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DOUGLAS RANGE FACTORIZATION THEOREM FOR REGULAR OPERATORS ON HILBERT C^*-MODULES

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ABSTRACT. In this paper, we aim to extend the Douglas range factorization theorem from the context of Hilbert spaces to the context of regular operators on a Hilbert C^*-module. In particular, we show that if t and s are regular operators on a Hilbert C^*-module E such that $\operatorname{ran}(t) \subseteq \operatorname{ran}(s)$ and if s has a generalized inverse $s^\dagger$, then $r = s^\dagger t$ is a densely defined operator satisfying $t = sr$. Moreover, if s is boundedly adjointable, then r is closed densely defined and its graph is orthogonally complemented in $E \oplus E$, and if t is boundedly adjointable, then r is boundedly adjointable.

1. Introduction. Douglas established relationships between the notations of majorization, factorization and range inclusion for bounded operators and closed densely defined operators on Hilbert spaces [**1**]. He observed that, if T and S are bounded operators on the Hilbert space H, then the following statements are equivalent:

(i) $\operatorname{range}(T) \subseteq \operatorname{range}(S)$.

(ii) $TT^* \leq \lambda^2 SS^*$ for some $\lambda \geq 0$.

(iii) There exists a bounded operator C on H so that $T = SC$.

Moreover, if t and s are closed densely defined operators on H, then:

(i) If $tt^* \leq ss^*$, then there is a contraction C so that $t \subseteq sC$.

(ii) If $\operatorname{range}(t) \subseteq \operatorname{range}(s)$, then there exists a densely defined operator C so that $t = sC$ and a number $M \geq 0$ so that $\|C(x)\|^2 \leq (\|x\|^2 + \|t(x)\|^2)$ for $x \in \operatorname{Dom}(C)$. Moreover, if t is bounded, then C is bounded and, if s is bounded, then C is closed.

We will refer to the Douglas results as the *Douglas factor decomposition theorem.* In [**9**], Zhang studied the Douglas factor decomposition theorem concerning boundedly adjointable operators on Hilbert

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C^*-modules. We extend the Douglas factor decomposition for closed densely defined operators on Hilbert spaces in the context of regular operators on Hilbert C^*-modules under suitable assumptions based on the work of Zhang and some results of [**2**]. Suppose that t and s are regular operators on the Hilbert C^*-module E and that s has the generalized inverse. If $\operatorname{ran}(t) \subseteq \operatorname{ran}(s)$, then there exists a densely defined operator r on E such that $t = sr$. Moreover, if s is boundedly adjointable, then r is closed densely defined and its graph is orthogonally complemented in $E \oplus E$, and if t is boundedly adjointable, then r is boundedly adjointable.

Let's now review some definitions and basic facts concerning operators on Hilbert C^*-modules, in particular regular operators and their generalized inverses.

Let A be a C^*-algebra. An A-inner product A-module is a right A-module E with compatible scalar multiplication, together with a map $(x, y) \mapsto \langle x, y\rangle : E \times E \to A$, which is A-linear in the second variable and has properties

(i) $\langle x, y\rangle^* = \langle y, x\rangle$ $(x, y \in E)$,

(ii) $\langle x, x\rangle \geq 0$; $\langle x, x\rangle = 0$ if and only if $x = 0$.

An A-inner product A-module E is called a Hilbert C^*-module if E is a Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. The basic properties of Hilbert C^*-modules can be found in [**6, 8**]. As a convention, throughout this paper, we assume that A is a C^*-algebra and E, F are Hilbert A-modules.

It is easy to see that $E \bigoplus F$ is complete with respect to the A-valued inner product $\langle (x, y), (\acute{x}, \acute{y})\rangle = \langle x, \acute{x}\rangle + \langle y, \acute{y}\rangle$, and so $E \bigoplus F$ is a Hilbert C^*-module over A.

Given a submodule M of E, we define

$$M^\perp = \{y \in E : \langle x, y\rangle = 0 \quad (x \in M)\}.$$

Then $M^\perp$ is a closed submodule, but in general, E is not equal to $M \bigoplus M^\perp$ even if M is closed (cf. [**6**]).

We call an A-linear operator T from E to F *boundedly adjointable* if there is a map $T^* : F \to E$ with the property that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x \in E$ and $y \in F$. We denote the space of all boundedly adjointable operators from E to F by $B(E, F)$. We use the notations $D(\cdot)$, $\ker(\cdot)$ and $\operatorname{ran}(\cdot)$ for domain, kernel and range, respectively.

