\} \times [0,1/m]$, $S_m = S \cup B_2 \cup \cdots \cup B_m$, $A_m = \{1/m\} \times [0,1/2m]$ and $p_m = (1/m,0) \in A_m$. We will need the following claim.

Claim 5. *Let* $\alpha : [0,1] \to C(X)$ *be a map and let* $m \in \mathbf{N}$ *be such that* $p_m p_{m+1} \not\subseteq \alpha(0)$ *(*$p_m p_{m+1}$ *denotes the shortest arc in* X *joining* p_m *and*

p_{m+1}) and, for each $t \in [0,1]$, $\{p_m, p_{m+1}\} \subset \alpha(t)$ and $S \not\subseteq \alpha(t)$. Then $p_m p_{m+1} \not\subseteq \alpha(1)$.

We prove Claim 5. Let $M = (\{-1,1\} \times [0,1]) \cup ([-1,1] \times \{1\}) \cup (([-1, 1/(m+1)] \cup [(1/m), 1]) \times \{0\})$. Let $J = \{t \in [0,1] : p_m p_{m+1} \subset \alpha(t)\}$ and $K = \{t \in [0,1] : M \subset \alpha(t)\}$. Then J and K are closed subsets of $[0,1]$ and $0 \notin J$. Since $p_m p_{m+1} \cup M = S$ and $S \not\subseteq \alpha(t)$ for any $t \in [0,1]$, $J \cap K = \varnothing$. Notice that each connected subset of X containing p_m and p_{m+1}, contains either $p_m p_{m+1}$ or M. Hence, $[0,1] = J \cup K$. The connectedness of $[0,1]$ implies that $J = \varnothing$, $1 \notin J$ and $p_m p_{m+1} \not\subseteq \alpha(1)$. This ends the proof of Claim 5. ◻

In order to prove that X has a unique hyperspace $C(X)$, let Y be a continuum such that $C(X)$ is homeomorphic to $C(Y)$. Then Y is a Peano continuum (see [**20**, Theorem 3.2]). Let $h : C(X) \to C(Y)$ be a homeomorphism.

Let $h_X : \mathrm{cl}_X(\mathcal{FA}(X)) \to \mathrm{cl}_{C(X)}(\mathfrak{P}^{\partial}(X))$, $h_Y : \mathrm{cl}_Y(\mathcal{FA}(Y)) \to \mathrm{cl}_{C(Y)}(\mathfrak{P}^{\partial}(Y))$ be homeomorphisms with the properties described in Theorem 27. Since X is almost meshed, $X = \mathrm{cl}_X(\mathcal{FA}(X))$. Since h is a homeomorphism, $h(\mathfrak{P}^{\partial}(X)) = \mathfrak{P}^{\partial}(Y)$ and $h(\mathrm{cl}_{C(X)}(\mathfrak{P}^{\partial}(X))) = \mathrm{cl}_{C(Y)}(\mathfrak{P}^{\partial}(Y))$. Thus, we can consider the map $g : X \to Y$ given by $g = h_Y^{-1} \circ h|(\mathrm{cl}_{C(X)}(\mathfrak{P}^{\partial}(X))) \circ h_X$. Then g is an embedding and $g(X) = \mathrm{cl}_Y(\mathcal{FA}(Y))$.

In order to prove that X and Y are homeomorphic, we are going to show that $Y = \mathrm{cl}_Y(\mathcal{FA}(Y))$. Suppose, to the contrary, that $Y \neq \mathrm{cl}_Y(\mathcal{FA}(Y))$. Note that $Y - \mathrm{cl}_Y(\mathcal{FA}(Y)) \subset \mathcal{P}(Y)$. We need to show the following claim.

Claim 6. *If $p \in X$ and $g(p) \in \mathcal{P}(Y)$, then $p \in \mathcal{P}(X)$.*

To prove Claim 6, let $y = g(p)$. Then $y \in \mathrm{cl}_Y(\mathcal{FA}(Y)) - \bigcup\{K^{\circ} : K \in \mathfrak{A}_E(Y)\}$. Thus, $h_Y(y) = \{y\}$. By Theorem 4, $\dim_{h_Y(y)}[C(Y)]$ is infinite. Then $\dim_{h^{-1}(h_Y(y))}[C(X)]$ is infinite. Applying again Theorem 4, we obtain that $h^{-1}(h_Y(y)) \cap \mathcal{P}(X) \neq \varnothing$. That is, $h_X(p) \cap \mathcal{P}(X) \neq \varnothing$. Given $J \in \mathfrak{A}_E(X)$, $J \cap \mathcal{P}(X) = \varnothing$. By the way the h_X was chosen as in Theorem 27, we have that $p \in \mathcal{P}(X)$. This completes the proof of Claim 6.

