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VECTOR-VALUED TREE MARTINGALES

TONG-JUN HE, QUAN YI AND YI SHEN

ABSTRACT. In this paper, we first introduce some interesting nontrivial tree martingale examples. Second, we show that, if $1 < a < 2$, then X-valued predictable tree martingale operators $S_t^{(a)}(f)$, $\sigma_t^{(a)}(f)$ can be dominated by X-valued predictable tree martingales f in a certain quasi-norm provided the space X is isomorphic to a $2a$-uniformly convex Banach space.

1. Preliminaries and definitions.

Definition 1.1. Let $\mathbf{T}$ be a countable, upward directed index set with respect to the partial ordering $\preceq$ satisfying the following two conditions:

(1) for every $t \in \mathbf{T}$, the set $\mathbf{T}^t := \{u \in \mathbf{T} : u \preceq t\}$ is finite;

(2) for every $t \in \mathbf{T}$, the set $\mathbf{T}_t := \{u \in \mathbf{T} : t \preceq u\}$ is linearly ordered.

Thus $\mathbf{T}$ is a tree set and every nonempty subset of $\mathbf{T}$ has at least one minimum. The succeeding element of $t \in \mathbf{T}$, namely, the minimum element of the set $\mathbf{T}_t - \{t\}$, is denoted by t^+. A tree $\mathbf{T}$ is also a special partially ordered set with respect to the partial ordering $\preceq$.

Let $(\Omega, \mathscr{S}, \mu)$ be a measure space $(\Omega, \mathscr{S})$, equipped with a finite measure μ, and let $L^1(\Omega, \mathscr{S}, \mu)$ be the space of integrable functions that are measurable relative to $\mathscr{S}$. With the help of positive contractive projections in $L^1(\Omega, \mathscr{S}, \mu)$ spaces, the definition of tree martingales shall be given as follows:

Definition 1.2. Let $(P_t, t \in \mathbf{T})$ be a family of positive contractive projections in $L^1(\Omega, \mathscr{S}, \mu)$ spaces. Then for every $f \in L^1(\Omega, \mathscr{S}, \mu)$, the

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family $(P_t f, t \in \mathbf{T})$ is called *tree martingales* if $s \preceq t$,

$$P_t P_s = P_s P_t = P_s \tag{1.1}$$

for every pair of comparable elements (s, t) (that is, $s \preceq t$ or $s \succeq t$) in $\mathbf{T}$, and

$$P_t P_s = P_s P_t = 0 \tag{1.2}$$

for every pair of non-comparable elements (s, t) (that is, $s \not\preceq t$ or $s \not\succeq t$) in $\mathbf{T}$.

It is clear that Definition 1.2 is equivalent to the definition of tree martingales introduced by Weisz [**19**] when the value space X of tree martingales is a scalar space. A set $\mathbf{T}_0$ is defined by

$$\mathbf{T}_0 \overset{\text{def}}{=} \{t_0 \in \mathbf{T} : \text{either } t_0 \preceq t \text{ or } t_0 \text{ is non-comparable with } t \text{ for every } t \in \mathbf{T}\} \tag{1.3}$$

for every nonempty tree set $\mathbf{T}$, then $\mathbf{T}_0$ is nonempty because of every nonempty subset of the tree set $\mathbf{T}$ having at least one minimum. Next, an element t_0^- is defined for which $t_0^- \prec t_0$ for every $t_0 \in \mathbf{T}_0$, and suppose $P_{t_0^-}$ is a common projection (i.e., $P_{t_0^-} = 0$) for which $P_{t_0} P_{t_0^-} = P_{t_0^-} P_{t_0} = P_{t_0^-}$. In this case, it is clear that $(t_0^-)^+ = t_0$. Let

$$\mathbf{T}_0^- \overset{\text{def}}{=} \{t_0^- : t_0^- \preceq t_0 \text{ for every } t_0 \in \mathbf{T}_0\} \text{ and } \overline{\mathbf{T}} \overset{\text{def}}{=} \mathbf{T} \cup \mathbf{T}_0^-. \tag{1.4}$$

Obviously, $\overline{\mathbf{T}}$ is still a tree set. That is, the common sigma algebras $(\mathscr{F}_{t_0^-}, t_0^- \in \mathbf{T}_0^-)$ of tree martingales can be defined as well as that of one-parameter martingales. Note that every common σ-algebra $\mathscr{F}_{t_0^-}$ only contains one element $\varnothing$, where every common σ-algebra $\mathscr{F}_{t_0^-}$ is replaced by a corresponding common projection $P_{t_0^-}$.

As one-parameter predictable martingales, we can define predictable tree martingales as follows:

Definition 1.3. We say that an X-valued tree martingale $Pf = (P_t f, t \in \mathbf{T})$ is predictable if there exists a family of $\lambda = (\lambda_t, t \in \mathbf{T})$ of nondecreasing, nonnegative and predictable functions such that

$$\|P_t f\| \leq \lambda_t \quad (t \in \mathbf{T}),$$

where nondecreasing functions mean that, for any comparable elements $s, t \in \mathbf{T}$, if $s \leq t$ then $\lambda_s \leq \lambda_t$; predictable functions mean that, for any $t \in \mathbf{T}$, λ_{t^+} is $\mathscr{F}_t$-measurable. Such a λ is called a prediction belonging to f.

Inspired by the Vilenkin system [**6, 12, 18, 20**], tree martingales and tree martingale transformation have been introduced by Schipp and Weisz. We refer to Weisz's exposition [**19**] and the successive papers on tree martingales [**7, 9–11**]. In the 1980's, Weisz and Schipp [**17, 19**] showed that Burkholder-Gundy's inequality of tree martingales holds if $2 < p < \infty$. Moreover, Weisz proved that the partial sums of the Vilenkin-Fourier series of an integrable function can be dominated by the maximal function of a suitable tree martingale transforms. However, because of the fact that tree martingale transforms cannot be defined as a one-parameter martingale and stopping times cannot be introduced for tree martingales, the study of tree martingales faces several difficulties. Weisz and Schipp [**17, 19**] obtained some results by using convexity methods. The question is: Are there efficient ways to overcome these difficulties? Here we study this problem.

By using graph-theoretic tricks, He and Shen [**10**] established a theorem on the structure of the index set $\mathbf{T}$ of tree martingales. For a tree $\mathbf{T}(V(\mathbf{T}), E(\mathbf{T}))$, the set of the vertex of $\mathbf{T}$ is denoted by $V(\mathbf{T})$, and the set of the arcs of $\mathbf{T}$ is denoted by $E(\mathbf{T})$.

Lemma 1.4. *A tree set $\mathbf{T}$ is isomorphic to a directed infinite locally finite forest. That is,*

$$\bigoplus_{i \in \mathbf{Z}^+} \{\mathbf{T}_i\} = \mathbf{T} \cong \mathbf{T}(V(\mathbf{T}), E(\mathbf{T})) = \bigoplus_{i \in \mathbf{Z}^+} \{\mathbf{T}_i(V(\mathbf{T}_i), E(\mathbf{T}_i))\},$$
$$\mathbf{T}_i \cong \mathbf{T}_i(V(\mathbf{T}_i), E(\mathbf{T}_i)), \quad i \in \mathbf{Z}^+,$$

where every $\mathbf{T}_i$ is a locally finite tree set, and $\mathbf{T}_i(V(\mathbf{T}_i), E(\mathbf{T}_i))$ is a directed infinite locally finite tree with root.

Based on Lemma 1.4, He and Shen [**10**] further established a decomposition of tree martingales as follows:

Lemma 1.5. *Let $Pf = (P_t f, t \in \mathbf{T})$ be a tree martingale. Then*

$$Pf = \bigoplus_{i=0}^{+\infty} P_i f, \tag{1.5}$$

where $P_i f = (P_t f, t \in \mathbf{T}_i)$ is a locally finite tree martingale and $\{\mathbf{T}_i\}_{i=0}^{\infty}$ is a sequence of locally finite tree sets for which

$$\mathbf{T} = \bigoplus_{i=0}^{+\infty} \mathbf{T}_i.$$

Lemma 1.5 shows that some inequalities for tree martingales can be obtained for locally finite tree martingales. Throughout this paper, we always assume that tree set $\mathbf{T}$ is a locally finite tree with root.

In [**7**], He characterized the collection of σ-filtrations $(\mathscr{F}_t, t \in \mathbf{T})$ on X-valued tree martingales by the positive contractive projections when the tree set $\mathbf{T}$ is a locally finite tree and X is a scalar-valued space. However, when X is a general Banach space and the tree set $\mathbf{T}$ is a locally finite tree, it is difficult to characterize the collection of σ-filtrations $(\mathscr{F}_t, t \in \mathbf{T})$ on X-valued tree martingales by the positive contractive projections, because the structure of contractive projections on a Lebesgue-Bochner space of X-valued functions $L^1(X)$ is a close connection to the geometric structure of the Banach space X (the structure of contractive projections is considerably complicated without any assumptions of σ-finiteness of $L^p(X)$-space or separability of the Hilbert space X (see [**16**]). Here, for simplicity, we assume directly that $(\Omega, \mathscr{F}_\infty, P)$ is a complete probability space and

$$\mathscr{F}_\infty \stackrel{\text{def}}{=} \bigvee_{t \in \mathbf{T}} \mathscr{F}_t = \sigma\bigg(\bigcup_{t \in \mathbf{T}} \mathscr{F}_t\bigg), \tag{1.6}$$

where $\mathscr{F} = (\mathscr{F}_t, t \in \mathbf{T})$ is a family of non-decreasing sub σ-algebras of $\mathscr{F}_\infty$ with respect to the partial ordering $\preceq$. Throughout this paper, unless otherwise stated, we let E_t be the conditional expectation operator with respect to $\mathscr{F}_t$, $(X, \|.\|)$ a Banach space, $L_1(X)$ the space of Bochner integrable measurable functions, any $A \in \mathscr{F}_\infty$, and the indicator function of a set A is denoted by χ_A.

