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THE NUMBER OF MINIMAL COMPONENTS AND HOMOLOGICALLY INDEPENDENT COMPACT LEAVES OF A WEAKLY GENERIC MORSE FORM ON A CLOSED SURFACE

I. GELBUKH

ABSTRACT. On a closed orientable surface M_g^2 of genus g, we consider the foliation of a weakly generic Morse form ω on M_g^2 and show that for such forms $c(\omega) + m(\omega) = g - (1/2)k(\omega)$, where $c(\omega)$ is the number of homologically independent compact leaves of the foliation, $m(\omega)$ is the number of its minimal components, and $k(\omega)$ is the total number of singularities of ω that are surrounded by a minimal component. We also give lower bounds on $m(\omega)$ in terms of $k(\omega)$ and the form rank $\operatorname{rk}\omega$ or the structure of $\ker[\omega]$, where $[\omega]$ is the integration map.

1. Introduction. Consider a closed connected orientable smooth two-dimensional manifold $M = M_g^2$ of genus g. Let ω be a Morse form on M, i.e., a closed 1-form with Morse singularities $\operatorname{Sing}\omega$, locally the differential of a Morse function. This form defines a foliation $\mathcal{F}_\omega$ on $M \setminus \operatorname{Sing}\omega$. A leaf $\gamma \in \mathcal{F}_\omega$ is called compactifiable if $\gamma \cup \operatorname{Sing}\omega$ is compact.

A Morse form is called *generic* if each of its non-compact compactifiable leaves is compactified by a unique singularity [**2**, Definition 9.1]. The set of such forms is dense in any cohomology class [**2**, Lemma 9.2]. The term *generic* introduced in [**2**] is somewhat misleading because the set of such forms is not open. We find it plausible that such forms are the "majority" of Morse forms and thus their properties are in a sense "typical," though we are not aware of any proof of this.

Our results hold for a wider class of forms, which we call *weakly generic*: the requirement for a leaf to be compactified by only one singularity is only applied to the leaves not surrounded by minimal components.

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The number $m(\omega)$ of minimal components and $c(\omega)$ of homologically independent compact leaves are important topological characteristics of the foliation. On M_g^2 it holds [**5**] that

$$0 \leq c(\omega) + m(\omega) \leq g \tag{1}$$

and all such combinations are possible on a given M [**4**]. In particular, if $c(\omega) = g$, then the foliation is compactifiable, i.e., $m(\omega) = 0$, though the converse is not true: there exist compactifiable foliations with $c(\omega) < g$.

In this paper, for weakly generic forms we give a precise expression for $c(\omega) + m(\omega)$ and better bounds on $m(\omega)$. A useful characteristic of a weakly generic form foliation is the number $k(\omega)$ of singularities that are surrounded by a minimal component; for a weakly generic form $k(\omega)$ is even (Corollary 7). Our main result states that, for such forms, the inequality (1) becomes

$$c(\omega) + m(\omega) = g - \frac{k(\omega)}{2} \tag{2}$$

(Theorem 5). In particular, for weakly generic forms on M_g^2, $g \neq 0$, the exact lower bound in (1) is

$$1 \leq c(\omega) + m(\omega) \leq g$$

(Corollary 6). On the other hand (2) gives a criterion for compactifiability for weakly generic forms [**11**]: $m(\omega) = 0$ if and only if $c(\omega) = g$.

Inequality (1) gives an upper bound on the number of minimal components: $m(\omega) \leq g$; this was also proved in [**9**]. For weakly generic forms, (2) gives a better upper bound:

$$m(\omega) \leq g - \frac{k(\omega)}{2}. \tag{3}$$

We are not aware, though, of any *lower* bound on $m(\omega)$ given in literature, except that, if $\operatorname{rk}\omega > g$ (the rank of the group of periods), then the foliation has minimal components: $m(\omega) > 0$ [**11**]. For weakly generic forms, we give a lower bound on $m(\omega)$, cf. (3):

$$m(\omega) \geq g - \frac{k(\omega)}{2} - h(\ker[\omega]) \tag{4}$$

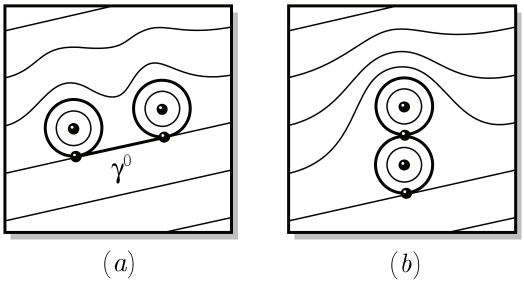

FIGURE 1. Foliations on T^2 with one minimal component. Form (a) is weakly generic, though not generic; form (b) is not.

