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A CONSTRUCTIVE PROOF OF THE BISHOP-PHELPS-BOLLOBÁS THEOREM FOR THE SPACE $C(K)$

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ABSTRACT. In 1961 Bishop and Phelps proved that every real Banach space is subreflexive; in other words, every continuous linear functional can be approximated by a norm-attaining functional of the same norm. In 1970, Bollobás refined the result to show that, if f is a norm one functional that "almost" attains its norm at a point x, then f can be approximated by a norm attaining norm one functional that attains its norm at a point close to x. The original proof of the Bishop-Phelps result is an existence argument. We give a constructive proof of the Bishop-Phelps-Bollobás theorem for the real space $C(K)$.

1. Introduction. In 1961, Bishop and Phelps [**1**] proved their well-known theorem that every real Banach space is subreflexive; in other words, the set of support functionals for a closed, bounded, convex subset S of a real Banach space X is norm dense in X^*. In 1970, Bollobás [**2**] showed the following refinement of the Bishop-Phelps result:

Theorem 1.1. *Denote by S and S' the unit spheres in a real Banach space E and its dual space E', respectively. Suppose $x \in S$, $f \in S'$ and $|f(x) - 1| \le \varepsilon^2/2$ $(0 < \varepsilon < 1/2)$. Then there exist $y \in S$ and $g \in S'$ such that $g(y) = 1$, $\|f - g\| \le \varepsilon$ and $\|x - y\| < \varepsilon + \varepsilon^2$.*

In other words, if f is a norm one functional that "almost" attains its norm at a point x, then f can be approximated by a norm attaining norm one functional that attains its norm at a point close to x.

The result that follows relates to Bollobás's formulation of the theorem. We begin by giving a constructive proof of the Bishop-Phelps-Bollobás theorem in the real space $C(K)$.

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2. A constructive proof of the Bishop-Phelps-Bollobás theorem for $C(K)$. In the original proof of the Bishop-Phelps theorem the result is proved by an existence argument, in part using the Hahn-Banach theorem. As a result, the exact structure of the desired functional is not given. The question then arises of whether, in particular spaces, it is possible to give a constructive proof of the Bollobás result. The following is a constructive proof of this result for the real spaces $C(K)$, where K is a compact set in a Banach space. (Note that, while the isometric result for the space $c = C(\mathbf{N} \cup \{\infty\})$ is captured in this case, the space c_0 is a proper subspace of c and as such is not a special case of the $C(K)$ result. A constructive proof for the space c_0 can be found in [**3**].)

The proof for the real space $C(K)$ is accomplished by recognizing that the space $C(K)^*$ is isometrically isomorphic to the space of all regular Borel measures on the set K. The duality is such that, for $\varphi \in C(K)^*$, there is a Borel measure ν on K such that

$$\varphi(h) = \int_K h \, d\nu$$

for every $h \in C(K)$. This is a classical result by Riesz.

Theorem 2.1. *Let $0 < \varepsilon < 1/4$. Let $f_0 \in C(K)$ and $\varphi \in C(K)^*$ be such that $\|f_0\| = \sup_{x\in K} |f_0(x)| = 1$, $\|\varphi\| = 1$ and $\varphi(f_0) > 1 - \varepsilon^2/2$. Then there exist $\widehat{\varphi} \in C(K)^*$ and $\widehat{f} \in C(K)$ such that $\|\widehat{\varphi}\| = \|\widehat{f}\| = \widehat{\varphi}(\widehat{f}) = 1$, $\|\varphi - \widehat{\varphi}\| < 5\varepsilon$ and $\|f_0 - \widehat{f}\| < \varepsilon$.*

Proof. The proof will be completed in three stages: the construction of $\widehat{f}$; the construction of $\widehat{\varphi}$; and finally a verification of the required relations.

(1) *Construction of $\widehat{f}$.* Let $S_1 = \{x \in K : |f_0(x)| < 1 - \varepsilon/3\}$, and let $S_2 = \{x \in K : |f_0(x)| \geq 1 - \varepsilon/3\} = K \backslash S_1$.