Since $\mathcal{P}(X) = \{\theta\}$, θ is the only point p in X for which $g(p) \in \mathcal{P}(Y)$. Thus, $g(X) \cap \mathcal{P}(Y) = \{g(\theta)\}$. Fix a point $y_0 \in Y - g(X)$, and let $\beta : [0,1] \to Y$ be a one-to-one map such that $\beta(0) = g(\theta)$ and $\beta(1) = y_0$. Let $t_0 = \max\{t \in [0,1] : \beta(t) \in g(X)\}$. Then $\beta(t_0) = g(\theta)$. Thus, $t_0 = 0$, $\beta((0,1]) \cap g(X) = \varnothing$ and $\operatorname{Im} \beta \subset \mathcal{P}(Y)$.

By Theorem 4, for each $m \geq 2$, $\dim_{S_m}[C(X)] = \infty$ and $S_m \in \operatorname{cl}_{C(X)}(\mathfrak{F}(X))$. Thus, $\dim_{h(S_m)}[C(Y)] = \infty$ and $h(S_m) \in \operatorname{cl}_{C(Y)}(\mathfrak{F}(Y))$. This implies that $h(S_m)$ is limit of subcontinua of Y contained in $Y - \mathcal{P}(Y)$ and $h(S_m) \cap \mathcal{P}(Y) \neq \varnothing$. Thus, $h(S_m) \subset g(X)$ and $g(\theta) \in h(S_m)$. Fix $m_0 \in \mathbf{N}$ such that $m_0 > 4$ and $h(S_{m_0}) \neq \{g(\theta)\}$. Then $h(S_{m_0}) \cap (Y - \mathcal{P}(Y)) \neq \varnothing$.

Let $\mathfrak{L} = \{E \in C(X), g(\theta) \in h(E)\}$. The uniform continuity of the map $\beta_0 : \mathfrak{L} \times [0,1] \to C(X)$ given by $\beta_0(E,t) = h^{-1}(h(E) \cup \beta([0,t]))$ implies that there exists $s_0 > 0$ such that, if $E \in \mathfrak{L}$ and $B_2 \cup B_3 \cup B_4 \subset E$, then for each $s \in [0, s_0]$, $A_2 \cup A_3 \cup A_4 \subset \beta_0(E,s)$. In particular, since $B_2 \cup B_3 \cup B_4 \subset S_{m_0}$, for each $s \in [0,s_0]$, $A_2 \cup A_3 \cup A_4 \subset h^{-1}(h(S_{m_0}) \cup \beta([0,s]))$. Let $Y_0 = h(S_{m_0}) \cup \beta([0,s_0])$ and $X_0 = h^{-1}(Y_0)$. Since $\beta(s_0) \in \mathcal{P}(Y) - g(X) \subset \operatorname{int}_Y(\mathcal{P}(Y))$, by Theorem 4, $Y_0 \in \operatorname{int}_{C(Y)}(C(Y) - \mathfrak{F}(Y))$. Hence, $X_0 \in \operatorname{int}_{C(X)}(C(X) - \mathfrak{F}(X))$. This implies that $S \not\subset X_0$. Then we can fix a point $z_0 \in S - X_0$. Since $A_2 \cup A_3 \cup A_4 \subset X_0$, we conclude that $p_2p_3 \subset X_0$ or $p_3p_4 \subset X_0$. We consider the case that $p_2p_3 \subset X_0$, the other one is similar. Note that $z_0 \notin p_2p_3$.