Let $t : D(t) \subseteq E \to F$ be a densely defined A-linear operator. We define a submodule $D(t^*)$ of F by

$$D(t^*) = \{y \in F : \text{ there exists } z \in E \text{ with } \langle t(x), y\rangle = \langle x, z\rangle \quad (x \in D(t))\}.$$

For y in $D(t^*)$ the element z is unique and is written $z = t^*(y)$. This defines an A-linear operator $t^* : D(t^*) \to E$ satisfying

$$\langle x, t^* y\rangle = \langle tx, y\rangle \quad (x \in D(t),\ y \in D(t^*)).$$

We denote the graph of a A-linear operator $t : D(t) \subseteq E \to F$ by $G(t)$ and define $G(t) = \{(x, t(x)) : x \in D(t)\}$. We say t is a closed operator if $G(t)$ is a closed submodule of $E \oplus F$. We define a regular operator from E to F to be a densely defined closed A-linear operator $t : D(t) \subseteq E \to F$ such that t^* is densely defined and $1 + t^*t$ has dense range. We denote the set of all regular operators from E to F by $R(E, F)$. We refer to [**6**, Chapters 9 and 10] for the basic properties of regular operators on Hilbert C^*-modules. Further investigation of regular operators can be found in [**3, 5**]. Define $Q_t = (1 + t^*t)^{-1/2}$ and $F_t = tQ_t$. Then $\operatorname{ran}(Q_t) = D(t)$, $0 \le Q_t \le 1$ in $B(E)$ and $F_t \in B(E, F)$ (cf. [**6**, Chapter 9]).

Frank and Sharifi in [**4**] defined the concept of generalized inverses of unbounded regular operators. Let $t \in R(E, F)$ be a regular operator between Hilbert A-modules E, F. A regular operator $t^\dagger \in R(F, E)$ is called the generalized inverse of t if $t^\dagger t t^\dagger = t^\dagger$, $t t^\dagger t = t$, $(t^\dagger t)^* = \overline{t^\dagger t}$, $(t t^\dagger)^* = \overline{t t^\dagger}$.

Theorem 3.1 of [**4**] states some necessary and sufficient conditions for the existence of generalized inverses of regular operators as follows. Let $t \in R(E, F)$. Then the following conditions are equivalent:

(i) t has a unique polar decomposition $t = V|t|$, where $V \in B(E, F)$ is a partial isometry for which $\ker(V) = \ker(t)$, $\ker(V^*) = \ker(t^*)$, $\operatorname{ran}(V) = \overline{\operatorname{ran}(t)}$, $\operatorname{ran}(V^*) = \overline{\operatorname{ran}(|t|)}$.

(ii) $E = \ker(|t|) \oplus \overline{\operatorname{ran}(|t|)}$ and $F = \ker(t^*) \oplus \overline{\operatorname{ran}(t)}$.

(iii) t and t^* have unique generalized inverses $t^\dagger$ and ${t^*}^\dagger$, respectively, which are adjoint to each other.

In this situation, $V^*V = \overline{t^*t^{*\dagger}}$ is the projection on $\overline{\mathrm{ran}\,(|t|)} = \overline{\mathrm{ran}\,(t^*)}$, and $VV^* = \overline{tt^\dagger}$ is the projection on $\overline{\mathrm{ran}\,(t)}$.

It follows from the proof of Theorem 3.1 of [**3**] that $D(t^\dagger) = \mathrm{ran}\,(t) \oplus \ker(t^*)$ and $t^\dagger(t(x_1+x_2)+x_3) = x_1$ if $x_1 \in D(t) \cap \overline{\mathrm{ran}\,(t^*)}$, $x_2 \in \ker(t)$ and $x_3 \in \ker(t^*)$. Moreover, we have $D(t^{\dagger^*}) = \mathrm{ran}\,(t^*) \oplus \ker(t)$ and $t^{\dagger^*}(t^*(y_1+y_2)+y_3) = y_1$ if $y_1 \in D(t^*) \cap \overline{\mathrm{ran}\,(t)}$, $y_2 \in \ker(t^*)$ and $y_3 \in \ker(t)$.

Note that if t is a regular operator on E with generalized inverse $t^\dagger$ $\ker(t^\dagger) = \ker(t^*)$ and $\mathrm{ran}\,(t^\dagger) = D(t) \cap \overline{\mathrm{ran}\,(t^*)}$. It follows also from [**4**] that $t \in R(E,F)$ has a boundedly adjointable generalized inverse $t^\dagger \in B(F,E)$ if and only if t has closed range.