(Theorem 10). Here, $\ker[\omega] = \langle z \in H_1(M) \mid \int_z \omega = 0 \rangle$ and $h(*)$ is the rank of a maximal subgroup consisting of non-intersecting cycles. We calculate the value of $h(\ker[\omega])$ (Lemma 8) and bound it in terms of $\operatorname{rk}\ker[\omega]$ (Corollary 9).

Bound (4) is not exact; however, it becomes exact together with a trivial observation that $m(\omega) > 0$ if $k(\omega) > 0$. All intermediate values are also reached, except for $m = 1$ when $k = 0$ and $h(\ker[\omega]) = g$; this combination is impossible [**6**]. Our account of the relationships between g, $k(\omega)$, $h(\ker[\omega])$ and $m(\omega)$ is complete: we build a (generic) form for any combination of these values within the corresponding bounds (Lemma 14).

Since it may be difficult to investigate the structure of $\ker[\omega]$, we give a weaker lower bound not involving $h(\ker[\omega])$:

$$m(\omega) \geq \operatorname{rk}\omega - g - \frac{k(\omega)}{2}$$

(Corollary 12), which can, though, be easier to calculate. This estimate is efficient only for large $\operatorname{rk}\omega$, specifically, for $\operatorname{rk}\omega \geq g$. However, this is the "majority" of all forms: the forms in general position have $\operatorname{rk}\omega = 2g$.

The paper is organized as follows. Section 2 introduces some necessary definitions and facts concerning a Morse form foliation. In Section 3 we prove our main result: $c(\omega) + m(\omega) = g - (1/2)k(\omega)$. Finally, in Section 4 we give the bounds on $m(\omega)$.

2. Definitions and basic facts. Let us introduce, for future reference, some necessary notions and facts about Morse forms and their foliations.

2.1. Morse form. A closed 1-form on M is called a *Morse form* if it is locally the differential of a Morse function. Let ω be a Morse form and $\operatorname{Sing}\omega = \{p \in M \mid \omega(p) = 0\}$ the set of its singularities; this set is finite since the singularities are isolated and M is compact.

By the Morse lemma, in a neighborhood of $p \in \operatorname{Sing}\omega$ on M_g^2 there exist local coordinates (x^1, x^2) such that $\omega(x) = \pm x^1 dx^1 + x^2 dx^2$. If the sign is positive, then p is a *center*; otherwise, p is a *conic singularity*. We denote the set of centers by Ω_0 and that of conic singularities by Ω_1, so that $\operatorname{Sing}\omega = \Omega_0 \cup \Omega_1$. By the Poincaré-Hopf theorem, the following holds

$$|\Omega_1| - |\Omega_0| = 2g - 2. \tag{5}$$

The *rank* of a closed 1-form ω is the rank of its group of periods:

$$\operatorname{rk}\omega = \operatorname{rk}_{\mathbf{Q}}\left\{ \int_{z_1} \omega, \dots, \int_{z_{2g}} \omega \right\},$$

where $z_1, \dots, z_{2g}$ is a basis of $H_1(M_g^2)$. For an exact form, $\operatorname{rk}\omega = 0$.

2.2. Morse form foliation. On $M \setminus \operatorname{Sing}\omega$, the form ω defines a foliation $\mathcal{F}_\omega$. A leaf $\gamma \in \mathcal{F}_\omega$ is *compactifiable* if $\gamma \cup \operatorname{Sing}\omega$ is compact (compact leaves are compactifiable); otherwise, it is *non-compactifiable*. If a foliation contains only compactifiable leaves, it is called *compactifiable*.

The foliation $\mathcal{F}_\omega$ defines a decomposition of M into mutually disjoint sets [**5**]; see Figure 2 (a), (c):

$$M = \left(\bigcup \mathcal{C}_i^{\max}\right) \cup \left(\bigcup \mathcal{C}_j^{\min}\right) \cup \left(\bigcup \gamma_k^0\right) \cup \operatorname{Sing}\omega. \tag{6}$$

The *maximal components* $\mathcal{C}_i^{\max}$ are connected components of the union of all compact leaves. On two-manifolds, the notion of maximal component coincides with the notion of periodic component [**10**]. If $\operatorname{Sing}\omega \neq \varnothing$, each maximal component is a cylinder over a compact leaf: $\mathcal{C}_i^{\max} \cong \gamma_i \times (0,1)$. Consider the group $H_\omega \subseteq H_{n-1}(M)$ generated by the homology classes of all compact leaves; $H_\omega = \langle [\gamma_i], \gamma_i \in \mathcal{F}_\omega \rangle$ [**3**]. We denote by $c(\omega) = \operatorname{rk} H_\omega$ the number of homologically independent compact leaves.