Let $\varepsilon > 0$ be such that, if $A \in C(X)$ and $H_X(A, X_0) < \varepsilon$, then $z_0 \notin A$. Let $\delta > 0$ be as in the definition of the uniform continuity of h^{-1} for the number ε. Let $x, y \in p_2p_3 - \{p_2, p_3\}$ be such that $x \neq y$, and let K be the subarc of p_2p_3 joining x and y; notice $K^{\circ} = K - \{x,y\}$. We choose x and y close enough to each other in such a way that $H_Y(h(S_{m_0} - K^{\circ}), h(S_{m_0})) < \delta$, we also ask that $h(S_{m_0} - K^{\circ}) \cap (Y - \mathcal{P}(Y)) \neq \varnothing$. Since $\theta \in S_{m_0} - K^{\circ}$, by Theorem 4, $\dim_{S_{m_0} - K^{\circ}}[C(X)]$ is infinite, so $\dim_{h(S_{m_0} - K^{\circ})}[C(Y)]$ is infinite and $h(S_{m_0} - K^{\circ}) \cap \mathcal{P}(Y) \neq \varnothing$. Hence, $g(\theta) \in h(S_{m_0} - K^{\circ})$.

Define $\alpha, \gamma : [0,1] \to C(X)$ by $\alpha(t) = h^{-1}(h(S_{m_0} - K^{\circ}) \cup \beta([0, ts_0]))$ and $\gamma(t) = h^{-1}(h(S_{m_0}) \cup \beta([0,ts_0]))$. Then α and γ are continuous, $\alpha(0) = S_m - K^{\circ}$, $\alpha(1) = h^{-1}(h(S_{m_0} - K^{\circ}) \cup \beta([0,s_0]))$, $\gamma(0) = S_{m_0}$ and $\gamma(1) = X_0$. Since $H_Y(h(S_{m_0} - K^{\circ}), h(S_{m_0})) < \delta$, $H_Y(h(S_{m_0} - K^{\circ}) \cup \beta([0,ts_0]), h(S_{m_0}) \cup \beta([0,ts_0])) < \delta$ for each $t \in [0,1]$. Thus,

$H_X(\alpha(t),\gamma(t))<\varepsilon$ for each $t\in[0,1]$. Hence, $H_X(\alpha(1),X_0)<\varepsilon$. This implies that $z_0\notin\alpha(1)$.

By the choice of s_0, since $B_2\cup B_3\cup B_4\subset S_{m_0}-K^{\circ}$, we obtain that $A_2\cup A_3\cup A_4\subset\alpha(t)$ for each $t\in[0,1]$. In particular, $\{p_2,p_3\}\subset\alpha(t)$ for each $t\in[0,1]$.

Given $t>0$, $\beta(ts_0)\in(h(S_{m_0}-K^{\circ})\cup\beta([0,ts_0]))\cap\operatorname{int}_Y(\mathcal{P}(Y))$. Theorem 4 implies that $(h(S_{m_0}-K^{\circ})\cup\beta([0,ts_0]))\in\operatorname{int}_{C(Y)}(C(Y)-\mathfrak{F}(Y))$. Hence, $\alpha(t)\in\operatorname{int}_{C(X)}(C(X)-\mathfrak{F}(X))$. If $S\subset\alpha(t)$, then there exists a sequence of elements in $C(X)$ which does not contain θ and converges to $\alpha(t)$, so $\alpha(t)\notin\operatorname{int}_{C(X)}(C(X)-\mathfrak{F}(X))$, a contradiction. Therefore, $S\not\subseteq\alpha(t)$.

We have shown that α satisfies the hypothesis in Claim 5, so $p_2p_3\not\subseteq\alpha(1)$. But z_0 is a point in S such that $z_0\notin p_2p_3$, $z_0\notin\alpha(1)$ and, since $p_2,p_3\in\alpha(1)$, we contradict the connectedness of $\alpha(1)$. This contradiction completes the proof that X has a unique hyperspace $C(X)$.

8. Dendrites not in class $\mathfrak{D}$ and hyperspace $C_2(X)$. For a dendrite W, it is known [**2, 13**] that $C(W)$ is unique if and only if W is in class $\mathfrak{D}$. This is not true for $C_2(W)$ as we see in this section. We prove that the continuum $Z_3=([-1,1]\times\{0\})\cup(\bigcup\{\{-1/m\}\times[0,1/m]:m\geq 2\})\cup(\bigcup\{\{\frac{1}{m}\}\times[0,1/m]:m\geq 2\})$ has unique hyperspace $C_2(Z_3)$. We emphasize that Z_3 does not have unique hyperspace $C(Z_3)$ (see [**2**] or Corollary 14). Let $\theta=(0,0)$.