Let $(\phi_t, t \in \mathbf{T})$ be a family of scalar complex-valued measurable functions with $|\phi| = 1$, and

$$P_t f = \phi_t E_t(f\overline{\phi_t}), \quad f \in L_1(X) \tag{1.7}$$

for each $t \in \mathbf{T}$. Then, from Definition 1.2, we see that $(P_t f, t \in \mathbf{T})$ is a family of tree martingales.

For an X-valued tree martingale, we are going to introduce a quasi-norm $\|\cdot\|_{\mathbf{M}^{pq}}$. Let $f = (f_t, t \in \mathbf{T})$ be a family of $\mathscr{F}_\infty$-measurable functions (not necessarily a tree martingale) defined on the complete probability space $(\Omega, \mathscr{F}_\infty, P)$. For any $y \geq 0$, we set

$$\nu_y^f = \inf\{t \in \mathbf{T} : \|f_t\| > y\}.$$

Then it is easy to see that

$$\{t \in \nu_y^f\} = \{\omega \in \Omega : \|f_t(\omega)\| > y, \|f_s(\omega)\| \leq y, \text{ for all } s < t\},$$

where $s < t$ means that $s \leq t$ but $s \neq t$. Note that ν_y^f is generally a subset of $\mathbf{T}$, and if $\mathbf{T}$ is a total set, then ν_y^f is a stopping time; but if $\mathbf{T}$ is a partial set, then ν_y^f is not a stopping time since, for a fixed $y > 0$, ν_y^f is not the only one. Now, we may introduce the definition of quasi-norm $\|.\|_{M^{pq}}$ by ν_y^f. For $0 < p, q < \infty$, let

$$\begin{gathered}\|f\|_{\mathbf{M}^{pq}} = \sup_{y>0} y\biggl(\int_\Omega \Bigl(\sum_{t\in\mathbf{T}} \chi_{\{t\in\nu_y^f\}}\Bigr)^{p/q}\biggr)^{1/p}, \\ \mathbf{M}^{pq} = \{f = (f_t, t \in \mathbf{T}) : \|f\|_{\mathbf{M}^{pq}} < \infty\},\end{gathered} \tag{1.8}$$

where $\chi_{\{t\in\nu_y^f\}}$ is the indicator function of the set $\{t \in \nu_y^f\}$.

Remark 1.6. Note that, for each fixed family of $f = (f_t, t \in \mathbf{T})$, the quasi-norm $\|f\|_{\mathbf{M}^{pq}}$ is decreasing with q increasing and increasing with p increasing. Therefore, the limit does exist as $q \to \infty$ and satisfies

$$\|f\|_{\mathbf{M}^{p\infty}} = \lim_{q\to\infty} \|f\|_{\mathbf{M}^{pq}} = \sup_{y>0} yP(f^* > y)^{1/p}, \quad 0 < p < \infty.$$

In [**19**], it is verified that the map $\|\cdot\|_{\mathbf{M}^{pq}}$ is a quasi-norm, namely, for any two families of functions $f = (f_t, t \in \mathbf{T})$ and $g = (g_t, t \in \mathbf{T})$ and for any $\lambda \in C$,

$$\|\lambda f\|_{\mathbf{M}^{pq}} = |\lambda| \|f\|_{\mathbf{M}^{pq}},$$
$$\|f + g\|_{\mathbf{M}^{pq}} \leq K_{pq}(\|f\|_{\mathbf{M}^{pq}} + \|g\|_{\mathbf{M}^{pq}}),$$

where $0 < p < \infty$, $0 < q \leq \infty$ and K_{pq} depends only on p and q. Moreover, the map $\|\cdot\|_{\mathbf{M}^{pq}}$ is nondecreasing in the following sense: if, for all $t \in \mathbf{T}$, the inequality $\|f_t\| \leq \|g_t\|$ holds, then

$$\|f\|_{\mathbf{M}^{pq}} \leq \|g\|_{\mathbf{M}^{pq}}, \quad 0 < p < \infty,\ 0 < q \leq \infty.$$

Remark 1.7. Note that if $\mathbf{T}$ is linearly ordered, then the sets $\{t \in \nu_y^f\}$ $(t \in \mathbf{T})$ are pairwise disjoint and $\sum_{t \in \mathbf{T}} \chi_{\{t \in \nu_y^f\}} = \chi_{\{f^* > y\}}$; in this case,

$$\|f\|_{\mathbf{M}^{pq}} = \sup_{y>0} y P(f^* > y)^{1/p}, \quad 0 < p < \infty,\ 0 < q \leq \infty.$$

Denote by $\mathbf{P}^{pq}$ the space of this kind of X-valued predictable tree martingale and endow it with the following quasi-norm:

$$\|f\|_{\mathbf{P}^{pq}} = \inf \|\lambda\|_{\mathbf{M}^{pq}} \quad (0 < p < \infty,\ 0 < q \leq \infty),$$

where the infimum is taken over all predictions $\lambda \in \mathbf{M}^{pq}$ belonging to f. For the reader's convenience, throughout this paper, we always let

$$\|(S_t^{(a)}(f), t \in T)\|_{\mathbf{M}^{pq}} = \|S^{(a)}(f)\|_{\mathbf{M}^{pq}},$$
$$\|(s_t^{(a)}(f), t \in T)\|_{\mathbf{M}^{pq}} = \|s^{(a)}(f)\|_{\mathbf{M}^{pq}}.$$

Next, some interesting tree martingale examples will be introduced in Section 2.

2. Some examples of tree martingales.

Example 2.1. Dyadic intervals are defined by

$$I_n(j) = (j2^{n-1}, (j+1)2^{n-1}], \quad j = 0, 1, 2, \ldots, 2^{n-1} - 1,\ n \geq 1;$$

furthermore, we define

$$\mathcal{I} = \{I_n(j),\ j = 0, 1, 2, \dots, 2^{n-1} - 1,\ n \geq 1\}.$$

Then the ordering $\preceq$ in $\mathcal{I}$ is defined by set inclusion and it is clear that the ordering $\preceq$ is a partial ordering. Define

$$\mathscr{F}_{I_n(j)} = \sigma\bigg(\bigcup_{\substack{I \subseteq I_n(j) \\ I \in \mathcal{I}}} I \bigg), \quad j = 0, 1, 2, \dots, 2^{n-1} - 1,\ n \geq 1.$$

Obviously, $(E(f|\mathscr{F}_{I_n(j)}), I_n(j) \in \mathcal{I})$ is a family of tree martingales for any $f \in L^1(X)$.

Example 2.2. Let $m = (m_k, k \in \mathbf{N})$ be a sequence of integers, each of them not less than 2. Let Z_{m_k} denote the m_kth discrete cyclic group. Z_{m_k} can be represented by the set $\{0, 1, \dots, m_k - 1\}$, where the group operation is the $\bmod m_k$ addition and every subset is open. The measure on Z_{m_k} is defined such that the measure of every singleton is $(k \in \mathbf{N})/m_k$. By means of these cyclic groups, we define the so-called Vilenkin group G_m as the complete direct product of these Z_{m_k}, i.e.,

$$G_m = \times_{k=0}^{\infty} Z_{m_k}. \tag{2.1}$$

Then G_m is a compact Abelian group with Haar measure 1, whose elements are of the form $(x_k, k \in \mathbf{N})$, with $x_k \in Z_{m_k} (k \in \mathbf{N})$. The group operation on G_m (denoted by $\dot{+}$) is the coordinatewise addition (the inverse operation is denoted by $\dot{-}$), the measure (denoted by μ) and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n \in \mathbf{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. A base for the neighborhoods of G_m can be given as follows: for $x \in G_m$, $n \in \mathbf{N} - \{0\}$,

$$\begin{aligned} &I_0(x) = G_m, \\ &I_n(x) = \{y = (y_k, k \in \mathbf{N}) \in G_m : y_k = x_k \text{ for } k < n\}. \end{aligned} \tag{2.2}$$

The set of intervals on G_m is denoted by

$$\mathcal{I} = \{I_n(x) : n \in \mathbf{N}, x \in G_m\}. \tag{2.3}$$

The Haar measure of $I_n(t)$ is M_n, where the generalized powers $M_n (n \in \mathbf{N})$ are defined in the following way:

$$M_0 = 1, \qquad M_n = \prod_{j=0}^{n-1} m_j \quad (n \in \mathbf{N} - \{0\}). \tag{2.4}$$

Then each natural number n can be uniquely expressed as

$$n = \sum_{j=0}^{\infty} n_j M_j, \quad 0 \le n_j < m_j, \ (n_j \in Z_{m_j}, j \in \mathbf{N}), \tag{2.5}$$

where only a finite number of n's may differ from zero. Notice that, under the identification of G_m with $[0,1)$, the interval $I_n(j)$ corresponds to the interval $[jM_n^{-1}, (j+1)M_n^{-1})$ provided $0 \le j < M_n$. Let

$$\mathscr{J} = \{I_n(j), 0 \le j < M_n, n \ge 1\},$$
$$\mathscr{F}_{I_n(j)} = \sigma\Bigg(\bigcup_{\substack{J \subseteq I_n(j) \\ J \in \mathscr{J}}} J\Bigg), \quad 0 \le j < M_n, \ n \ge 1.$$

Obviously, $(E(f|\mathscr{F}_{I_n(j)}), I_n(j) \in \mathscr{J})$ is a family of tree martingales for any $f \in L^1(X)$. When $m_k = 2 (k \in \mathbf{N})$, $(E(f|\mathscr{F}_{I_n(j)}), I_n(j) \in \mathscr{J})$ is a family of tree martingales as that in Example 2.1.