As f_0 is continuous on K, it is uniformly continuous, and so there exists $\delta > 0$ such that $x, y \in K$, $\|x - y\| < \delta$ gives $|f_0(x) - f_0(y)| < \varepsilon/3$. Cover the closed, and hence compact, set S_2 with a finite number of open balls $\{I_i : i = 1, \dots, n\}$, where each I_i has diameter at most δ and a non-empty intersection with the set S_2. Let $\mathcal{I} = \cup_{i=1}^n I_i$.

Note that $|f_0(x)| \geq 1 - (2\varepsilon)/3$ for all $x \in \overline{\mathcal{I}}$ and $|f_0(x)| < 1 - \varepsilon/3$ for all $x \in K \backslash \mathcal{I}$.

Let $\mathcal{J} = \{x \in K : |f_0(x)| < 1 - \varepsilon\}$. Then $\overline{\mathcal{I}} \cap \overline{\mathcal{J}} = \varnothing$.

Define $\widehat{f}$ as follows:

$$\widehat{f}(x) = \begin{cases} 1 & \text{if } x \in \overline{\mathcal{I}} \text{ and } f_0(x) > 0 \\ -1 & \text{if } x \in \overline{\mathcal{I}} \text{ and } f_0(x) < 0 \\ f_0(x) & \text{if } x \in \overline{\mathcal{J}}. \end{cases}$$

Then $\widehat{f}$ is continuous on $\overline{\mathcal{I}} \cup \overline{\mathcal{J}}$, so by Tietze's extension theorem, we can extend $\widehat{f}$ continuously to all of K. Also, for all $x \in K \backslash \mathcal{J}$, $|f_0(x)| \geq 1 - \varepsilon$, so we have that $1 - \varepsilon \leq |f_0(x)| \leq 1$ for all x on the boundary of each component of $K \backslash (\overline{\mathcal{I}} \cup \overline{\mathcal{J}})$. Thus, we can extend $\widehat{f}$ in such a way that $1 - \varepsilon \leq |\widehat{f}(x)| \leq 1$ for all $x \in K \backslash (\overline{\mathcal{I}} \cup \overline{\mathcal{J}})$.

(2) *Construction of* $\widehat{\varphi}$. Let ν be the unique regular Borel measure on K such that $\varphi(h) = \int_K h\, d\nu$ for every $h \in C(K)$. So $|\nu|(K) = 1$ as $\|\varphi\| = 1$. By the Hahn decomposition theorem, there exist subsets P_ν and N_ν of K such that:

(a) P_ν is a positive set with respect to ν, N_ν is a negative set with respect to ν, and

(b) $P_\nu \cup N_\nu = K$ and $P_\nu \cap N_\nu = \varnothing$.

Let $P_{f_0} = \{x \in K : f_0(x) \geq 0\}$ and $N_{f_0} = K \backslash P_{f_0}$. Define:

$$A = (P_\nu \cap P_{f_0}) \cup (N_\nu \cap N_{f_0})$$
$$B = (P_\nu \cap N_{f_0}) \cup (N_\nu \cap P_{f_0}).$$

Then $A \cup B = K$ and $A \cap B = \varnothing$.

Define the measure λ on K as follows: For every ν-measurable set $C \subset K$, let

$$\lambda(C) = \nu(C \cap A).$$

We will first prove three claims that we will use to construct $\widehat{\varphi}$.

Claim 1: $|\nu|(B) \leq \varepsilon^2$. Suppose not. Then $|\nu|(B) > \varepsilon^2$ and

$|\nu|(A) \leq 1 - \varepsilon^2$. So:

$$\begin{aligned}1 - \frac{\varepsilon^2}{2} < \varphi(f_0) &= \int_A f_0 \, d\nu + \int_B f_0 \, d\nu \\ &= \int_A |f_0| \, d|\nu| - \int_B |f_0| \, d|\nu| \leq \|f_0\| |\nu|(A) - 0 \\ &\leq 1 \cdot (1 - \varepsilon^2),\end{aligned}$$

which is a contradiction.