Example 39. The continuum Z_3 has unique hyperspace $C_2(Z_3)$.

Note that $Z_3\notin\mathfrak{D}$. We see that Z_3 has a unique hyperspace $C_2(Z_3)$.

Suppose that Y is a continuum such that $C_2(Z_3)$ and $C_2(Y)$ are homeomorphic. Let $h:C_2(Z_3)\to C_2(Y)$ be a homeomorphism. By [**16**, Theorem 4.1], Y is not a finite graph.

Let $J,K\in\mathfrak{A}_S(Z_3)$. Notice that $\theta\notin J,K$ and J and K are arcs. By Theorem 4, $\dim_J[C_2(Z_3)]$ and $\dim_K[C_2(Z_3)]$ are finite. By the first paragraph in the proof of Lemma 32, there exist $L,M\in\mathfrak{A}_S(Y)$ such that $h(\langle J^{\circ},K^{\circ}\rangle)=\langle L^{\circ},M^{\circ}\rangle$. Thus, $h(\operatorname{cl}_{C_2(Z_3)}(\langle J^{\circ},K^{\circ}\rangle))=\operatorname{cl}_{C_2(Y)}(\langle L^{\circ},M^{\circ}\rangle)$. Since $L\cup M\in\operatorname{cl}_{C_2(Y)}(\langle L^{\circ},M^{\circ}\rangle)$, there exists an

$A \in \mathrm{cl}_{C_2(Z_3)}(\langle J^{\mathrm{o}}, K^{\mathrm{o}}\rangle)$ such that $h(A) = L \cup M$. Since $A \subset J \cup K$, by Theorem 4, $\dim_A[C_2(Z_3)]$ is finite. Thus, $\dim_{L\cup M}[C_2(Y)]$ is finite and $(L \cup M) \cap \mathcal{P}(Y) = \varnothing$. By Theorem 4 there exists a finite graph D in Y such that $L \cup M \subset \mathrm{int}_Y(D)$. This implies that $\mathrm{Fr}_Y(L) \subset \mathrm{cl}_Y(\mathcal{FA}(Y) - L)$ and $\mathrm{Fr}_Y(M) \subset \mathrm{cl}_Y(\mathcal{FA}(Y) - M)$. Since $\mathrm{Fr}_{Z_3}(J) \subset \mathrm{cl}_{Z_3}(\mathcal{FA}(Z_3) - J)$ and $\mathrm{Fr}_{Z_3}(K) \subset \mathrm{cl}_{Z_3}(\mathcal{FA}(Z_3) - K)$, we can apply Theorem 34. In particular, if $J = K$, then $L = M$ and L is an arc; moreover, for each $p \in J - J^{\mathrm{o}}$, $h(\{p\})$ is a one-point set and $h(\{p\}) \subset L - L^{\mathrm{o}}$. By continuity, $h(\{\theta\})$ is also a one-point set in $Y - \bigcup\{M^{\mathrm{o}} : M \in \mathfrak{A}_S(Y)\}$.

We define a map $g : Z_3 \to Y$. Let $F = Z_3 - \bigcup\{L^{\mathrm{o}} : L \in \mathfrak{A}_S(Z_3)\}$. Given $p \in F$, let $g(p) \in Y$ be such that $h(\{p\}) = \{g(p)\}$, which exists by Theorem 34. Given $J \in \mathfrak{A}_S(Z_3)$, let $K_J \in \mathfrak{A}_S(Y)$ be such that $h(\langle J^{\mathrm{o}}\rangle) = \langle K_J^{\mathrm{o}}\rangle$. Note that J is not a simple closed curve.