Example 2.3. Let $\mathbf{T} = (V(\mathbf{T}), E(\mathbf{T}))$ be a tree. If $e_t \in E(\mathbf{T})$ is an arc of $\mathbf{T}$, $e_t = (t^+, t)$, t^+ is called the trail of e_t and t is called the head of e_t. Now, we put i.i.d. random variables with mean zero on the edges, that is,

$$\xi(e_t) = \xi_t, \quad t \in \mathbf{T}$$

is an independent random variable with mean zero for every arc $e_t \in E(\mathbf{T})$. Therefore, a directed network $N = (\xi(e_t), e_t \in E(\mathbf{T}))$ is obtained. Furthermore, note that

$$\mathbf{T}^{t^+} = (V(\mathbf{T}^{t^+}), E(\mathbf{T}^{t^+}))$$

is a directed locally finite subtree of the tree $\mathbf{T}$; the weight of each vertex in the tree $\mathbf{T}$ is defined by

$$f_{t^+} \overset{\text{def}}{=} \sum_{e_t \in E(\mathbf{T}^{t^+})} \xi(e_t) = \sum_{\substack{t \in \mathbf{T}^{t^+} \\ t \ne t^+}} \xi_t \tag{2.6}$$

for every element $t \in \mathbf{T}$. We define

$$\mathscr{F}_{t^+} \stackrel{\text{def}}{=} \sigma(f_{t^+}), \quad t \in \mathbf{T},$$

which is a family of σ-algebras generated by the weighted function f_{t^+} of every vertex in the tree $\mathbf{T}$; then a family $(f_{t^+}, t \in \mathbf{T})$ of tree martingales is obtained. Obviously,

$$f_t = (f_{t^+}|\mathscr{F}_t) \quad \text{and} \quad f_{t^+} = (f_{t^+}|\mathscr{F}_{t^+}),$$

namely, $(f_t, t \in \mathbf{T})$ is a family of tree martingales with respect to the family $(\mathscr{F}_t, t \in \mathbf{T})$ of σ-algebras.

3. The main results. For any X-valued tree martingales $f = (f_t, t \in \mathbf{T})$, $f = (f_s, s \in \mathbf{T}_t)$ is a sequence of X valued one-parameter martingales. Their maximal functions are defined, respectively, by

$$f_t^* \stackrel{\text{def}}{=} \sup_{s \in \mathbf{T}_t} \|f_s\|, \qquad f^* \stackrel{\text{def}}{=} \sup_{t \in \mathbf{T}} \|f_t\|.$$

X-valued tree martingale differences are defined by

$$d_t f \stackrel{\text{def}}{=} P_{t^+} f - P_t f, \quad t \in \mathbf{T}; \tag{3.1}$$

furthermore, for $1 < a < \infty$, we define

(3.2)
$$S_t^{(a)}(f) = \left(\sum_{s \in \mathbf{T}_t} \|d_s f\|^a \right)^{1/a}, \qquad s_t^{(a)}(f) = \left(\sum_{s \in \mathbf{T}_t} E_s \|d_s f\|^a \right)^{1/a},$$
$$S^{(a)}(f) \stackrel{\text{def}}{=} \sup_{t \in \mathbf{T}} S_t^{(a)}(f), \qquad s^{(a)}(f) \stackrel{\text{def}}{=} \sup_{t \in \mathbf{T}} s_t^{(a)}(f).$$

When $a = 2$, the two operators $S^{(a)}(f)$ and $s^{(a)}(f)$ are, respectively, extensions of quadratic and conditional quadratic variations of one-parameter martingales in the tree martingale case. For the two operators, Weisz [**19**] established an analogue of Burkholder-Davis-Gundy's inequalities for tree martingales when X is a scalar space. With the help of a family of positive contractive projections in L^1 spaces, He [**7**] proved the tree martingale Doob's inequality provided X is a scalar space.

Bourgain [**1**] and Burkholder [**2, 3**] have shown that there are some relations between X-valued martingale inequalities and the convexity of martingale's valued space X if X is a Banach space. Next, Piser [**15**] and Kwapien [**14**] further point out that, if Burkholder-Gundy's inequality for X valued one-parameter martingales holds, provided X is a Banach space, then the Banach space X is a Hilbert space. So the research of quadratic variation and conditional quadratic variation for general Banach space X-valued martingales is trivial. Piser [**15**] proved the following two theorems for X-valued one-parameter martingales $f = (f_s, s \in \mathbf{T}_t)$.

Lemma 3.1. *Assume that X is a Banach space, $2 \le a < \infty$, $1 < p < \infty$. Then the following conditions are equivalent*:

(1) *X is isomorphic to a a-uniformly convex space*;

(2) *there exists a constant $C > 0$ depending only on a and p such that*

$$\|S_t^{(a)}(f)\|_p \le C\|f_t^*\|_p, \tag{3.3}$$

$$\|s_t^{(a)}(f)\|_p \le C\|f_t^*\|_p, \tag{3.4}$$

for any X-valued one-parameter martingales $f = (f_s, s \in \mathbf{T}_t)$.

Lemma 3.2. *Assume that X is a Banach space, $1 \le a < 2$, $1 < p < \infty$. Then the following conditions are equivalent*:

(1) *X is isomorphic to a a-uniformly convex space*;

(2) *there exists a constant $C > 0$ depending only on a and p such that*

$$C\|f_t^*\|_p \le \|S_t^{(a)}(f)\|_p, \tag{3.5}$$

$$C\|f_t^*\|_p \le \|s_t^{(a)}(f)\|_p, \tag{3.6}$$

for any X-valued one-parameter martingales $f = (f_s, s \in \mathbf{T}_t)$.

He and Shen [**9**] tried to investigate the tree martingale transform operator and have shown that the maximal operators of X-valued tree martingale transforms are norm-bounded in $L^p(X)$, provided X is a UMD space. In [**8**], He and Hou have shown that, assuming that X is isomorphic to an a-uniformly convex space, $(2 \le a < \infty)$. Then, for an X-valued predictable tree martingale $f = (f_t, t \in \mathbf{T})$ and $p \geq 1, \max(a,p) \leq q < \infty$, we have

$$\|(S_t^{(a)}(f), t \in \mathbf{T})\|_{\mathbf{M}^{p\infty}} \leq C_{pq}\|f\|_{\mathbf{P}^{pq}}, \tag{3.7}$$

$$\|(s_t^{(a)}(f), t \in \mathbf{T})\|_{\mathbf{M}^{p\infty}} \leq C_{pq}\|f\|_{\mathbf{P}^{pq}}, \tag{3.8}$$

where C_{pq} depends only p and q. It is well known that if the measure space is *granular* in the sense that one has a lower bound $\mu(E) \geq C > 0$ for all sets E of positive measure, then functions are prohibited from being arbitrarily narrow, and lower norms control higher norms:

$$\|f\|_q \leq \|f\|_p c^{(1/q)-(1/p)} \text{ whenever } 0 < p \leq q \leq \infty.$$

Now, suppose a probability space $(\Omega, \mathscr{F}_\infty, P)$ is *granular* and $f \in L^1(X)$ with $f > c > 0$. Then

$$\inf_{t \in \mathbf{T}} E(f|\mathscr{F}_t) \geq c.$$

Since Lemma 3.2 shows that if $1 < a < 2$ then $L^p(X)$-norms of the operators $S_t^{(a)}(f), s_t^{(a)}(f)$ control $L^p(X)$-norms of maximal function, but $L^p(X)$-norms of maximal function do not control the $L^p(X)$-norms of the operators $S_t^{(a)}(f), s_t^{(a)}(f)$ in the one-parameter martingale case, as some functions in a *granular* probability space, for some X-valued tree martingales $f = (f_t, t \in \mathbf{T})$ with

$$\inf_{t \in \mathbf{T}} \|f_t)\| \geq c > 0,$$

do (3.7) and (3.8) still hold when $1 < a \leq 2$? In this paper, we shall investigate this problem.