2. Main results. To obtain our main theorems, we will need to recall some results from [**2**].

Lemma 2.1. *Suppose $S : E \to F$ is a bounded A-linear operator and $D(S^*)$ is dense in F. Then S is boundedly adjointable.*

Proof. We need to verify that $D(S^*) = F$. Let $y \in F$ and choose the sequence $\{y_n\}$ in $D(S^*)$ which converges to $y \in F$ in norm. We then have

$$\begin{aligned}\|S^*(y_n) - S^*(y_m)\| &= \sup\{\|\langle x, S^*(y_n - y_m)\rangle\|; x \in E; \|x\| \le 1\}\\ &= \sup\{\|\langle S(x), y_n - y_m\rangle\|; x \in E; \|x\| \le 1\}\\ &\le \|S\|\|y_n - y_m\|.\end{aligned}$$

Thus, $\{S^*(y_n)\}$ is Cauchy so that $S^*(y_n) \to z$ for some $z \in E$. It follows that

$$\begin{aligned}\langle S(x), y\rangle &= \lim\langle S(x), y_n\rangle = \lim\langle x, S^*(y_n)\rangle\\ &= \langle x, z\rangle, \quad \text{for all } x \in E.\end{aligned}$$

So we have shown that $y \in D(S^*)$. $\square$

The following corollary is an immediate consequence of Lemma 2.1.

Corollary 2.2. *Suppose $t : D(t) \subseteq E \to F$ is a regular operator. Then t is boundedly adjointable if and only if $D(t) = E$.*

We recall that the products of A-linear operators on Hilbert C^*-modules are defined just as for Hilbert space operators. Suppose t and s are two A-linear operators from E to F with domains $D(t)$ and $D(s)$, respectively. Then we define

$$D(ts) = \{x \in D(s); s(x) \in D(t)\}$$

and

$$(ts)(x) = t(s(x)) \quad (x \in D(ts)).$$

Lemma 2.3. *Let $t : D(t) \subseteq F \to F$ be a regular operator, and let $T : E \to F$ be a boundedly adjointable operator. If $D(tT) = E$, then tT is boundedly adjointable.*

Proof. Since t is closed and T is bounded, operator tT has to be closed. A simple calculation shows $D(t^*) \subseteq D((tT)^*)$, so $D((tT)^*)$ is dense in F. Since, by supposition, $D(tT) = E$, operator tT has to be bounded on E by general Banach space properties of linear operators. It follows from Lemma 2.1 that tT is boundedly adjointable. $\square$

Theorem 2.4. *Let t and s be regular operators on E satisfying $D(ts) = D(s)$. Then the graph of ts is orthogonally complemented in $E \oplus E$ if ts is closed.*

Proof. Recall that $D(s) = \operatorname{ran}(Q_s)$ and $F_s = sQ_s$ (see the proof of Lemma 9.2 of [**6**]). We have

$$\begin{aligned} G(ts) &= \{(x, ts(x)) : x \in D(ts)\} = \{(x, ts(x)) : x \in D(s)\} \\ &= \{(Q_s(x), tsQ_s(x)) : x \in D(s)\} = \{(Q_s(x), tF_s(x)) : x \in E\}. \end{aligned}$$

Note that $D(tF_s) = D(tsQ_s) = E$. Lemma 2.3 implies that tF_s is boundedly adjointable. Let us define an $A-$linear operator $V : E \to E \oplus E$ by $V(x) = (Q_s(x), tF_s(x))$. Since tF_s and Q_s are boundedly adjointable, V is boundedly adjointable. Obviously, $\operatorname{ran}(V) = G(ts)$ so the range of V is closed. Therefore, [**6**, Theorem 3.2] implies that $\operatorname{ran}(V) = G(ts)$ is orthogonally complemented in $E \oplus E$. $\square$

Remark 2.5. It must be emphasized that a densely defined closed operator does not necessarily have a complemented graph. To see this, we recall the following example from [**7**].

Let $A = C[0,1]$ and $E = C[0,1] \otimes L^2(0,1)$. Let

$$\begin{aligned} D &= \{f \in L^2(0,1) : f \text{ is absolutely continuous, } f' \in L^2(0,1)\}, \\ D_0 &= \{f \in L^2(0,1) : f \text{ is absolutely continuous,} \\ &\qquad f' \in L^2(0,1),\ f(0) = f(1)\}, \\ D_{00} &= \{f \in L^2(0,1) : f \text{ is absolutely continuous,} \\ &\qquad f' \in L^2(0,1),\ f(0) = f(1) = 0\}. \end{aligned}$$

Define T on D by $T(f) = if'$. Let $T_0 = T|_{D_0}$. Now define an operator t on E as follows:

$$\begin{aligned} D(t) = \{f \in E : f_0 \in D_{00}, f_\pi \in D_0 \text{ for } 0 \leq \pi \leq 1, \\ \pi \longmapsto f'_\pi \text{ continuous}\}, \quad (tf)(\pi) = if'_\pi. \end{aligned}$$

Proposition 2.2 of [**7**] and Proposition 9.5 of [**6**], t is a closed densely defined operator whose graph is not orthogonally complemented in $E \oplus E$.

Suppose s, t are regular operators on E. Recall that $tt^* \leq ss^*$ if $D(ss^*) \subseteq D(tt^*)$ and $\langle tt^*(x), x\rangle \leq \langle ss^*(x), x\rangle$ for all $x \in D(ss^*)$.