If $J \in \mathfrak{A}_E(Z_3)$, let q_J and p_J be the end points of J, where $p_J \in J^{\mathrm{o}}$. Then q_J is the only point in J such that $J - \{q_J\}$ is open in Z_3. By Theorem 34, $K_J \in \mathfrak{A}_E(Y)$. Note that $q_J \in F$ and $g(q_J) \in Y - \bigcup\{K^{\mathrm{o}} : K \in \mathfrak{A}_S(Y)\}$. Thus, $\{q_J\} \in \mathrm{cl}_{C_2(Z_3)}(\langle J^{\mathrm{o}}\rangle)$ and $\{g(q_J)\} \in \mathrm{cl}_{C_2(Y)}(\langle K_J^{\mathrm{o}}\rangle)$. Hence, $g(q_J) \in K_J - K_J^{\mathrm{o}}$. Therefore, $g(q_J)$ is the only point in K_J such that $K_J - \{g(q_J)\}$ is open in Y. Fix a homeomorphism $g_J : J \to K_J$ such that $g_J(q_J) = g(q_J)$. If J is an arc and $J \notin \mathfrak{A}_E(X)$, let q_J and p_J be the end points of J. Then q_J and p_J are the only points in J such that $J - \{p_J, q_J\}$ is open in X. By Theorem 34, K_J is an arc in $\mathfrak{A}_S(Y) - \mathfrak{A}_E(Y)$. Proceeding as before, $g(p_J)$ and $g(q_J)$ are the only points in the arc K_J such that $K_J - \{g(p_J), g(q_J)\}$ is open in Y. Hence, $g(p_J)$ and $g(q_J)$ are the end points of K_J. Fix a homeomorphism $g_J : J \to K_J$ such that $g_J(p_J) = g(p_J)$ and $g_J(q_J) = g(q_J)$.

Now define $g : Z_3 \to Y$ as the common extension of g (defined in F) and the maps g_J for $J \in \mathfrak{A}_S(Z_3)$. Proceeding as in Theorem 35, it can be shown that g is a well-defined embedding from Z_3 into Y. Given $J \in \mathfrak{A}_S(Z_3)$, $g(J) \subset \mathrm{cl}_Y(\mathcal{FA}(Y))$. Then $g(Z_3) = g(\mathrm{cl}_{Z_3}(\mathcal{FA}(Z_3))) \subset \mathrm{cl}_Y(g(\mathcal{FA}(Z_3))) \subset \mathrm{cl}_Y(\mathcal{FA}(Y))$. Hence, $g(Z_3) \subset \mathrm{cl}_Y(\mathcal{FA}(Y))$. Given $K \in \mathfrak{A}_S(Y)$, fix a point $q \in K^{\mathrm{o}}$. Then $\{q\} \in \mathfrak{P}_2(Y)$ and $h^{-1}(\{q\}) \in \mathfrak{P}_2(Z_3)$. Hence, there exist $J, L \in \mathfrak{A}_S(Z_3)$ such that $h^{-1}(\{q\}) \in \langle J^{\mathrm{o}}, L^{\mathrm{o}}\rangle$. If $J \neq L$, proceeding as in the first paragraph of the proof of Theorem 32 and using Theorem 34, we obtain that there exist $M, N \in \mathfrak{A}_S(Y)$ such that $M \neq N$ and $h(\langle J^{\mathrm{o}}, L^{\mathrm{o}}\rangle) = \langle M^{\mathrm{o}}, N^{\mathrm{o}}\rangle$. Thus,

$\{q\} \in \langle M^{\circ}, N^{\circ}\rangle$, a contradiction. Hence, $J = L$ and $K = K_J$. This proves that $K \subset g(Z_3)$, for every $K \in \mathfrak{A}_S(Y)$. Hence, $\mathrm{cl}_Y(\mathcal{FA}(Y)) \subset g(Z)$. Therefore, $g(Z) = \mathrm{cl}_Y(\mathcal{FA}(Y))$.

In order to prove that Z_3 and Y are homeomorphic, we are going to show that $Y = \mathrm{cl}_Y(\mathcal{FA}(Y))$. Suppose to the contrary that $Y \neq \mathrm{cl}_Y(\mathcal{FA}(Y))$. Note that $Y - \mathrm{cl}_Y(\mathcal{FA}(Y)) \subset \mathcal{P}(Y)$.

We need to show the following claim.