The *minimal components* $\mathcal{C}_j^{\min}$ of the foliation are connected components of the set covered by all non-compactifiable leaves. A foliation consisting of exactly one minimal component (and no maximal components) is called *minimal*. Each non-compactifiable leaf is dense in its minimal component [**1, 8**]. We denote by $m(\omega)$ the number of minimal components. *Par abus de langage*, we say that a minimal component $\mathcal{C}^{\min}$ *contains* a leaf or singularity, or the leaf or singularity is *inside* the minimal component, if it belongs to $\operatorname{int}(\overline{\mathcal{C}^{\min}})$. We denote by $k(\omega) = \sum_{i=1}^{m(\omega)} |\operatorname{int}(\overline{\mathcal{C}_i^{\min}}) \cap \operatorname{Sing}\omega|$ the number of singularities inside minimal components; in Figure 5, $k(\omega) = 2$.

The components $\mathcal{C}_i^{\max}$ and $\mathcal{C}_j^{\min}$ are open; their boundaries lie in the union $(\bigcup_k \gamma_k^0) \cup \operatorname{Sing}\omega$ of non-compact compactifiable leaves and singularities. The number of components, as well as the number of non-compact compactifiable leaves γ_k^0, is finite.

2.3. Weakly generic Morse form. While a foliation $\mathcal{F}_\omega$ is defined on $M \setminus \operatorname{Sing}\omega$, a *singular foliation* $\overline{\mathcal{F}}_\omega$ is defined on the whole M: two points $p, q \in M$ belong to the same *leaf* of $\overline{\mathcal{F}}_\omega$ if there exists a path $\alpha : [0,1] \to M$ with $\alpha(0) = p$, $\alpha(1) = q$ and $\omega(\dot{\alpha}(t)) = 0$ for all t [**2**]. A *singular leaf* contains a singularity.

On $M \setminus \operatorname{Sing}\omega$, $\overline{\mathcal{F}}_\omega$ differs from $\mathcal{F}_\omega$ only by possibly merging together some of its leaves: indeed, non-singular leaves of $\overline{\mathcal{F}}_\omega$ are leaves of $\mathcal{F}_\omega$; the number of singular leaves of $\overline{\mathcal{F}}_\omega$ is finite, and each such leaf consists of a finite number of non-compact leaves of $\mathcal{F}_\omega$ and singularities.

A Morse form is called *generic* if each of its singular leaves contains a unique singularity [**2**]. On M_g^2 this means that each non-compact compactifiable leaf is compactified by only one singularity. The set of generic forms is dense in any cohomology class [**2**].

We call a form *weakly generic* if its non-compact compactifiable leaves not lying inside a minimal component are compactified by only one singularity, while those inside a minimal component can form segments, as γ^0 in Figure 1 (a). On $M \setminus \bigcup_{i=1}^{m(\omega)} \operatorname{int}(\overline{\mathcal{C}_i^{\min}})$ a weakly generic foliation is generic: all its compact singular leaves are either centers or figures of eight, and connected components of the boundaries of minimal components are single-leaf circles; see Figure 2.

2.4. Foliation graph. The configuration formed by the *maximal* components in the decomposition (6) is described by the *foliation graph.* Rewrite (6) as

$$M = \left(\bigcup \mathcal{C}_i^{\max}\right) \cup \left(\bigcup P_j\right),$$

where P_j are connected components of the union $P = (\bigcup \mathcal{C}_j^{\min}) \cup (\bigcup \gamma_k^0) \cup \operatorname{Sing}\omega$ of all non-compact leaves and singularities.

Since $\partial\mathcal{C}_i^{\max} \subseteq P$ consists of one or two connected components, each $\mathcal{C}_i^{\max}$ adjoins one or two of P_j. This allows representing M as a connected graph Γ with edges $\mathcal{C}_i^{\max}$ and vertices P_j: an edge $\mathcal{C}_i^{\max}$ is incident to a vertex P_j if $\partial\mathcal{C}_i^{\max} \cap P_j \neq \varnothing$; see Figure 2.

We call those vertices P_j^I that consist solely of compactifiable leaves and singularities I-vertices, see Figure 2 (b); II-vertices P_j^{II} contain minimal components, such as P_2 in Figure 2 (d). Note that I-vertices are compact singular leaves (including center singularities). A II-vertex can contain several minimal components separated by compactifiable leaves.

3. Total number of homologically independent compact leaves and minimal components.

Lemma 1. *Let P be an* I*-vertex. Then* $\deg P = 1$ *if and only if P is a center.*

Proof. If P is a center, in its neighborhood the manifold foliates into circles. Thus, a unique cylinder adjoins P, and so $\deg P = 1$.