Theorem 3.3. *Let $f = (f_t, t \in \mathbf{T})$ be a family of X-valued predictable tree martingales for which $E[S^{(a)}(f)] < \infty$, $E[s^{(a)}(f)] < \infty$ and $\inf_{t \in \mathbf{T}}\|f_t\| \geq c > 0$. $\lambda = (\lambda_t, t \in \mathbf{T})$ is a prediction belonging to f, and*

$$\alpha_t \stackrel{\text{def}}{=} \chi_{\{\lambda_t > y,\ \lambda_s \leq y,\ s<t\}} \quad (s, t \in \mathbf{T}).$$

Suppose X is isomorphic to a $2a$-uniformly convex space $(1 < a \leq 2)$. If there exists a constant $\alpha_0 \in (1, \infty)$ such that $\sum_{t \in \mathbf{T}} (E\alpha_t)^{1/\alpha_0} < \infty$, *then for any* $p \geq 1$ *and* $\max(a,p) \leq q < \infty$,

(3.9)
$$\|(S_t^{(a)}(f), t \in \mathbf{T})\|_{\mathbf{M}^{p\infty}} \leq C_{pq}(\|f\|_{\mathbf{P}^{qp}} + \|f\|_{\mathbf{P}^{qp}}^{(p/q)^2} + \|f\|_{\mathbf{P}^{(q+p)q}}^{(p+q)/p}),$$

(3.10)
$$\|(s_t^{(a)}(f), t \in \mathbf{T})\|_{\mathbf{M}^{p\infty}} \leq C_{pq}(\|f\|_{\mathbf{P}^{qp}} + \|f\|_{\mathbf{P}^{qp}}^{(p/q)^2} + \|f\|_{\mathbf{P}^{(q+p)p}}^{(p+q)/p}),$$

where C_{pq} *depends only on* p *and* q.

4. Proof of the main results. Our proof of Theorem 3.3 requires a series of preliminary lemmas. Let $f = (f_t, t \in \mathbf{T})$ be an X-valued tree martingale and $\varepsilon = (\varepsilon_t, t \in \mathbf{T})$ a sequence of functions for which each ε_t is $\mathscr{F}_t$-measurable. Define

$$S_{\varepsilon,t,s}^{(a)}(f) = \Bigg(\sum_{t \leq r < s} \|\varepsilon_r d_r f\|^a\Bigg)^{1/a},$$
(4.1)
$$s_{\varepsilon,t,s}^{(a)}(f) = \Bigg(\sum_{t \leq r < s} E_r\|\varepsilon_r d_r f\|^a\Bigg)^{1/a},$$

and

$$S_{\varepsilon,t}^{(a)}(f) = \sup_{s \geq t} S_{\varepsilon,t,s}^{(a)}(f), \;\; S_{\varepsilon}^{(a)} = \sup_{t \in \mathbf{T}} S_{\varepsilon,t}^{(a)}(f), \tag{4.2}$$

$$s_{\varepsilon,t}^{(a)}(f) = \sup_{s \geq t} S_{\varepsilon,t,s}^{(a)}(f), \;\; s_{\varepsilon}^{(a)} = \sup_{t \in \mathbf{T}} S_{\varepsilon,t}^{(a)}(f). \tag{4.3}$$

He and Hou in [**11**] extended Weisz's result [**19**] to the following case.

Lemma 4.1. *Suppose that* $f = (f_t, t \in \mathbf{T})$ *is an* X*-valued predictable tree martingale and that* $\lambda = (\lambda_t, t \in \mathbf{T})$ *is a prediction belonging to* f. *Let* $\varepsilon_t = \chi_{\{x < \lambda_{t+} \leq 2x\}}$ $(t \in \mathbf{T})$ *for any real number* $x > 0$ *and* $\varepsilon = (\varepsilon_t, t \in \mathbf{T})$. *If* $1 \leq a < \infty$, *then*

$$S_{\varepsilon}^{(a)}(f) \leq \sup_{t \in \mathbf{T}} \alpha_t S_{\varepsilon,t}^{(a)}(f) + 4x\chi_{\{\lambda^* > x\}}, \tag{4.4}$$

$$s_{\varepsilon}^{(a)}(f) \leq \sup_{t \in \mathbf{T}} \alpha_t s_{\varepsilon,t}^{(a)}(f) + 4x\chi_{\{\lambda^* > x\}}, \tag{4.5}$$

where $(\alpha_t, t \in \mathbf{T})$ *is defined as in Theorem* 3.3 *and* $\lambda^* = \sup_{t\in\mathbf{T}} \lambda_t$ *is the maximal function of* λ.

Since $\sum_{t\in\mathbf{T}} E\alpha_t$ can be viewed as a nonnegative function on $\mathbf{R}^+$, i.e., $\sum_{t\in\mathbf{T}} E\alpha_t$ is a nonnegative function depending on y, some subsets on $\mathbf{R}^+$ are defined by

$$\mathbf{A}_1 = \Big\{y \in \mathbf{R}^+ : 1 < \sum_{t\in\mathbf{T}} E\alpha_t \le \infty\Big\},$$
$$\mathbf{A}_2 = \Big\{y \in \mathbf{R}^+ : \sum_{t\in\mathbf{T}} E\alpha_t < 1\Big\},$$
$$\mathbf{A}_3 = \Big\{y \in \mathbf{R}^+ : \sum_{t\in\mathbf{T}} E\alpha_t = 1\Big\}.$$

It is clear that

$$\mathbf{A}_i \cap \mathbf{A}_j = \varnothing (i \ne j;\ i, j = 1, 2, 3) \quad \text{and} \quad \cup_{i=1}^3 \mathbf{A}_i = \mathbf{R}^+. \tag{4.6}$$

We can also define their indicator functions in the form

$$\chi_{\mathbf{A}_i}(y) = 1, y \in \mathbf{A}_i \quad \text{and} \quad \chi_{\mathbf{A}_i}(y) = 0, y \notin \mathbf{A}_i, \quad i = 1, 2, 3.$$

Lemma 4.2. *Let* $f = (f_t, t \in \mathbf{T})$, $\lambda = (\lambda_t, t \in \mathbf{T})$ *and* $(\alpha_t, t \in \mathbf{T})$ *be defined as in Lemma* 4.1 *and* $1 < p \le q < \infty$. *If there exists a constant* $\alpha_0 \in (1, \infty)$ *for which* $\sum_{t\in\mathbf{T}} (E\alpha_t)^{1/\alpha_0} < \infty$, *then there also exists a constant* $\beta > 1$ *such that*

$$\sum_{t\in\mathbf{T}} (\chi_{\mathbf{A}_1} E\alpha_t)^{1/\beta} \le \left(\sum_{t\in\mathbf{T}} \chi_{\mathbf{A}_1} E\alpha_t\right)^{1+(q/p)}, \tag{4.7}$$

$$\sum_{t\in\mathbf{T}} (\chi_{\mathbf{A}_2} E\alpha_t)^{1/\beta} \le \left(\sum_{t\in\mathbf{T}} \chi_{\mathbf{A}_2} E\alpha_t\right)^{p/q}. \tag{4.8}$$

Proof. Using Hölder's inequality, we can derive that, for any $a, b > 1$ with $(1/a) + (1/b) = 1$,

$$E\alpha_t \le (E\alpha_t)^{1/a} \quad (t \in \mathbf{T}). \tag{4.9}$$

Let $Q(x)=\sum_{t\in\mathbf{T}}(E\alpha_t)^{1/x}$ $(x>1)$; then, by (4.9), we see that

$$\sum_{t\in\mathbf{T}} E\alpha_t \le Q(x) \quad (x>1). \tag{4.10}$$

On the other hand, since $0\le E\alpha_t\le 1(t\in\mathbf{T})$, if $1<x_1<x_2$ then $(E\alpha_t)^{1/x_1}<(E\alpha_t)^{1/x_2}$ for any $t\in\mathbf{T}$. Thus,

$$\sum_{t\in\mathbf{T}}(E\alpha_t)^{1/x_1} \le \sum_{t\in\mathbf{T}}(E\alpha_t)^{1/x_2}, \text{ i.e., } Q(x_1)\le Q(x_2). \tag{4.11}$$

Moreover, it follows from the assumption (there exists a constant $\alpha_0\in(1,\infty)$ such that $\sum_{t\in\mathbf{T}}(E\alpha_t)^{1/\alpha_0}<\infty$) and (4.10), (4.11) that $Q(x)$ satisfies on interval $[1,\alpha_0]$

$$\sum_{t\in\mathbf{T}} E\alpha_t \le Q(x) \le Q(\alpha_0) = \sum_{t\in\mathbf{T}}(E\alpha_t)^{1/\alpha_0} < \infty. \tag{4.12}$$

Therefore, by (4.11) and (4.12), we see that $Q(x)$ is a finite increasing function with respect to x on interval $[1,\alpha_0]$. And, by (4.12), we further see that

$$\lim_{x\to 1^+} Q(x) = \lim_{x\to 1^+}\sum_{t\in\mathbf{T}}(E\alpha_t)^{1/x} = \sum_{t\in\mathbf{T}} E\alpha_t, \tag{4.13}$$

that is, the series $Q(x)=\sum_{t\in\mathbf{T}}(E\alpha_t)^{1/x}$ converges to $\sum_{t\in\mathbf{T}} E\alpha_t$ as $x\to 1^+$ on interval $[1,\alpha_0]$. If $y\in\mathbf{A}_1$, then $1<\sum_{t\in\mathbf{T}}(\chi_{\mathbf{A}_1}E\alpha_t)<\infty$. It is easy to see that

$$\sum_{t\in\mathbf{T}}(\chi_{\mathbf{A}_1}E\alpha_t) < \left[\sum_{t\in\mathbf{T}}(\chi_{\mathbf{A}_1}E\alpha_t)\right]^{1+(q/p)} < \infty. \tag{4.14}$$

Also, it follows from (4.13) that

$$\lim_{x\to 1^+}\sum_{t\in\mathbf{T}}(\chi_{\mathbf{A}_1}E\alpha_t)^{1/x} = \sum_{t\in\mathbf{T}}\chi_{\mathbf{A}_1}E\alpha_t < \infty. \tag{4.15}$$

