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TWIST POINTS OF A JORDAN DOMAIN

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1. Introduction. In this paper we will describe how the twist points of a Jordan domain are distributed about each other. Our description will indicate in what sense Ostrowski's condition fails at a twist point. We first introduce the background material and notation. Many of the definitions are found in McMillan's papers.

Let D be a bounded Jordan domain and J its boundary. On $D \cup J$ we define the relative distance d_D between two points as the infimum of the Euclidean diameters of curves lying in D and joining these two points. Any limits involving boundary points will be with respect to the metric, d_D.

Let $f(z)$ be a one-to-one conformal map of the unit disk onto D. It is well known that $f(z)$ can be extended to a homeomorphism of the closed unit disk onto $D \cup J$. A subset $N \subset J$ is said to be a D-conformal null set if $\{e^{i\theta} : f(e^{i\theta}) \in N\}$ has measure zero. This definition is independent of f. Let $T \subset J$ denote the set of points where the inner tangent to J exists. That is, if $a \in T$, then there is a unique $v(a)$, $0 \leq v(a) < 2\pi$, such that, for each $\varepsilon > 0$, $\varepsilon < \pi/2$, there exists a $\delta > 0$ such that

$$\Delta = \{a + \rho e^{i\varphi} : 0 < \rho < \delta,\ |\varphi - v(a)| < \pi/2 - \varepsilon\} \subset D, \qquad \text{and } d_D(w, a) \to 0 \text{ as } |w - a| \to 0,\ w \in \Delta.$$

Let R be those $a \in J$ such that

$$\liminf_{\substack{w \to a \\ w \in D}} \arg(w - a) = -\infty \quad \text{and} \quad \limsup_{\substack{w \to a \\ w \in D}} \arg(w - a) = +\infty,$$

where $\arg(w - a)$ is defined and continuous in D. It has been shown [**4**, page 44] that $J = T \cup R \cup N$, where N is a D-conformal null set. There are examples of domains D such that $J = R \cup N$. See [**4**, pages 65–67] and [**6**, pages 736–738]. The set R is called the set of twist points of D.

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Our main theorem describes how the points of R are distributed about each other. In [**3**] a similar examination was done on the set T.

Theorem 1. *Let S be any subset of R. Except for a D-conformal null subset of S, each $a \in S$ has the property that, for any $\upsilon \in [0, 2\pi)$, there exists a sequence $\{w_n\} \subset S$, tending to a, satisfying*

$$\arg(w_n - a) \bmod 2\pi \longrightarrow \upsilon \quad \textit{as } n \to \infty.$$

Corollary. *Except for a D-conformal null set of S, for each $a \in S$ and any line through a whose argument with the real axis is υ, $\upsilon \in [0, \pi)$, there exist sequences $\{w_n'\} \subset S$ and $\{w_n''\} \subset S$, tending to a, satisfying*

$$\arg(w_n' - a) \bmod 2\pi \longrightarrow \upsilon \quad \textit{and} \quad \arg(w_n'' - a) \bmod 2\pi \longrightarrow \upsilon + \pi.$$

Theorem 2. *Except for a D-conformal null subset of R, for each $a \in R$ there exists an $\eta(a)$, $0 \leq \eta(a) < \pi$, such that for any line through a whose argument with the real axis is $\upsilon \in [0, \pi) \setminus \{\eta(a)\}$, and any pair of sequences $\{w_n'\} \subset J$ and $\{w_n''\} \subset J$, tending to a and satisfying $\arg(w_n' - a) \bmod 2\pi \to \upsilon$ and $\arg(w_n'' - a) \bmod 2\pi \to \upsilon + \pi$, then either*

$$\liminf_{n \to \infty} \frac{|w_{n+1}' - a|}{|w_n' - a|} = 0$$

or

$$\liminf_{n \to \infty} \frac{|w_{n+1}'' - a|}{|w_n'' - a|} = 0.$$

We make a few remarks concerning Theorems 1 and 2 and the corollary. We briefly turn our attention to the inner tangent points T. For each $a \in T$ we say that Ostrowski's condition holds provided there exist sequences $\{w_n'\}$, $\{w_n''\}$ of points in J tending to a having the properties (i′) $\arg(w'_n - a) \bmod 2\pi \to \upsilon(a) + \pi/2$ and $\arg(w''_n - a) \bmod 2\pi \to \upsilon(a) - \pi/2$.

(ii″) $\lim_{n\to\infty} \frac{|w'_{n+1}-a|}{|w'_n-a|} = 1$ and $\lim_{n\to\infty} \frac{|w''_{n+1}-a|}{|w''_n-a|} = 1$.