Conversely, if P is not a center, then $P = (\bigcup_i \gamma_i^0) \cup (\bigcup_j s_j)$, where γ_i^0 are non-compact compactifiable leaves and $s_j \in \Omega_1$. In a neighborhood of P the form is exact: $\omega = df$, $f(P) = 0$. The components covering the areas $\{f > 0\}$ and $\{f < 0\}$ are locally distinct. Since P is a I-vertex, these have to be maximal components, which means $\deg P \geq 2$. ◻

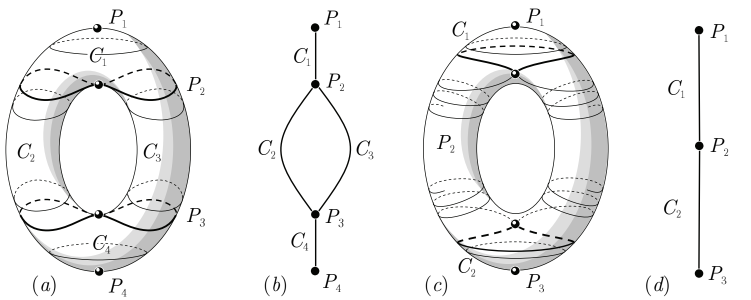

FIGURE 2 (a), (c). Examples of the decomposition (6). (b) Vertices of Γ can include singularities, non-compact compactifiable leaves, and (d) whole minimal components.

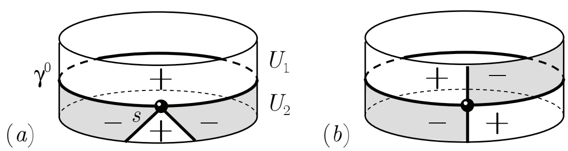

FIGURE 3. Possible (a) and impossible (b) configuration of the leaves adjoining the singularity s. Areas with different sign of f are shown in different colors.

Lemma 2. *Let $\gamma^0 \in \mathcal{F}_\omega$ be a non-compact compactifiable leaf such that $\gamma^0 \cup s$ is compact for some $s \in \operatorname{Sing}\omega$. Then, in any neighborhood of $\overline{\gamma^0} = \gamma^0 \cup s$, there exists a compact leaf $\gamma \in \mathcal{F}_\omega$.*

Proof. Similarly, consider a small cylindrical neighborhood U of $\overline{\gamma^0}$ such that $U \cap \operatorname{Sing}\omega = \{s\}$. In this neighborhood, $\omega = df$; let $f(\gamma^0) = 0$. The set $U \setminus \overline{\gamma^0}$ has two connected components U_1, U_2. Locally, there are exactly four (non-compact) leaves adjoining s, and f changes sign when crossing a leaf. Since $U \cap \operatorname{Sing}\omega = \{s\}$, the function f has a constant sign in one of U_i (see Figure 3); let $f > 0$ in U_1. Then there exists a $t > 0$ such that a connected component γ of $f^{-1}(t)$ is a compact leaf and lies in U. ◻

The condition of Lemma 2 requires the leaf to be compactified only by one singularity. For leaves compactified by more than one singularity

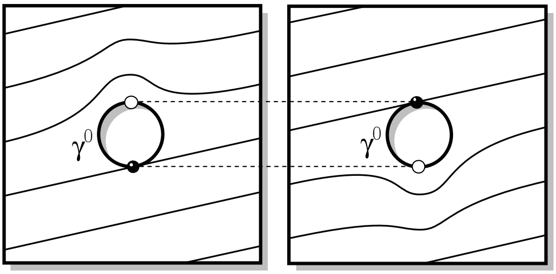

FIGURE 4. Foliation on $M_2^2 = T^2 \sharp T^2$ (connected sum) with a compactifiable leaf γ^0, two minimal components, and without compact leaves.

the conclusion of Lemma 2 may not hold: there exist non-compact compactifiable leaves without compact leaves in their neighborhood; see Figure 4.

Proposition 3. *Let P be an* I*-vertex of a weakly generic form. Then either P is a center or* $\deg P = 3$.

Proof. If P is not a center, then $P = S^1 \vee_s S^1$, $s \in \Omega_1$. As in Lemma 2, in a small neighborhood of P the form is exact, so leaves of the foliation are levels of a Morse function. Since P contains a unique singularity, close levels have one and two connected components, correspondingly. Thus, $\deg P = 3$. □

Proposition 4. *Let P be a* II*-vertex of a weakly generic form. Then*:

(i) *P contains a unique minimal component $\mathcal{C}^{\min}$*;

(ii) *each connected component of $\partial\overline{\mathcal{C}^{\min}}$ locally attaches to $\mathcal{C}^{\min}$ exactly one maximal component*;

(iii) $\deg P = |\partial\overline{\mathcal{C}^{\min}} \cap \operatorname{Sing}\omega|$.