Therefore, from (4.14) and (4.15), one can see that there exists a constant $\beta'\in(1,\alpha_0]$ such that

$$\sum_{t\in\mathbf{T}}(\chi_{\mathbf{A}_1}E\alpha_t)^{1/\beta'} \le \left(\sum_{t\in\mathbf{T}}\chi_{\mathbf{A}_1}E\alpha_t\right)^{1+(q/p)}.$$

If $y \in \mathbf{A}_2$, then $0 \leq \sum_{t\in\mathbf{T}}(\chi_{\mathbf{A}_2} E\alpha_t) < 1$. Furthermore, since $0 < (p/q) < 1$, we have

$$\sum_{t\in\mathbf{T}} \chi_{\mathbf{A}_2} E\alpha_t < \Bigg(\sum_{t\in\mathbf{T}} \chi_{\mathbf{A}_2} E\alpha_t\Bigg)^{p/q} < 1.$$

In the same way, we can show that there is a constant $\beta'' \in (1, \alpha_0]$ such that

$$\sum_{t\in\mathbf{T}} (\chi_{\mathbf{A}_2} E\alpha_t)^{1/\beta''} \leq \Bigg(\sum_{t\in\mathbf{T}} \chi_{\mathbf{A}_2} E\alpha_t\Bigg)^{p/q}.$$

Set $\beta = \min\{\beta', \beta''\}$. Then β is a constant satisfying (4.7) and (4.8). □

A decomposition of martingale differences of an X-valued tree martingale $f = (f_t, t \in \mathbf{T})$ is expressed in terms of

$$u^{(k)} = (u_t^{(k)},\ t \in \mathbf{T}) = (\chi_{\{2^k < \|d_t f\| \leq 2^{k+1}\}},\ t \in \mathbf{T}), \quad k \in \mathbf{Z}. \tag{4.16}$$

It is clear that, for an X-valued tree martingale $f = (f_t, t \in \mathbf{T})$, we have

$$|d_t f|^{2a} = \sum_{k\in Z} u_t^{(k)} |d_t f|^{2a}, \qquad |d_t f|^a = \sum_{k\in Z} u_t^{(k)} |d_t f|^a. \tag{4.17}$$

Lemma 4.3. *Let $f = (f_t, t \in \mathbf{T})$, $\lambda = (\lambda_t, t \in \mathbf{T})$, and $\varepsilon = (\varepsilon_t, t \in \mathbf{T})$ be defined as in Lemma 4.1. If $\|S^{(a)}(f)\|_\alpha < \infty$, $\|s^{(a)}(f)\|_\alpha < \infty$, $\inf_{t\in\mathbf{T}}\|f_t\| \geq c > 0$ and $1 < a < p$, then*

$$E\Bigg[E_t\Bigg(\sum_{r\geq t}\sum_{k\leq -1} \varepsilon_r u_r^{(k)} \|d_r f\|^a\Bigg)^{p/a}\Bigg] \leq Cx^{2a}, \tag{4.18}$$

$$E\Bigg[E_t\Bigg(\sum_{r\geq t}\sum_{k\leq -1} u_r^{(k)} E_r\|\varepsilon_r d_r f\|^a\Bigg)^{p/a}\Bigg] \leq Cx^{2a}. \tag{4.19}$$

Proof. *Case* 1: $x < c/2$. From the definition of a predictable tree martingale $f = (f_t, t \in \mathbf{T})$, we easily see that

$$\|f_{r^+}\| \leq \lambda_{r^+}, \qquad \|f_r\| \leq \lambda_r \leq \lambda_{r^+} \quad (r \geq t; r, t \in \mathbf{T}), \tag{4.20}$$

and by $\varepsilon_r = \chi_{\{x<\lambda_{r^+}\le 2x\}}$ and (4.20), one can further derive that

$$\|f_{r^+}\| + \|f_r\| \le 4x. \tag{4.21}$$

Using (4.21) and $\inf_{t\in\mathbf{T}}\|f_t\| \ge c > 0$, we can derive that $x \ge c/2$; this is an antinomy with the assumption $x < c/2$. Therefore,

$$\varepsilon_r = \chi_{\{x<\lambda_{r^+}\le 2x\}} \le \chi_{\{\|f_{r^+}\|+\|f_r\|\le 4x\}} = 0. \tag{4.22}$$

We easily get the following equality from (4.22)

$$E\Bigg[E_t\bigg(\sum_{r\ge t}\sum_{k\le -1} u_r^{(k)}\|\varepsilon_r d_r f\|^a\bigg)^{p/a}\Bigg] = 0.$$

This implies (4.18) as desired.

Case 2: $x \ge c/2$. By [**19**, Theorem 2.10, page 22] and $\|s^{(a)}(f)\|_p < \infty$, we can also derive that

$$\begin{aligned} E\Bigg[E_t\bigg(\sum_{r\ge t}\sum_{k\le -1} u_r^{(k)}E_r\|\varepsilon_r d_r f\|^a\bigg)^{p/a}\Bigg] & \\ &\le C_q^p E\Bigg[\bigg(\sum_{r\ge t}\sum_{k\le -1} \varepsilon_r u_r^{(k)}\|d_r f\|^a\bigg)^{p/a}\Bigg] \\ &\le C_q^p\|s^{(a)}(f)\|_p^p \\ &\le Cx^{2p}. \end{aligned}$$

In the same way, we can show that (4.19) holds. The proof is now complete. $\square$

Lemma 4.4. *Suppose that the conditions of Theorem* 3.3 *are met, and let* $\varepsilon = (\varepsilon_t, t \in \mathbf{T})$ *be defined as in Lemma* 4.1. *If* $p \ge 1$, $\max(a,p) \le q < \infty$, *then for all real numbers* $z, x \in \mathbf{R}^+$,

$$P(Y > zx,\ \mathbf{A}_1) \le C_{pq} z^{-q} x^{-p}\|\lambda\|_{\mathbf{M}^{(p+q)q}}^{(p+q)}, \tag{4.23}$$

$$P(Y > zx,\ \mathbf{A}_2) \le C_{pq} z^{-q} x^{q-(p^3/q^2)}\|\lambda\|_{\mathbf{M}^{pq}}^{p^3/q^2}, \tag{4.24}$$

where Y *denotes one of the functions* $\sup_{t\in\mathbf{T}}\alpha_t S_{\varepsilon,t}^{(a)}(f)$, $\sup_{t\in\mathbf{T}}\alpha_t s_{\varepsilon,t}^{(a)}(f)$.

Proof. Let $Y = \sup_{t\in\mathbf{T}} \alpha_t(y) s_{\varepsilon;t}(f)$. It follows from $|\overline{\phi}| = 1$ that

$$
\begin{aligned}
s^{(a)}_{\varepsilon;t}(f) &= \sup_{s\geq t}\left(\sum_{t\leq r<s} E_r\|\varepsilon_r(d_r f)\|^a\right)^{1/a} \\
&\leq \left(\sum_{r\geq t}\varepsilon_r E_r\|d_r f\overline{\phi}_t\|^a\right)^{1/a}.
\end{aligned}
\tag{4.25}
$$

Substituting (4.25) into this and by [**19**, Theorem 2.10, page 22], we have

$$
\begin{aligned}
E_t[(s^{(a)}_{\varepsilon;t}(f))^\theta] &\leq E_t\left[\left(\sum_{r\geq t}\varepsilon_r E_r\|d_r f\overline{\phi}\|^a\right)^{\theta/a}\right] \\
&\leq C_\theta E_t\left[\left(\sum_{r\geq t}\varepsilon_r\|d_r f\|^a\right)^{\theta/a}\right]
\end{aligned}
\tag{4.26}
$$

for any $\theta \geq a$. By the definition of $u^{(k)}_r$, we can derive that

$$
\begin{aligned}
\|d_r f\|^a &= \sum_{k\leq -1} u^{(k)}_r\|d_r f\|^a + \sum_{k\geq 0} u^{(k)}_r\|d_r f\|^a \\
&\leq \sum_{k\leq -1} u^{(k)}_r\|d_r f\|^a + \sum_{k\geq 0} u^{(k)}_r\|d_r f\|^{2a}.
\end{aligned}
\tag{4.27}
$$