Claim 7. *If $p \in Z_3$ and $g(p) \in \mathcal{P}(Y)$, then $p \in \mathcal{P}(Z_3)$.*

To prove Claim 7, let $y = g(p)$. Then $y \in \mathrm{cl}_Y(\mathcal{FA}(Y)) - \bigcup\{K^{\circ} : K \in \mathfrak{A}_E(Y)\}$. Thus, $p \in Z_3 - \bigcup\{J^{\circ} : J \in \mathfrak{A}_E(Z_3)\}$. Hence, $h(\{p\}) = \{g(p)\} = \{y\}$. By Theorem 4, $\dim_{h(\{p\})}[C_2(Y)]$ is infinite. So $\dim_{\{p\}}[C_2(Z_3)]$ is infinite. Thus, $p \in \mathcal{P}(Z_3)$. So Claim 7 is proved. □

Since $\mathcal{P}(Z_3) = \{\theta\}$, θ is the only point p in X for which $g(p) \in \mathcal{P}(Y)$. Thus, $g(Z_3) \cap \mathcal{P}(Y) = \{g(\theta)\}$. This implies that $\mathcal{P}(Y)$ is a subcontinuum of Y.

We are going to obtain a contradiction by proving that the set $\mathfrak{T}_{Z_3} = \mathrm{int}_{C_2(Z_3)}(C_2(Z_3) - \mathfrak{F}_2(Z_3))$ is disconnected and the set $\mathfrak{T}_Y = \mathrm{int}_{C_2(Y)}(C_2(Y) - \mathfrak{F}_2(Y))$ is pathwise connected.

Take $A \in \mathfrak{T}_{Z_3}$. Then $\theta \in A$. If A is connected, then A is the limit of elements A_m in $C_2(Z_3)$ such that $\theta \notin A_m$. This implies that $A_m \in \mathfrak{F}_2(Z_3)$ and $A \notin \mathrm{int}_{C_2(Z_3)}(C_2(Z_3) - \mathfrak{F}_2(Z_3))$. This contradiction proves that A has two components: A_1 and A_2. We may assume that $\theta \in A_1$. Let $\pi : Z_3 \to [-1, 1]$ be the projection on the first coordinate. Then $\mathfrak{T}_{Z_3} \subset \{A_1 \cup A_2 \in C_2(X) : A_1, A_2 \in C(Z_3),\ A_1 \cap A_2 = \varnothing,\ \theta \in A_1 \text{ and } \pi(A_2) \subset [-1, 0)\} \cup \{A_1 \cup A_2 \in C_2(X) : A_1, A_2 \in C(Z_3),\ A_1 \cap A_2 = \varnothing,\ \theta \in A_1 \text{ and } \pi(A_2) \subset (0, 1]\}$. It follows that $\mathfrak{T}_{Z_3}$ is disconnected.

Take $B \in \mathfrak{T}_Y - \{Y\}$. If $B \not\subseteq g(Z_3)$, then $B \cap \mathrm{int}_Y(\mathcal{P}(Y)) \neq \varnothing$. Let $\alpha : [0, 1] \to C_2(Y)$ be an order arc from B to Y. Then, for each $t \in [0, 1]$, $\alpha(t) \cap \mathrm{int}_Y(\mathcal{P}(Y)) \neq \varnothing$. This implies that $\alpha(t) \in \mathfrak{T}_Y$. Therefore, B can be connected to Y by a path in $\mathfrak{T}_Y$. Now suppose that $B \subset g(Z_3)$. Since $\dim_B[C_2(Y)]$ is infinite, $B \cap \mathcal{P}(Y) \neq \varnothing$. Thus, $g(\theta) \in B$. Let $\beta : [0, 1] \to C(Y)$ be an order arc from $\{g(\theta)\}$ to

$\mathcal{P}(Y)$. Let $\alpha : [0,1] \to C_2(Y)$ be given by $\alpha(t) = B \cup \beta(t)$. Then α is continuous, $\alpha(0) = B$, $\alpha(1) = B \cup \mathcal{P}(Y)$ and, for each $t > 0$, $\varnothing \neq \beta(t) \cap \text{int}_Y(\mathcal{P}(Y)) \subset \alpha(t) \cap \text{int}_Y(\mathcal{P}(Y))$. Hence, $\alpha(t) \in \mathfrak{T}_Y$. Therefore, B can be connected to $B \cup \mathcal{P}(Y)$ by a path in $\mathfrak{T}_Y$. Since $\mathcal{P}(Y) \cap \text{int}_Y(\mathcal{P}(Y)) \neq \varnothing$, we have reduced the problem to the first case. Hence, $\mathfrak{T}_Y$ is pathwise connected.