Proof. Since P is a II-vertex, it contains a minimal component $\mathcal{C}^{\min}$. Each connected component ∂_i of $\partial\overline{\mathcal{C}^{\min}}$ is compact and includes exactly one $s \in \operatorname{Sing}\omega$, which adjoins at least one non-compactifiable leaf and at least one non-compact compactifiable leaf γ^0, which adjoins only this singularity. Thus, $\partial_i = \gamma^0 \cup s$. By Lemma 2, there is exactly one maximal component $\overline{\mathcal{C}_i^{\max}}$ glued to $\overline{\mathcal{C}^{\min}}$ by ∂_i; see Figure 3 (a). Therefore, P consists of $\overline{\mathcal{C}^{\min}}$ with $|\partial\overline{\mathcal{C}^{\min}} \cap \operatorname{Sing}\omega|$ maximal components locally attached to it (globally they can be different ends of the same cylinder). $\square$

Now we are ready to prove our main theorem:

Theorem 5. *Let ω be a weakly generic Morse form on M_g^2. Then*

$$c(\omega) + m(\omega) = g - \frac{k(\omega)}{2}.$$

Proof. Denote by n_i the number of vertices of degree i of the foliation graph Γ; $n_i = n_i^{\mathrm{I}} + n_i^{\mathrm{II}}$, where n_i^{I}, n_i^{II} are the corresponding numbers for I- and II-vertices. Similarly, denote Ω_1^{I} and Ω_1^{II} to be the sets of conic singularities belonging to the vertices of each type.

Consider n_i^I. By Lemma 1, it holds that $n_1^I = |\Omega_0|$; Proposition 3 gives $n_3^I = |\Omega_1^I|$ and $n_i^I = 0$ for $i \neq 1, 3$.

Consider n_i^{II}. By Proposition 4 (i), each II-vertex contains a unique minimal component, so $\sum_i n_i^{II} = m(\omega)$. Denote $k_j = |\operatorname{int}(\overline{\mathcal{C}_j^{\min}}) \cap \operatorname{Sing}\omega|$. By Proposition 4 (iii), $|\Omega_1^{\mathrm{II}}| = \sum_i i n_i^{\mathrm{II}} + \sum_j k_j = \sum_i i n_i^{\mathrm{II}} + k(\omega)$.

For the cycle rank $m(\Gamma) = (1/2)\sum_i (i-2)n_i + 1$ [**7**] we have

$$\begin{aligned} 2m(\Gamma) &= -n_1^{\mathrm{I}} + n_3^{\mathrm{I}} + \sum_i i n_i^{\mathrm{II}} - 2\sum_i n_i^{\mathrm{II}} + 2 \\ &= -|\Omega_0| + |\Omega_1^{\mathrm{I}}| + |\Omega_1^{\mathrm{II}}| - k(\omega) - 2m(\omega) + 2. \end{aligned}$$

Since $m(\Gamma) = c(\omega)$ [**5**] and, by (5), this proves the theorem. $\square$

Corollary 6. *For weakly generic forms on M_g^2, $g \neq 0$, the following holds*

$$1 \leq c(\omega) + m(\omega) \leq g;$$

for a given M_g^2, the bounds are exact and all combinations of $c(\omega)$ and $m(\omega)$ within these bounds are possible in the class of generic forms.

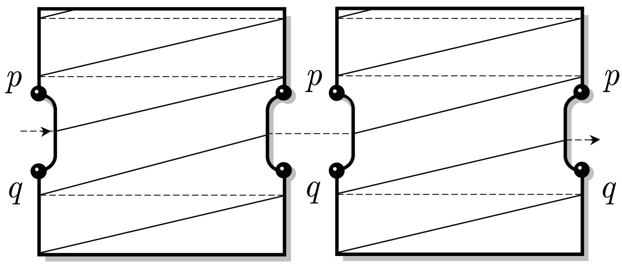

FIGURE 5. Minimal foliation on $M_2^2 = T^2 \sharp\, T^2$.

Proof. If $c(\omega) + m(\omega) = 0$, then $m(\omega) = 0$, and thus $k(\omega) = 0$; Theorem 5 gives $g = 0$. That all intermediate values are reached for generic forms was shown in [**4**]. In particular, on any M_g^2, $g \neq 0$, there exists a minimal foliation [**4**], see Figure 5, which shows the exactness of the lower bound; the upper bound is reached on $\omega = df$. $\square$

The condition for the form to be weakly generic in Corollary 6 is important: on every M_g^2, there exist not weakly generic forms with $c(\omega) + m(\omega) = 0$; see Figure 6.