Choose a constant $\alpha > 1$ such that $1/\alpha + 1/\beta = 1$ (where β is required in Lemma 4.2). Since $2 \geq a > 1$, we must have $2a > 2$. Substituting (4.26) and (4.27) into this, and applying Lemma 3.1 , [**19**, Theorem 2.10, page 22] and Jensen's inequality to the one parameter X-valued martingale $(f_r\overline{\phi}_t, r \geq t)$ $(r, t \in \mathbf{T})$, one can further derive that, for any $\theta \geq a$,

$$
\begin{aligned}
(E_t[(s^{(a)}_{\varepsilon;r}(f))^\theta])^\alpha &\leq C_\theta\left(E_t\left[\left(\sum_{r\geq t}\varepsilon_r\|d_r f\|^a\right)^{\theta/a}\right]\right)^\alpha \\
&\leq C^\alpha_\theta E_t\left[\left(\sum_{r\geq t}\varepsilon_r\|d_r f\|^a\right)^{(\theta\alpha)/a}\right]
\end{aligned}
\tag{4.28}
$$

$$
\begin{aligned}
&\le C_\theta^\alpha E_t\Bigg[\Bigg(\sum_{r\ge t}\varepsilon_r\Bigg(\sum_{k\le -1}u_r^{(k)}\|d_rf\|^a \\
&\quad+\sum_{k\ge 0}u_r^{(k)}\|d_rf\|^{2a}\Bigg)\Bigg)^{(2\theta\alpha)/(2a)}\Bigg] \\
&\le C_\theta^\alpha E_t\Bigg[\Bigg(\sum_{r\ge t}\sum_{k\le -1}\varepsilon_r u_r^{(k)}\|d_rf\|^a\Bigg)^{(\theta\alpha)/a}\Bigg] \\
&\quad+E_t\Bigg[\Bigg(\sum_{r\ge t}\varepsilon_r\|d_rf\|^{2a}\Bigg)^{(2\theta\alpha)/2a}\Bigg] \\
&\le C_\theta^\alpha E_t\Bigg[\Bigg(\sum_{r\ge t}\sum_{k\le -1}\varepsilon_r u_r^{(k)}\|d_rf\|^a\Bigg)^{(\theta\alpha)/a}\Bigg] \\
&\quad+C_\theta^\alpha E_t\Bigg[\|\sum_{r\ge t}\varepsilon_r(E_{r^+}f-E_rf)\|^{2\theta\alpha}\Bigg] \\
&= C_\theta^\alpha E_t\Bigg[\Bigg(\sum_{r\ge t}\sum_{k\le -1}\varepsilon_r u_r^{(k)}\|d_rf\|^a\Bigg)^{(\theta\alpha)/a}\Bigg] \\
&\quad+C_\theta^\alpha E_t\Bigg[\|\sum_{r\ge t}\varepsilon_r(f_{r^+}-f_r)\|^{2\theta\alpha}\Bigg].
\end{aligned}
$$

Set

$$
r_0=\min(r\ge t:\lambda_{r^+}>x),\qquad \eta=\min(r\ge t:\lambda_{r^+}>2x).
$$

By the definition of ε_r and (4.28), we have

$$
\begin{aligned}
(E_t[(s_{\varepsilon;r}^{(a)}(f))^\theta])^\alpha &\le C_\theta^\alpha E_t\Bigg[\Bigg(\sum_{r\ge t}\sum_{k\le -1}\varepsilon_r u_r^{(k)}\|d_rf\|^a\Bigg)^{(\theta\alpha)/a}\Bigg] \\
&\quad+C_\theta^\alpha x^{2\theta\alpha}.
\end{aligned}\tag{4.29}
$$

Taking the expectation and substituting (4.18) of Lemma 4.3 into this yields that

$$
\begin{aligned}
E[(E_t(s_{\varepsilon;r}^{(a)}(f))^\theta)^\alpha] &\le C_\theta^\alpha E\Bigg[E_t\Bigg(\sum_{r\ge t}\sum_{k\le -1}\varepsilon_r u_r^{(k)}\|d_rf\|^a\Bigg)^{(\theta\alpha)/a}\Bigg] \\
&\quad+C_\theta^\alpha x^{2\theta\alpha} \\
&\le C_\theta x^{2\theta\alpha}+C_\theta^\alpha x^{2\theta\alpha}\le C_\theta x^{2\theta\alpha}.
\end{aligned}\tag{4.30}
$$

On the other hand, by Chebyshev's inequality and [**19**, Theorem 2.10, page 22], for any $\theta \geq q$, we have

$$
\begin{aligned}
P(Y > zx) &\leq (zx)^{-q} E \sup_{t \in \mathbf{T}} \alpha_t (s_{\varepsilon;t}^{(a)}(f))^q \\
&\leq C_q (zx)^{-q} E\biggl(\sum_{t \in \mathbf{T}} \alpha_t (s_{\varepsilon;t}^{(a)}(f))^\theta \biggr)^{q/\theta} \\
&\leq C_q (zx)^{-q} E\biggl(\sum_{t \in \mathbf{T}} \alpha_t E_t (s_{\varepsilon;t}^{(a)}(f))^\theta \biggr)^{q/\theta}.
\end{aligned}
\tag{4.31}
$$

Since $0 < q/\theta < 1$, using Jensen's inequality we get

$$
\begin{aligned}
E\biggl(\sum_{t \in \mathbf{T}} \alpha_t E_t (s_{\varepsilon;t}^{(a)}(f))^\theta \biggr)^{q/\theta} &\leq \biggl(E \sum_{t \in \mathbf{T}} \alpha_t E_t (s_{\varepsilon;t}^{(a)}(f))^\theta \biggr)^{q/\theta} \\
&\leq \biggl(\sum_{t \in \mathbf{T}} E[\alpha_t E_t (s_{\varepsilon;t}^{(a)}(f))^\theta] \biggr)^{q/\theta},
\end{aligned}
\tag{4.32}
$$

and since $1/\alpha + 1/\beta = 1$, using Hölder's inequality and (4.30), we have

$$
\begin{aligned}
E[\alpha_t E_t (s_{\varepsilon;t}^{(a)}(f))^\theta] &\leq (E\alpha_t^\beta)^{1/\beta} (E[(E_t (s_{\varepsilon;r}^{(a)}(f))^\theta)^\alpha])^{1/\alpha} \\
&\leq C_\theta x^{2\theta} (E\alpha_t)^{1/\beta}.
\end{aligned}
\tag{4.33}
$$

Set $\theta = q^2/p$; substituting (4.32) and (4.33) into this yields that

$$
\begin{aligned}
E\biggl(\sum_{t \in \mathbf{T}} \alpha_t E_t (s_{\varepsilon;t}^{(a)}(f))^\theta \biggr)^{q/\theta} &\leq \biggl(\sum_{t \in \mathbf{T}} C_\theta x^{2\theta} (E\alpha_t)^{1/\beta} \biggr)^{q/\theta} \\
&\leq \biggl(C_\theta x^{2\theta} \sum_{t \in \mathbf{T}} (E\alpha_t)^{1/\beta} \biggr)^{q/\theta}.
\end{aligned}
\tag{4.34}
$$

Next, since $(p+q)/p > 1$, by Jensen's inequality, (4.7) of Lemma 4.2

and (4.34), one can show that

$$
\begin{aligned}
(4.35)\quad E\Big(\sum_{t\in\mathbf{T}}\chi_{\mathbf{A}_1}\alpha_t E_t(s^{(a)}_{\varepsilon;t}(f))^\theta\Big)^{q/\theta}
&\le C_\theta x^{2q}\bigg(\Big(\sum_{t\in\mathbf{T}}\chi_{\mathbf{A}_1}E\alpha_t\Big)^{(p+q)/p}\bigg)^{q/\theta}\\
&\le C_\theta x^{2q}\chi_{\mathbf{A}_1}\Big[E\Big(\sum_{t\in\mathbf{T}}\alpha_t\Big)\Big]^{(p+q)/p}\\
&\le C_\theta x^{2q}\chi_{\mathbf{A}_1}E\Big(\sum_{t\in\mathbf{T}}\alpha_t\Big)^{(p+q)/q}.
\end{aligned}
$$

By the definition of the quasi-norm $\|.\|_{\mathbf{M}^{(p+q)q}}$, one can derive that

$$
\begin{aligned}
(4.36)\quad E\Big(\sum_{t\in\mathbf{T}}\alpha_t\Big)^{(p+q)/q} &= E\Big(\sum_{t\in\mathbf{T}}\chi_{\{\lambda_t>x,\lambda_s\le x,s\le t\}}\Big)^{(p+q)/q}\\
&\le x^{-(p+q)}\|\lambda\|^{(p+q)}_{\mathbf{M}^{(p+q)q}}.
\end{aligned}
$$

Using (4.31) and substituting (4.35) and (4.36) into this, one can further show that

$$
\begin{aligned}
(4.37)\quad P(Y>zx,\ \mathbf{A}_1) &\le C_\beta(zx)^{-q}E\Big(\sum_{t\in\mathbf{T}}\chi_{\mathbf{A}_1}\alpha_t E_t(s^{(a)}_{\varepsilon;t}(f))^\theta\Big)^{q/\theta}\\
&\le C_{pq}z^{-q}x^q\chi_{\mathbf{A}_1}E\Big(\sum_{t\in\mathbf{T}}\alpha_t\Big)^{(p+q)/q}\\
&\le C_{pq}z^{-q}x^{-p}\|\lambda\|^{p+q}_{\mathbf{M}^{(p+q)q}}.
\end{aligned}
$$

On the other hand, since $q/p>1$, by using Jensen's inequality, (4.8) of Lemma 4.2 and (4.34), one can show that

$$
\begin{aligned}
(4.38)\quad E\Big(\sum_{t\in\mathbf{T}}\chi_{\mathbf{A}_2}\alpha_t E_t(s^{(a)}_{\varepsilon;t}(f))^\theta\Big)^{q/\theta}
&\le C_\theta x^{2q}\bigg(\Big(\sum_{t\in\mathbf{T}}\chi_{\mathbf{A}_2}E\alpha_t\Big)^{p/q}\bigg)^{q/\theta}\\
&\le C_\theta x^{2q}\bigg[\chi_{\mathbf{A}_2}E\Big(\sum_{t\in\mathbf{T}}\alpha_t\Big)\bigg]^{(p/q)^2}\\
&\le C_\theta x^{2q}\chi_{\mathbf{A}_2}\bigg[\bigg(E\Big(\sum_{t\in\mathbf{T}}\alpha_t\Big)\bigg)^{q/p}\bigg]^{(p/q)^3}\\
&\le C_\theta x^{2q}\chi_{\mathbf{A}_2}\bigg[E\Big(\sum_{t\in\mathbf{T}}\alpha_t\Big)^{q/p}\bigg]^{(p/q)^3}.
\end{aligned}
$$