We can now state and prove our main results.

Theorem 2.6. *Let t, s be regular operators on E. If s has a boundedly adjointable generalized inverse and $tt^* \leq \lambda ss^*$ for some $\lambda > 0$, then there exists a boundedly adjointable operator Q on E such that $Qs^* \subseteq t^*$.*

Proof. First we show that $D(s^*) \subseteq D(t^*)$. Let $y \in D(s^*)$. Then there is a sequence $\{x_n\}$ in $D(ss^*)$ such that $x_n \to x$ and $s^*(x_n) \to s^*(x)$ since, by [**6**, Lemma 9.2], $D(ss^*)$ is a core for s^*. Therefore, it follows from $tt^* \leq \lambda ss^*$ that $\|t^*(x_n))\| \leq \lambda^{1/2}\|s^*(x_n)\|$. Hence, $\{t^*(x_n)\}$ is a Cauchy sequence and so $t^*(x_n) \to z$ for some $z \in E$. This shows that $x \in D(t^*)$ and $\|t^*(x)\| \leq \lambda^{1/2}\|s^*(x)\|$.

Next, we define $Q_1 : s^*(x) \to t^*(x)$ from $\operatorname{ran}(s^*)$ to E. The above argument shows that Q_1 is well defined and bounded. We shall show that Q_1 is boundedly adjointable. To see this, first note that, for $x \in D(s^*)$, we have $Q_1 s^*(x) = Q_1 s^*(s^*)^\dagger s^*(x) = t^*(s^*)^\dagger s^*(x)$. This shows that $Q_1(x) = t^* {s^*}^\dagger(x)$. Since we assume that $\operatorname{ran}(s^*)$ is closed since, by hypothesis, $(s^*)^\dagger$ is boundedly adjointable. Also $D(t^* {s^*}^\dagger) = \operatorname{ran}(s^*)$ since $\operatorname{ran}({s^*}^\dagger) \subseteq D(s^*)$ and $D(s^*) \subseteq D(t^*)$. Therefore, it follows from Corollary 2.2 that $Q_1 : \operatorname{ran}(s^*) \to E$ is boundedly adjointable.

Note that $\operatorname{ran}(s^*)$ is orthogonally complemented in E since S has a boundedly adjointable generalized inverse. So we can define $Q : E \to E$ by $Q(x) = Q_1(x)$ if $x \in \operatorname{ran}(s^*)$ and $Q(x) = 0$ if $x \in \operatorname{ran}(s^*)^\perp$. It is evident that Q is boundedly adjointable and $Qs^* \subseteq t^*$. This completes the proof. □

Theorem 2.7. *Suppose t and s are regular operators, and s has a generalized inverse. If* $\operatorname{ran}(t) \subseteq \operatorname{ran}(s)$, *then there exists a densely defined operator r of E such that $t = sr$. Moreover, if s is boundedly adjointable, then r is closed densely defined and its graph is orthogonally complemented in $E \oplus E$, and, if t is boundedly adjointable, then r is boundedly adjointable.*

Proof. We have $D(s) = \ker(s) \oplus \ker(s)^\perp \cap D(s)$ since by [**4**, Theorem 3.1], $\operatorname{Ker}(s) = \overline{\operatorname{ran}(s^*)}^\perp$ is orthogonally complemented in E. So, for $x \in D(t)$, there is a unique $y \in \ker(s)^\perp \cap D(s)$ such that $t(x) = s(y)$. This defines an A-linear operator $r : D(t) \subseteq E \to E$ by $r(x) = y$. It is clear that r is a densely defined A-linear map with $D(r) = D(t)$, $\operatorname{ran}(r) \subseteq \ker(s)^\perp$ and $t = sr$. Also $\overline{s^\dagger s}$ is the projection onto $\ker(s)^\perp$. So, for each $x \in D(t)$, we have $s^\dagger t(x) = s^\dagger sr(x) = r(x)$. Therefore r can be written as $r = s^\dagger t$.

Suppose s is boundedly adjointable. First we show that r is closed. To see this, let $\{x_n\}$ be a sequence in $D(r) = D(t)$ such that $x_n \to x$ and $r(x_n) \to y$. Then $t(x_n) = sr(x_n) \to s(y)$ since s is bounded and closedness of t implies that $x \in D(t) = D(r)$ and $t(x) = s(y)$. Note that $r(x_n) \in \ker(s)^\perp$ so $r(x) = y$. This shows that r is closed. Moreover, we have shown that $r = s^\dagger t$. Thus, $D(s^\dagger t) = D(r)$. It follows from Theorem 2.4 that the graph of r is orthogonally complemented in

$E \oplus E$. Finally, if t is boundedly adjointable, then by Lemma 2.3, r is boundedly adjointable. □

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