In (i′) observe that $\upsilon(a)$ is the unique value given in the definition of T and in (ii″) the liminf of the ratios is 1. McMillan observes in [**5**, page 73] that, with the exception of a set of linear measure zero (that is, a D-conformal null set), for each $a \in T$, Ostrowski's condition holds. Comparing (i′) and (ii″) with Theorems 1 and 2, it follows that, for each $a \in T$, we can always find sequences $\{w_n\}$ approaching a along a *unique* "tangent" line with the property that successive terms of the sequence are *relatively close* to one another; whereas, for each $a \in R$, we can always find sequences $\{w_n\}$ approaching a along *any* given line with the property that, except for one line, successive terms of one of the sequences are *relatively far apart* from one another. So, our theorems indicate in what sense Ostrowski's condition fails on R. The proof of Theorem 1 will use a rectifiable curve construction and a result of Kaufman and Wu [**1**, page 268]; the proof of Theorem 2 will use a result of this author [**2**, pages 148–149]. We state these two results respectively, calling them Lemma 1 and Lemma 2.

Lemma 1. *Let D be a simply connected domain, $a \in D$. Let K be a subset of ∂D, and suppose that Γ is a rectifiable quasi-smooth curve in the plane such that $K \subset \Gamma$. Then K has linear measure zero implies that the harmonic measure of K, with respect to D and a, is zero.*

Linear measure is a one-dimensional Hausdorff measure. The definition of harmonic measure is given in Kaufman and Wu's paper and can be found in most complex variable texts. See [**7**] for a more complete discussion of these two measures. A rectifiable Jordan curve Γ is called quasismooth if there exists an $M < \infty$ so that, for any x, y in Γ the linear measure of the arc in $\Gamma \setminus \{x, y\}$ with smaller diameter is less than $M|x - y|$.

Given $a \in R$ and positive numbers υ, δ', δ'', ε, with $\delta' < \delta''$ and $\varepsilon < \pi/2$, we define $A(a, \upsilon, \delta', \delta'', \pi/2 - \varepsilon) = \{a + \rho e^{i\varphi} : |\varphi - \upsilon| < \pi/2 - \varepsilon, \delta' < \rho < \delta''\}$.

Lemma 2. *Except for a D-conformal null set of R, for each $a \in R$, there exist a $\upsilon(a)$, $0 \le \upsilon(a) \le 2\pi$, and a sequence of pairwise disjoint disks $\{O_n\}$ with radii r_n and center w_n such that, for each $\varepsilon > 0$, $\varepsilon < \pi/2$, there exist sequences $\{\delta_n'\}$, $\{\delta_n''\}$ of positive real numbers such that*

1) $A(a, \upsilon(a), \delta_n', \delta_n'', \pi/2 - \varepsilon) \subset O_n \subset D$ *for all n sufficiently large.*

2) $d_D(w_n, a) \to 0$ *as $n \to \infty$. Consequently,* $r_n, \delta_n', \delta_n'' \to 0$ *as $n \to \infty$.*

3) $\lim_{n\to\infty} \frac{\delta_n'' - \delta_n'}{r_n} = 2\cos(\pi/2 - \varepsilon)$ *and* $\lim_{n\to\infty} \frac{\delta_n'' - \delta_n'}{\delta_n''} = 1$.

We close this section by making some brief remarks on rectifiable curve constructions and Lemmas 1 and 2. One of the earliest uses of a rectifiable curve construction can be found in Lusin-Privaloff's uniqueness theorem [**8**, page 320] which was published around 1925. In McMillan's twist point theorem [**4**, pages 44, 63–65], published around 1970, a rectifiable curve construction is used to show that a set $N \subset T$ is D-conformal null if and only if N has one-dimensional Hausdorff measure zero; that is, linear measure zero. Now any D-conformal null set N corresponds under f to a set of Lebesgue measure zero on the unit circle, and since harmonic measure is conformal invariant, we can conclude that the harmonic measure of N with respect to D is also zero. Hence, McMillan's result is an earlier version of Kaufman and Wu's result (Lemma 1) restricted to the point set T.

In the rectifiable curve constructions used by McMillan and by Lusin-Privaloff, a subdomain H of D is constructed having a rectifiable boundary. Then the measure (harmonic, D-conformal null, Lebesgue) of some point set $F \subset \partial H \cap \partial D$ with respect to D is compared to its measure with respect to H. In this paper the construction is used in a nontypical way: a domain H is constructed having a rectifiable boundary, and it will not be a subdomain of D.