Claim 2: $|\nu|(S_2) > 1 - 3\varepsilon/2$. Now,

$$\begin{aligned}1 - \frac{\varepsilon^2}{2} < \varphi(f_0) &\leq \left| \int_{S_1} f_0 \, d\nu \right| + \left| \int_{S_2} f_0 \, d\nu \right| \\ &\leq \left(1 - \frac{\varepsilon}{3}\right) |\nu|(S_1) + 1 \cdot |\nu|(S_2) \\ &= \left(1 - \frac{\varepsilon}{3}\right)(1 - |\nu|(S_2)) + |\nu|(S_2) \\ &= 1 - \frac{\varepsilon}{3} + \frac{\varepsilon}{3} |\nu|(S_2).\end{aligned}$$

Thus,

$$\frac{\varepsilon}{3} - \frac{\varepsilon^2}{2} < \frac{\varepsilon}{3} |\nu|(S_2),$$

i.e.,

$$|\nu|(S_2) > 1 - \frac{3\varepsilon}{2}.$$

Claim 3: $|\lambda|(S_2) = M > 1 - (7\varepsilon)/4 > 9/16$. (In particular, $M \neq 0$.)

By Claim 1, $|\nu|(B) \leq \varepsilon^2$ so $|\nu|(A) > 1 - \varepsilon^2$ which gives

$$|\lambda|(K) = |\nu|(K \cap A) = |\nu|(A) > 1 - \varepsilon^2.$$

By Claim 2, $|\nu|(S_1) < 3\varepsilon/2$. Thus,

$$\begin{aligned}M = |\lambda|(S_2) &= |\lambda|(K) - |\lambda|(S_1) > 1 - \varepsilon^2 - |\nu|(S_1) \\ &> 1 - \varepsilon^2 - \frac{3\varepsilon}{2} > 1 - \frac{7\varepsilon}{4} \quad \text{as } \varepsilon < \frac{1}{4}.\end{aligned}$$

So $|\lambda|(S_2) > 1 - (7\varepsilon)/4 > 9/16$ as $\varepsilon < 1/4$.

Now, for every $f \in C(K)$, define

$$\widehat{\varphi}(f) = \frac{1}{M}\int_{S_2} f\,d\lambda,$$

where $M = |\lambda|(S_2) \neq 0$. Then $\widehat{\varphi} \in C(K)^*$.

(3) *Verification of norms and relations.* First we have that

$$\|\widehat{\varphi}\| = \sup_{\|f\|=1}\left|\int_{S_2} f\,d\lambda\right| \cdot \frac{1}{M} \leq |\lambda|(S_2)\cdot\frac{1}{M} = 1.$$

Also note that $S_2 \subset \overline{\mathcal{I}}$, so $|\widehat{f}(x)| = 1$ for all $x \in S_2$. Thus,

$$\begin{aligned}
\widehat{\varphi}(\widehat{f}) &= \frac{1}{M}\int_{S_2}\widehat{f}\,d\lambda \\
&= \frac{1}{M}\left(\int_{S_2\cap A}\widehat{f}\,d\lambda + \int_{S_2\cap B}\widehat{f}\,d\lambda\right) \\
&= \frac{1}{M}\left(|\nu|(S_2\cap A) + 0\right) \quad \text{as } \lambda(S_2\cap B) = \nu(S_2\cap B\cap A) = 0 \\
&= \frac{1}{M}|\lambda|(S_2) = 1.
\end{aligned}$$

So $\|\widehat{\varphi}\| = \widehat{\varphi}(\widehat{f}) = 1$, and $\widehat{\varphi}$ is norm attaining. Clearly $\|\widehat{f}\| = 1$.

Next, we have:

$$\begin{aligned}
\|f_0 - \widehat{f}\| &= \max_{x\in K}|f_0(x) - \widehat{f}(x)| \\
&= \max\Big\{\max_{x\in\overline{\mathcal{I}}}|f_0(x)-\widehat{f}(x)|, \max_{x\in\overline{\mathcal{J}}}|f_0(x)-\widehat{f}(x)|, \\
&\qquad \max_{x\in K\setminus(\overline{\mathcal{I}}\cup\overline{\mathcal{J}})}|f_0(x)-\widehat{f}(x)|\Big\} \\
&\leq \max\left\{\frac{2\varepsilon}{3}, 0, \varepsilon\right\} = \varepsilon.
\end{aligned}$$