Therefore, $\mathfrak{T}_{Z_3}$ is disconnected and $\mathfrak{T}_Y$ is connected. This contradicts the fact that h is a homeomorphism. This contradiction completes the proof that Z_3 and Y are homeomorphic. Therefore, Z_3 has unique hyperspace $C_2(Z_3)$.

Problem 40. *Characterize dendrites X with unique hyperspace $C_2(X)$.*

Problem 41. *Does there exist a Peano continuum X such that X has unique hyperspace $C(X)$ but X does not have unique hyperspace $C_2(X)$?*

Problem 42. *Let X be an almost meshed Peano continuum such that $X - \mathcal{P}(X)$ is connected. Does X have unique hyperspace $C(X)$?*

9. Other examples.

Example 43. Let $Z_1 = Z_3 \cup (\{0\} \times [0,1])$. Then Z_1 does not have unique hyperspace $C_2(Z_1)$. To see this, notice that the point $(0,0)$ satisfies the conditions of Corollary 25. Recall that, by Example 39, Z_3 has unique hyperspace $C_2(Z_3)$.

Example 44. Let X be a dendrite that contains a homeomorphic copy of dendrite F_ω. Suppose that there is a point $q \in F_\omega$ such that $F_\omega - \{q\}$ is open in X. Then X does not have a unique hyperspace $C_n(X)$ for any $n \in \mathbf{N}$. To see this, notice that the vertex of F_ω satisfies the conditions of Corollary 25.

Example 45. Let X be a local dendrite. Suppose that X contains a homeomorphic copy of dendrite F_ω. Then X does not have unique hyperspace $C_n(X)$ for any $n \in \mathbf{N}$.

Proof. Let d be a metric for X. Let $F_\omega = \bigcup\{\theta p_m : m \in \mathbf{N}\}$, where $\theta, p_m \in X$, each θp_m is an arc in X, joining θ and p_m, $\lim \theta p_m = \{\theta\}$ (in $C(X)$) and $\theta p_m \cap \theta p_k = \{\theta\}$, if $m \neq k$. In order to apply Theorem 22, we only need to prove that $X - \{\theta\}$ has infinitely many components. Suppose, to the contrary, that $X - \{\theta\}$ has only finitely many components. Then we may suppose that there exists a component W of $X - \{\theta\}$ such that $\theta p_m - \{\theta\} \subset W$ for each $m \in \mathbf{N}$. Let M be a dendrite in X such that $\theta \in M^o$, and let $\varepsilon > 0$ be such that $B(2\varepsilon, \theta) \subset M$. We may assume that $F_\omega \subset B(\varepsilon, \theta)$ and $W - M \neq \varnothing$. Fix a point $w \in W - M$. Given $m \in \mathbf{N}$, since W is arcwise connected, there exists an arc $\alpha_m \subset W$ which joins p_m and w. Then we can choose a point $q_m \in \alpha_m$ such that $d(\theta, q_m) = \varepsilon$ and the subarc β_m of α_m joining p_m and q_m is contained in $\{x \in X : d(x, \theta) \leq \varepsilon\}$. We may assume that $\lim q_m = q$ for some $q \in X$ such that $d(\theta, q) = \varepsilon$. Let U be an open connected subset of X such that $q \in U \subset M$ and $\theta \notin U$. Let $m_0 \in \mathbf{N}$ be such that $q_{m_0}, q_{m_0+1} \in U$. Then there exists an arc γ in U joining q_{m_0} and q_{m_0+1}. Thus, p_{m_0} and p_{m_0+1} can be joined by a path in $\beta_{m_0} \cup \gamma \cup \beta_{m_0+1} \subset M - \{\theta\}$. This is a contradiction since the unique arc in M joining p_{m_0} and p_{m_0+1} is $\theta p_{m_0} \cup \theta p_{m_0+1}$. Therefore, $X - \{\theta\}$ has infinitely many components. $\square$

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