By the definition of the quasi-norm $\|.\|_{\mathbf{M}^{pq}}$ we can also derive that

$$
\begin{aligned}
(4.39)\quad \bigg[E\Big(\sum_{t\in\mathbf{T}}\alpha_t\Big)^{q/p}\bigg]^{(p/q)^3} &= \bigg[E\Big(\sum_{t\in\mathbf{T}}\chi_{\{\lambda_t>x,\lambda_s\le x,s\le t\}}\Big)^{q/p}\bigg]^{(p/q)^3}\\
&\le x^{-p^3/q^2}\|\lambda\|^{p^3/q^2}_{\mathbf{M}^{pq}}.
\end{aligned}
$$

Using (4.31) and substituting (4.38) and (4.39) into this, one can further show that

$$
\begin{aligned}
(4.40)\quad P(Y>zx,\ \mathbf{A}_2) &\le C_q(zx)^{-q}\chi_{\mathbf{A}_2}E\Big(\sum_{t\in\mathbf{T}}\alpha_t E_t(\sigma^p_{\varepsilon;t}(f))^\theta\Big)^{q/\theta}\\
&\le C_{pq}z^{-q}x^q\chi_{\mathbf{A}_2}\bigg[E\Big(\sum_{t\in\mathbf{T}}\alpha_t\Big)^{q/p}\bigg]^{(p/q)^3}\\
&\le C_{pq}z^{-q}x^{q-(p^3/q^2)}\|\lambda\|^{p^3/q^2}_{\mathbf{M}^{qp}}.
\end{aligned}
$$

The proof is complete. □

Proof of Theorem 3.3.

Proof. To prove Theorem 3.3 we need to verify the following inequality

$$y^pP(s^{(a)}(f) > 9y) \le C_{pq}(\|\lambda\|^p_{\mathbf{M}^{qp}} + \|\lambda\|^{p^3/q^2}_{\mathbf{M}^{qp}} + \|\lambda\|^{(p+q)}_{\mathbf{M}^{(p+q)q}}). \tag{4.41}$$

We divide the proof of (4.41) into several steps.

Step I. We need to verify that, for any $y \in \mathbf{R}^+$, choose $j \in \mathbf{Z}$ such that $2^j < y \le 2^{j+1}$. Then

$$y^pP(s^{(a)}(f) > 9y) \le \|\lambda\|^p_{\mathbf{M}^{p\infty}} + y^pP\bigg(\sum_{k\le j} Y_k > 2^j\bigg). \tag{4.42}$$

Equation (4.42) has been proved in Theorem 2.2 ([**11**, page 226] or [**19**, page 151]).

Step II. We shall discuss this step by dividing into the following three cases.

Case 1: $y \in \mathbf{A}_1$. Applying (4.23) of Lemma 4.4 to

$$Y_k = \sup_{t\in\mathbf{T}} \alpha_t^k s^{(a)}_{\varepsilon^{(k)},t}(f), \quad k \in \mathbf{Z}, \tag{4.43}$$

we get for any $k \in \mathbf{Z}$ and $z_k > 0$ with $x = 2^k$

$$P(Y_k > z_k 2^k,\ \mathbf{A}_1) \le C_{pq} z_k^{-q} 2^{-pk} \|\lambda\|^{(p+q)}_{\mathbf{M}^{(p+q)q}}. \tag{4.44}$$

By (4.42) and (4.44), we can show that the following inequality holds in the same way as in the proof of Theorem 2.2 ([**8**, pages 225–227]):

$$y^pP(Y > 9y,\ \mathbf{A}_1) \le C_{pq} \|\lambda\|^{(p+q)}_{\mathbf{M}^{(p+q)q}}. \tag{4.45}$$

Case 2: $y \in \mathbf{A}_2$. Applying (4.24) of Lemma 4.4 to

$$Y_k = \sup_{t\in\mathbf{T}} \alpha_t^k s^{(a)}_{\varepsilon^{(k)},t}(f), \quad k \in \mathbf{Z}, \tag{4.46}$$

we get, for any $k \in \mathbf{Z}$ and $z_k > 0$ with $x = 2^k$,

$$P(Y_k > z_k 2^k,\ \mathbf{A}_2) \le C_{pq} z_k^{-q} 2^{(q-(p^3/q^2))k} \|\lambda\|_{\mathbf{M}^{qp}}^{p^3/q^2}. \tag{4.47}$$

And it follows from (4.42) that

$$y^p P(Y > 9y,\ \mathbf{A}_2) \le \|\lambda\|_{\mathbf{M}^{p\infty}}^p + y^p P\left(\sum_{k\le j} Y_k > 2^j,\ \mathbf{A}_2\right). \tag{4.48}$$

(i) $j > 0$. For any $k \in \mathbf{Z}$, when $k < 0$, we shall consider the following equation

$$c_{\theta_1} \sum_{k<\min\{j,0\}} 2^{\theta_1(k-j)} = \frac{1}{2}, \tag{4.49}$$

where $\theta_1 > 0$ and $c_{\theta_1} = (1-2^{-\theta_1})/2$. Set

$$z_k 2^k := 2^j c_{\theta_1} 2^{\theta_1(k-j)} = c_{\theta_1} 2^{(\theta_1-1)(k-j)} 2^k. \tag{4.50}$$

Then

$$z_k^{-q} 2^{(q-(p^3/q^2))k} \le C_{\theta_1} y^{-p} 2^{[q-p(p/q)^2-q(\theta_1-1)]k+[p+q(\theta_1-1)]j}. \tag{4.51}$$

Next, we shall consider the following inequality

$$\begin{cases} q - p(p/q)^2 - q(\theta_1 - 1) > 0 \\ [q - p(p/q)^2 - q(\theta_1 - 1)]/\gamma = -[p + q(\theta_1 - 1)]. \end{cases}$$

We obtain by solving the inequality

$$\begin{cases} \theta_1 < 2 - (p/q)^3 \\ \theta_1 = \frac{\gamma-2}{\gamma-1} + \frac{1}{\gamma-1}(\frac{p}{q})^3 - \frac{\gamma}{\gamma-1}\frac{p}{q} \quad (\gamma \in \mathbf{Z}^+); \end{cases}$$

moreover, there exists a θ_1 such that the inequality above holds, because for a fixed pair p, q $(q > p > 1)$, there exists a $\gamma_0 \in \mathbf{Z}^+$ such that

$$2 - \left(\frac{p}{q}\right)^3 > \theta_1 = \frac{\gamma_0 - 2}{\gamma_0 - 1} + \frac{1}{\gamma_0 - 1}\left(\frac{p}{q}\right)^3 - \frac{\gamma_0}{\gamma_0 - 1}\frac{p}{q} > 0,$$

and

$$\begin{aligned}\left[q-p\left(\frac{p}{q}\right)^2-q(\theta_1-1)\right]k+[p+q(\theta_1-1)]j \\ <\frac{1}{\gamma_0}\left[q-p\left(\frac{p}{q}\right)^2-q(\theta_1-1)\right]k \\ +[p+q(\theta_1-1)]j \\ =\frac{1}{\gamma_0-1}\left[q+p\left(1-\left(\frac{p}{q}\right)^2\right)\right](k-j).\end{aligned}\tag{4.52}$$

Substituting (4.52) into (4.51) yields that

$$z_k^{-q}2^{(q-(p^3/q^2))k}\le C_{\theta_1}y^{-p}2^{1/(\gamma_0-1)[q+p(1-(p/q)^2)](k-j)}.\tag{4.53}$$

By using (4.53), one can derive that

$$\begin{aligned}\sum_{k<\min\{j,0\}} z_k^{-q}2^{(q-(p^3/q^2))k}\le C_{pq}y^{-p}\sum_{k<\min\{j,0\}}2^{1/(\gamma_0-1)[q+p(1-(p/q)^2)](k-j)} \\ \le C_{pq}y^{-p}.\end{aligned}\tag{4.54}$$

Substituting (4.53) into (4.47) yields that

$$P(Y_k>z_k2^k,\ \mathbf{A}_2)\le C_{pq}y^{-p}2^{1/(\gamma_0-1)[q+p(1-(p/q)^2)](k-j)}\|\lambda\|_{\mathbf{M}^{qp}}^{p^3/q^2},\tag{4.55}$$

Combining (4.54) and (4.55) yields that

$$\sum_{k<\min\{j,0\}}P(Y_k>z_k2^k,\ \mathbf{A}_2)\le C_{pq}y^{-p}\|\lambda\|_{\mathbf{M}^{qp}}^{p^3/q^2}.\tag{4.56}$$

For any $k\in Z$, when $k\ge 0$, we shall consider the following equation

$$c_{\theta_2}\sum_{0\le k\le j}2^{\theta_2(k-j)}=\frac{1}{2},\tag{4.57}$$

where $\theta_2>0$ and $c_{\theta_2}=(1/2)(1-2^{-\theta_2})/(1-2^{-(j+1)\theta_2})$. Set

$$z_k2^k:=2^jc_{\theta_2}2^{\theta_2(k-j)}=c_{\theta_2}2^{(\theta_2-1)(k-j)}2^k;\tag{4.58}$$

then

$$z_k^{-q} 2^{(q-(p^3/q^2))k} \leq C_{\theta_2} y^{-p} 2^{[q-p(p/q)^2-q(\theta_2-1)]k+[p+q(\theta_2-1)]j}. \tag{4.59}$$