Finally, for every $f \in C(K)$ such that $\|f\| = 1$, we have:

$$
\begin{aligned}
|(\varphi - \widehat{\varphi})(f)| &= \left| \int_K f\,d\nu - \frac{1}{M}\int_{S_2} f\,d\lambda \right| \\
&= \left| \int_K f\,d\nu - \int_{S_2\cap A} f\,d\nu + \int_{S_2\cap A} f\,d\nu \right. \\
&\quad \left. - \frac{1}{M}\int_{S_2\cap A} f\,d\nu \right| \\
&\le \left| \int_{S_1} f\,d\nu + \int_{S_2\cap B} f\,d\nu \right| \\
&\quad + \left| 1 - \frac{1}{M} \right| \left| \int_{S_2\cap A} f\,d\nu \right| \\
&\le |\nu|(S_1) + |\nu|(S_2\cap B) + \left| \frac{M-1}{M} \right| |\nu|(S_2\cap A) \\
&\le |\nu|(S_1) + |\nu|(B) + \frac{1-M}{M}|\nu|(S_2\cap A) \\
&< \frac{3\varepsilon}{2} + \varepsilon^2 + \frac{28}{9}\varepsilon \quad \text{by Claims 2, 1 and 3} \\
&< 5\varepsilon.
\end{aligned}
$$

So

$$\|\varphi - \widehat{\varphi}\| < 5\varepsilon,$$

and this completes the proof. □

The following corollary shows that, if we start with a measure that is singular (respectively, absolutely continuous) with respect to a positive Borel measure, μ, then the construction process yields a functional defined by a measure that is also singular (respectively, absolutely continuous) with respect to μ.

Corollary 2.2. *Let f_0 and φ be as in the statement of Theorem* 2.1. *Suppose*

$$\varphi(f) = \int_K f\,d\nu$$

for all $f \in C(K)$ *and for some Borel measure* ν. *Let* $\widehat{\varphi}$ *and* $\widehat{f}$ *be constructed as in Theorem* 2.1, *and suppose that*

$$\widehat{\varphi}(f) = \int_K f \, d\widehat{\nu}$$

for some Borel measure $\widehat{\nu}$. *Let* μ *denote a positive Borel measure on* K.

(1) *If* ν *is absolutely continuous with respect to* μ, *then so is* $\widehat{\nu}$.

(2) *If* ν *is singular with respect to* μ, *then so is* $\widehat{\nu}$.

Proof. (1) Suppose that ν is absolutely continuous with respect to μ. Then, by the Radon-Nikodym theorem, there exists a μ-measurable function g such that

$$\varphi(f) = \int_K f \, d\nu = \int_K f g \, d\mu$$

for all $f \in C(K)$. In the proof of Theorem 2.1, notice that A and S_2 are both measurable sets, so $A \cap S_2$ is also measurable. Also notice that $\widehat{\varphi}$ can be written as:

$$\widehat{\varphi}(f) = \frac{1}{M} \int_{S_2 \cap A} f \, d\nu = \frac{1}{M} \int_K f \cdot \chi_{S_2 \cap A} \cdot g \, d\mu$$

where χ is the characteristic function. So, if we let $\widehat{g} = \frac{1}{M} \chi_{S_2 \cap A} \cdot g$, then $\widehat{g}$ is a μ-measurable function and, as

$$\widehat{\varphi}(f) = \int_K f \, d\widehat{\nu} = \int_K f \widehat{g} \, d\mu$$

for all $f \in C(K)$, we have that $\widehat{\nu}$ is absolutely continuous with respect to μ.

(2) Suppose that ν is singular with respect to μ. Then there exists a set $E \subset K$ such that $\operatorname{supp}(\nu) \subset E$ and $\mu(E) = 0$. Note that if

$$\widehat{\nu}(C) = \frac{1}{M} \nu(C \cap A \cap S_2),$$

then
$$\int_K f\, d\widehat{\nu} = \frac{1}{M}\int_{A\cap S_2} f\, d\nu = \frac{1}{M}\int_{S_2} f\, d\lambda = \widehat{\varphi}(f)$$
and $\operatorname{supp}(\widehat{\nu}) \subset E \cap A \cap S_2 \subset E$. In other words, the support of $\widehat{\nu}$ is contained in a set of μ-measure zero. Thus, $\widehat{\nu}$ is singular with respect to μ. $\square$

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