Theorem 5 and Corollary 6 give:

Corollary 7. *For a weakly generic form on M_g^2, $k(\omega)$ is even. In addition,*

$$0 \le k(\omega) \le 2g - 2$$

if $g \neq 0$; otherwise, $k(\omega) = 0$. On a given M_g^2, the bounds are exact and all (*even*) *intermediate values are possible in the class of generic forms.*

4. Bounds on the number of minimal components. The inequality (1) gives an upper bound on the number of minimal components of a Morse form: $m(\omega) \le g$; this fact was also proven in [**9**]. We obtain a lower bound and a better upper bound on $m(\omega)$ for weakly generic Morse forms.

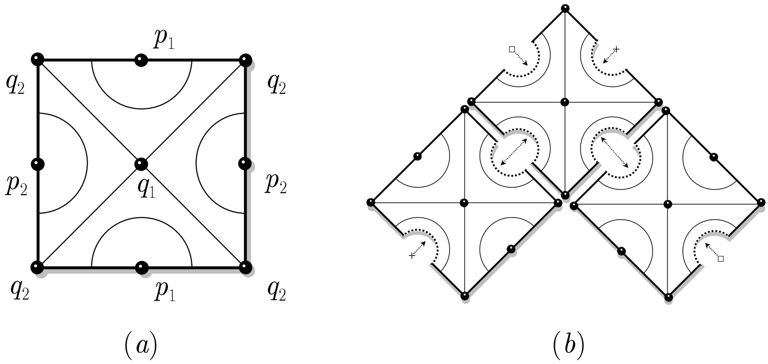

FIGURE 6. Compactifiable foliation with $c(\omega) = 0$ on (a) T^2, (b) $M_g^2 = \sharp\, T_i^2$.

Consider on $H_1(M_g^2)$ the intersection of cycles:

$$\cdot : H_1(M_g^2) \times H_1(M_g^2) \to \mathbf{Z};$$

it is skew-symmetric and non-degenerated. A subgroup $H \subset H_1(M_g^2)$ is called *isotropic* with respect to the intersection $\cdot$ if, for any $z, z' \in H$, it holds that $z \cdot z' = 0$ [**12**]. For an isotropic subgroup, $\operatorname{rk} H \leq g$.

For $G \subseteq H_1(M_g^2)$, denote $h(G) = \operatorname{rk} H$, where $H \subseteq G$ is a maximal isotropic subgroup. For higher-dimensional manifolds M, this value would depend on the choice of H; the maximal rank of an isotropic subgroup is an important topological invariant of a manifold denoted $h(M)$ [**3, 12**]; $h(M_g^2) = h(H_1(M)) = g$ [**13**]. For M_g^2, though, this definition does not depend on the choice of H:

Lemma 8. *Let* $G \subseteq H_1(M_g^2)$. *Then*

$$h(G) = \operatorname{rk} G - \frac{\operatorname{rk} \|z_i \cdot z_j\|}{2},$$

where $\{z_i\}$ *is a basis of* G.

Proof. Obviously, $\operatorname{rk} \|z_i \cdot z_j\|$ does not depend on the choice of the basis $\{z_i\}$. Let $H \subseteq G$ be a maximal isotropic subgroup; denote $n = \operatorname{rk} G$,

$h = \operatorname{rk} H$. Choose a basis $\{z_i\}$ such that $z_i \in H$ for $i \leq h$. Consider $A = \|z_i \cdot z_j\|$:

$$\begin{array}{|ccc|c|}
\multicolumn{1}{c}{1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{h} & \multicolumn{1}{c}{} \\
\hline
0 & \cdots & 0 & \\
\vdots & & \vdots & B \\
0 & \cdots & 0 & \\
\hline
\multicolumn{3}{|c|}{C} & \\
\hline
\end{array}$$

Since H is maximal, the $n-h$ columns of B are independent, and so are the rows of $C = -B^T$ and thus some $n-h$ of its columns. The corresponding $2(n-h)$ columns of A are independent, and no greater system of columns is independent. Thus, $\operatorname{rk} A = 2(n-h)$. □

Corollary 9. *The following holds*:

$$\frac{\operatorname{rk} G}{2} \leq h(G) \leq \min\{\operatorname{rk} G, g\}.$$

Consider the subgroup $\ker[\omega] = \{z \in H_1(M_g^2) \mid \int_z \omega = 0\}$; obviously, $\operatorname{rk} \ker[\omega] = 2g - \operatorname{rk} \omega$, and thus

$$g - \frac{\operatorname{rk} \omega}{2} \leq h(\ker[\omega]) \leq \min\{2g - \operatorname{rk} \omega, g\}. \tag{7}$$