And let $\theta_2 = 1-(p/q)^3$. Then we can derive that

$$\begin{gathered}\left[q-p\left(\frac{p}{q}\right)^2-q(\theta_2-1)\right]k = qk,\\ [p+q(\theta_2-1)]j = p\left[1-\left(\frac{p}{q}\right)^2\right]j,\\ z_k^{-q} 2^{(q-(p^3/q^2))k} \leq C_{\theta_2} y^{-p} 2^{qk+p[1-(p/q)^2]j},\end{gathered} \tag{4.60}$$

and by (4.60) we can further derive that

$$\sum_{0\leq k\leq j} z_k^{-q} 2^{(q-(p^3/q^2))k} \leq C_{pq} y^{-p} \frac{2^{p[1-(p/q)^2]j}[2^{q(j+1)}-1]}{2^q-1}. \tag{4.61}$$

On the other hand, since $E[s^{(a)}(f)] < \infty$, we must have $s^{(a)}(f) < \infty$ almost everywhere. Then $P(s^{(a)}(f) = \infty) = 0$. Therefore, there exist a constant $y_0 \in R$ and $0 < y_0 < \infty$ such that

$$\|s^{(a)}(f)\|_{\mathbf{M}^{p\infty}} = \sup_{y>0} yP(Y>y)^{1/p} = y_0 P(Y>y_0)^{1/p}.$$

Furthermore, there is a fixed $j_0 \in \mathbf{Z}$ such that $2^{j_0} < y_0 \leq 2^{j_0+1}$. If $j > j_0$ and $j \in \mathbf{Z}$, then for any y with $2^j < y \leq 2^{j+1}$, we have $y > y_0$ and

$$yP(Y>y,\ \mathbf{A}_2)^{1/p} \leq y_0 P(Y>y_0,\ \mathbf{A}_2)^{1/p}.$$

In this case, if we can show that the following inequality holds

$$y_0 P(Y>y_0,\ \mathbf{A}_2)^{1/p} \leq C_{pq}(\|\lambda\|_{\mathbf{M}^{qp}} + \|\lambda\|_{\mathbf{M}^{qp}}^{(p/q)^2}),$$

then the following inequality must also hold:

$$yP(Y>y,\ \mathbf{A}_2)^{1/p} \leq C_{pq}(\|\lambda\|_{\mathbf{M}^{qp}} + \|\lambda\|_{\mathbf{M}^{qp}}^{(p/q)^2}).$$

Consequently, we only need to verify that if $j \leq j_0$ and $2^j < y \leq 2^{j+1}$, then the following inequality holds:

$$yP(Y>y,\ \mathbf{A}_2)^{1/p} \leq C_{pq}(\|\lambda\|_{\mathbf{M}^{qp}} + \|\lambda\|_{\mathbf{M}^{qp}}^{(p/q)^2}). \tag{4.62}$$

In fact, it is not difficult to show that (4.62) holds by the results above. For any $j \le j_0$, by (4.61) we can derive that

$$\sum_{0\le k\le j} z_k^{-q} 2^{(q-(p^3/q^2))k} \le C_{pq} y^{-p} \frac{2^{p[1-(p/q)^2]j_0}[2^{q(j_0+1)}-1]}{2^q-1} \le C_{pq} y^{-p}. \tag{4.63}$$

Substituting (4.47) and (4.63) into (4.62) yields that

$$\begin{aligned}\sum_{0\le k\le j} P(Y_k > z_k 2^k,\ \mathbf{A}_2) & \\ &\le C_{pq} y^{-p} \sum_{0\le k\le j} z_k^{-q} 2^{(q-(p^3/q^2))k} \|\lambda\|_{\mathbf{M}^{qp}}^{p^3/q^2} \\ &\le C_{pq} y^{-p} \|\lambda\|_{\mathbf{M}^{qp}}^{p^3/q^2}.\end{aligned} \tag{4.64}$$

On the other hand, it follows from (4.49) and (4.57) that

$$c_{\theta_1} \sum_{k<0} 2^{\theta_1(k-j)} + c_{\theta_2} \sum_{0\le k\le j} 2^{\theta_2(k-j)} = 1. \tag{4.65}$$

Combining (4.65), (4.50) and (4.58) yields that

$$2^j = \sum_{k<0} c_{\theta_1} 2^{\theta_1(k-j)} 2^j + \sum_{0\le k\le j} c_{\theta_2} 2^{\theta_2(k-j)} 2^j = \sum_{k\le j} z_k 2^k. \tag{4.66}$$

That is, if $\sum_{k\le j} Y_k > 2^j$, then $\sum_{k\le j} Y_k > \sum_{k\le j} z_k 2^k$. In addition, $Y_k \ge 0$ and $z_k 2^k \ge 0$; thus, there exist $k_0 \in \mathbf{Z}$ and $k_0 \le j$ at least such that $Y_{k_0} > z_{k_0} 2^{k_0}$. Therefore,

$$\begin{aligned} P\Big(\sum_{k\le j} Y_k > 2^j,\ \mathbf{A}_2\Big) &\le P(Y_{k_0} > z_{k_0} 2^{k_0},\ \mathbf{A}_2) \\ &\le \sum_{k\le j} P(Y_k > z_k 2^k,\ \mathbf{A}_2).\end{aligned} \tag{4.67}$$

Substituting (4.56) and (4.64) into (4.67) yields that

$$\begin{aligned} P\Big(\sum_{k\le j} Y_k > 2^j,\ \mathbf{A}_2\Big) &\le \sum_{k<\min\{j,0\}} P(Y_k > z_k 2^k,\ \mathbf{A}_2) \\ &\quad + \sum_{0\le k\le j} P(Y_k > z_k 2^k,\ \mathbf{A}_2) \\ &\le C_{pq} y^{-p} \|\lambda\|_{\mathbf{M}^{qp}}^{p^3/q^2}.\end{aligned} \tag{4.68}$$

(ii) $j \le 0$. For any $k \in \mathbf{Z}$, since $k \le j$, we must have $k < 0$. In this case, we shall consider the following equation:

$$c_\theta \sum_{k \le j} 2^{\theta(k-j)} = 1, \tag{4.69}$$

where $\theta > 0$, $c_\theta = 1 - 2^{-\theta}$. By (4.69), we can show that the inequality (4.68) holds by using the method in the proof of Theorem 4.7 [**19**, pages 150–152]).

Finally, we substitute (4.68) into (4.48) and note that $\|\lambda\|^p_{\mathbf{M}^{p\infty}} \le \|\lambda\|^p_{\mathbf{M}^{qp}}$ (refer to Remark 1). Therefore,

$$y^p P(Y > 9y,\ \mathbf{A}_2) \le C_{pq}(\|\lambda\|^p_{\mathbf{M}^{qp}} + \|\lambda\|^{p^3/q^2}_{\mathbf{M}^{qp}}). \tag{4.70}$$

Case 3. $y \in \mathbf{A}_3$. By Jensen's inequality ($q > p > 1$), we can derive that

$$1 = \sum_{t \in \mathbf{T}} E\alpha_t = \left(\sum_{t \in \mathbf{T}} E\alpha_t\right)^{q/p} \le E\left(\sum_{t \in \mathbf{T}} \alpha_t\right)^{q/p}.$$

Thus, it follows from $y \in \mathbf{R}^+$ that

$$y \le y\left[E\left(\sum_{t \in \mathbf{T}} \alpha_t\right)^{q/p}\right]^{1/q}. \tag{4.71}$$

By the definition of $\|\lambda\|_{\mathbf{M}^{qp}}$ and (4.71), one can further derive that

$$\begin{aligned} \sup_{y \in \mathbf{A}_3} yP(Y > 9y)^{1/p} &\le \sup_{y \in \mathbf{A}_3} y \\ &\le \sup_{y \in \mathbf{A}_3} y\left[E\left(\sum_{t \in \mathbf{T}} \alpha_t\right)^{q/p}\right]^{1/q} \le \|\lambda\|_{\mathbf{M}^{qp}}. \end{aligned} \tag{4.72}$$

Because of (4.72), one sees that

$$y^p P(Y > 9y, \mathbf{A}_3) \le C_{qp}\|\lambda\|^p_{M^{qp}}. \tag{4.73}$$

Step III. By (4.45), (4.70) and (4.73), we can show that (4.41) holds.

Finally, taking the minimum over the quasi-norm $\|\lambda\|_{\mathbf{M}^{qp}}$ of all predictions $\lambda = (\lambda_t, t \in \mathbf{T})$ belonging to f by (4.41), we have

$$\|(s_t^{(p)}(f), t \in \mathbf{T})\|_{\mathbf{M}^{p\infty}} \le C_{pq}(\|f\|_{\mathbf{P}^{qp}} + \|f\|_{\mathbf{P}^{qp}}^{(p/q)^2}) + \|f\|_{\mathbf{P}^{(q+p)q}}^{(p+q)/p}),$$

as required. Now (3.10) of Theorem 3.3 is proved. We can also show that (3.9) of Theorem 3.3 holds in the same way. The whole proof is now complete. □

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School of Automation, Huazhong University of Science and Technology, Wuhan 430074, China
Email address: hetongjun@163.com

School of Automation, Huazhong University of Science and Technology, Wuhan 430074, China
Email address: yinquan@mail.hust.edu.cn

School of Automation, Huazhong University of Science and Technology, Wuhan 430074, China
Email address: shenyi@hust.edu.cn