In particular,

$$0 \leq h(\ker[\omega]) \leq g. \tag{8}$$

Since $H_\omega \subseteq \ker[\omega]$,

$$c(\omega) \leq h(\ker[\omega]). \tag{9}$$

It can be shown [**6**] that, if $k(\omega) = 0$ and $m(\omega) \leq 1$, then

$$h(\ker[\omega]) = c(\omega) = g - m(\omega). \tag{10}$$

A lower bound on $m(\omega)$ can be given in terms of the structure of $\ker[\omega]$. Theorem 5, (9) and (10) give:

Theorem 10. *For weakly generic forms ω on M_g^2, the following holds*:

$$g - \frac{k(\omega)}{2} - h(\ker[\omega]) \leq m(\omega) \leq g - \frac{k(\omega)}{2}. \tag{11}$$

In addition,

(i) $m(\omega) > 0$ *if* $k(\omega) > 0$;

(ii) $m(\omega) \neq 1$ *if* $k(\omega) = 0$ *and* $h(\ker[\omega]) = g$.

On a given M_g^2, the bounds given by system (11) *and* (i) *are exact, and all intermediate values are reached except for the case specified in* (ii).

Exactness of the bounds and existence of all intermediate values are shown in Lemma 14 below.

Note that, if $k(\omega) = 0$, then the left side of (11) is non-negative (can be zero) and the bound given by (11) alone is exact. However, if $k(\omega) > 0$, then the left side of (11) can be zero or even negative and (i) can give a better bound. As an example, consider the foliation in Figure 5, assuming the periods $(1, \sqrt{2})$ in each torus; then $h(\ker[\omega]) = 1$ and the left side of (11) is zero. Assuming the periods $(1, \sqrt{2})$ and $(1, -\sqrt{2})$, we have $h(\ker[\omega]) = 2$ and the left side of (11) negative.

Note also that, if $k(\omega) = 0$ and $h(\ker[\omega]) = g$, then $m(\omega) = 0, \cancel{1}, 2, 3, \dots, g$.

Corollary 11. *For a weakly generic form on M_g^2, $m(\omega) = 0$ implies $h(\ker[\omega]) = g$.*

The converse is not true; a counterexample is a connected sum $T^2 \sharp T^2$ with windings with the periods $(1, \sqrt{2})$ and $(1, -\sqrt{2})$, correspondingly.

Since $H \subseteq \ker[\omega]$ implies $\operatorname{rk} H \leq 2g - \operatorname{rk}\omega$, Theorem 10 gives:

Corollary 12. *For weakly generic forms ω on M_g^2, the following holds*:

$$m(\omega) \geq \operatorname{rk}\omega - g - \frac{k(\omega)}{2}.$$

Though this bound is weaker than (11), it is easier to calculate. This bound is efficient for forms with large $\operatorname{rk}\omega$, which are the "majority" of all forms: a form in general position has $\operatorname{rk}\omega = 2g$. In the general case, a Morse form with $\operatorname{rk}\omega = 2g$ (i.e., $\ker[\omega] = 0$) has $c(\omega) = 0$ [**5**] and $m(\omega) \geq 1$ [**3**]. For weakly generic forms, Theorem 10 gives an exact value:

Corollary 13. *For weakly generic forms ω on M_g^2 such that $rk\omega = 2g$, the following holds*:

$$m(\omega) = g - \frac{k(\omega)}{2}.$$

Note that, for $c(\omega)$, (7) and (9) give a bound not involving $k(\omega)$:

$$c(\omega) \leq h(\ker[\omega]) \leq 2g - \operatorname{rk}\omega.$$

The following lemma shows that we have given a complete account of the relations between g, $k(\omega)$, $h(\ker[\omega])$, and $m(\omega)$:

Lemma 14. *For any $g \geq 0$, k, m, and h satisfying the constraints of Corollary* 7*, Theorem* 10 *and* (8)*, on M_g^2 there exists a generic form ω such that $k(\omega) = k$, $m(\omega) = m$, and $h(\ker[\omega]) = h$.*

Proof. Consider g, k, h and m satisfying the constraints:

$$\begin{aligned} & 0 \leq g, \\ \text{Corollary 7}: \quad & 0 \leq k \leq 2g-2 (k=0 \text{ if } g=0);\ k \text{ is even}, \\ \text{Theorem 10}: \quad & 0 \leq m \leq g - \frac{1}{2}k, \\ \text{Theorem 10, (8)}: \quad & c \leq h \leq g;\ h < g \text{ if } k=0 \text{ and } m=1, \end{aligned}$$

where $c = g - (1/2)k - m$. If $g = 0$, then $k = m = 0$, and the statement trivially holds, so we assume $g > 0$. In the rest of the proof we assume that all unspecified periods of ω are incommensurable.

Let $k = 0$ and $m \leq 1$; then, $h = c$. An example is a connected sum $\sharp_{j=1}^c T_j$ of tori with a compact foliation each plus, if $m = 1$, a torus with a minimal foliation. By (10), $h(\ker[\omega]) = h$.

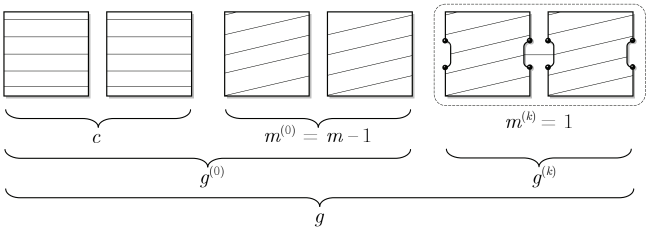

FIGURE 7. Construction of the foliation in Lemma 14.

Let $k = 0$ and $2 \le m \le g$. Consider a connected sum $\sharp$ of m tori $T_i^{(m)}$ with a minimal foliation and $c = g - m$ tori $T_j^{(c)}$ with a compact foliation. Complete H_ω to a maximal isotropic subgroup $H \subseteq \ker[\omega]$ such that $\operatorname{rk} H = h$. Namely, denote $h^{(m)} = h - c$; obviously, $0 \le h^{(m)} \le m$.

(i) Let $h^{(m)} = 0$. Then just choose all incommensurable periods in all $T_i^{(m)}$.

(ii) Let $h^{(m)} = 1$. Choose the periods $(1, \sqrt{2})$ in $T_1^{(m)}$ and $(1, \sqrt{3})$ in $T_2^{(m)}$. Then, $\ker[\omega|_{\sharp T_i^{(m)}}] = \langle z_{11} - z_{21}\rangle$, where z_{i1}, z_{i2} are the basic cycles of $T_i^{(m)}$ corresponding to these periods.

(iii) Let $h^{(m)} = 2$. Similarly, choose the periods $(1, \sqrt{2})$ and $(\sqrt{2}, 1)$ in the first two $T_i^{(m)}$. Then $\ker[\omega|_{\sharp T_i^{(m)}}] = \langle z_{11} - z_{22}, z_{12} - z_{21}\rangle$ is isotropic.

(iv) Let $h^{(m)} = 3$. Choose the periods $(1, \sqrt{2})$, $(\sqrt{2}, -1)$ and $(\sqrt{2} - 1, 2\sqrt{2})$ in the first three $T_i^{(m)}$. By Lemma 8, the isotropic subgroup $\langle z_{11} - z_{21} + z_{31}, z_{12} + z_{22} - z_{31}, z_{12} + z_{21} - z_{32}\rangle$ of $\ker[\omega|_{\sharp T_i^{(m)}}]$ is maximal.

(v) Let $h^{(m)} = 2n$, $n \in \mathbf{N}$. Consider n pairs of tori with periods $(\alpha_i, \alpha_i\sqrt{2})$ and $(\alpha_i\sqrt{2}, \alpha_i)$, so that each pair behaves as in (iii) above, but different pairs are incommensurable.

(vi) Let $h^{(m)} = 2n + 1$. Choose $n - 1$ pairs as in (v) and a triple as in (iv).

By construction, we obtain $h(\ker[\omega]) = c + h^{(m)} = h$.

Now let $k \geq 2$; thus, $g \geq (1/2)k+1$. Construct a manifold $M^{(k)}$ with $g^{(k)} = (1/2)k + 1$, $m(\omega^{(k)}) = 1$, $k(\omega^{(k)}) = k$ as shown in Figure 5 and a manifold $M^{(0)}$ with $g^{(0)} = g - g^{(k)}$, $m(\omega^{(0)}) = m - 1$, $k(\omega^{(0)}) = 0$ as discussed above; see Figure 7. Then $M^{(k)} \sharp M^{(0)}$ has the desired properties. To obtain $h(\ker[\omega]) = h$, $M^{(0)}$ is to be constructed with $h^{(0)} = \min(h, g^{(0)})$ and in $M^{(k)}$, the periods are constructed as in (i)–(vi) above with $h^{(k)} = h - h^{(0)}$ if positive. $\square$

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Department of Mathematics, Moscow State University, Russia
Email address: gelbukh@member.ams.org