Finally, if one compares Lemma 2 with the inner tangent property of T, it is clear that the sets $A(a, \upsilon(a), \delta_n', \delta_n'', \pi/2 - \varepsilon)$ are trying to mimic the property of the Δ's.

2. Proof of Theorem 1.

For each $\upsilon \in [0, 2\pi)$, $\varepsilon > 0$, $\varepsilon < \pi/2$ and $\delta > 0$, let $\Delta(a, \delta, \varepsilon, \upsilon) = \{a + \rho e^{i\varphi} : 0 < \rho < \delta, |\varphi - \upsilon| < \varepsilon\}$. We prove the following assertion:

Except for a D-conformal null set of S, each $a \in S$ has the property that, for all positive δ, ε and υ, $S \cap \Delta(a, \delta, \varepsilon, \upsilon) \neq \varnothing$.

Let F be those $a \in S$ such that there exist $\delta(a)$, $\varepsilon(a)$ and $\upsilon(a)$ with the property $S \cap \Delta(a, \delta(a), \varepsilon(a), \upsilon(a)) = \varnothing$. Our assertion will follow provided F is shown to be D-conformal null. Suppose to the contrary that F is not D-conformal null. For each $a \in F$ one can find rational numbers $\delta_r(a)$, $\varepsilon_r(a)$ and $\upsilon_r(a)$ such that $S \cap \Delta(a, \delta_r(a), \varepsilon_r(a), \upsilon_r(a)) = \varnothing$. Hence, there exists a subset F' of F which is not D-conformal null such that, for all $a \in F'$, $\delta_r(a) = \delta_0$, $\varepsilon_r(a) = \varepsilon_0$ and $\upsilon_r(a) = \upsilon_0$. By replacing F with F' we can assume without loss of generality that, for all $a \in F$, $S \cap \Delta(a, \delta_0, \varepsilon_0, \upsilon_0) = \varnothing$. Again associate for each $a \in F$ a straight line $L(a)$ such that

1) $L(a)$ intersects the segment $\{a + \rho e^{i\upsilon} : \upsilon = \upsilon_0, 0 < \rho < \delta_0\}$ at right angles,

2) the Euclidean distance from $L(a)$ to the origin is a rational number,

3) one of the components of $\Delta(a, \delta_0, \varepsilon_0, \upsilon_0) \setminus \{L(a)\}$, which we denote by $\Delta(a, L(a))$, is a triangular region.

Since $\{L(a)\}$ is a countable set, there exists an $F' \subset F$, which is not D-conformal null such that, for all $a \in F'$, $L(a) = L_0$. By replacing F with F' we can assume without loss of generality that, for all $a \in F'$, $L(a) = L_0$. Now consider the set $\cup_{a \in F} \Delta(a, L_0)$. It has at most countable many components, one of which has the form $\cup_{a \in F'} \Delta(a, L_0)$, where F' is not D-conformal null. By replacing F with F', we can further assume without loss of generality that $H = \cup_{a \in F} \Delta(a, L_0)$ is connected. It follows that the boundary of H, which we denote by Γ, is a rectifiable Jordan curve. Note that part of Γ is a closed line segment lying on L_0 and the rest of Γ is contained in one of the half planes determined by $C \setminus L_0$. This part of Γ satisfies a Lipschitz condition and contains the set F, which is not D-conformal null. The Lipschitz condition implies that Γ is also quasi-smooth. Using McMillan's twist point theorem and Makarov's compression theorem [**7**, page 127] it readily follows that the linear measure of R, and thus F, is zero. By Lemma 1, we conclude that the harmonic measure of F is also zero.

This implies that F corresponds under f to a set of measure zero on the unit circle, and so F is D-conformal null. This contradiction establishes our assertion.

If we fix $\upsilon \in [0, 2\pi)$ and let δ_n, ε_n be sequences of positive numbers approaching zero, it follows from the assertion that, for each n, there exists a $w_n \in S$ such that $|\arg(w_n - a) \bmod 2\pi - \upsilon| < \varepsilon_n$ and $|w_n - a| < \delta_n$. The latter condition indicates that $d_D(w_n, a) \to 0$ as $n \to \infty$. Thus, the sequence $\{w_n\}$ satisfies the conditions of Theorem 1.

The corollary is a direct consequence of Theorem 1.

3. Proof of Theorem 2. From Lemma 2, there exists a D-conformal null subset N of R such that on $R \setminus N$ the results of Lemma 2 hold. For each $a \in R \setminus N$, we define $\eta(a)$ using the $\upsilon(a)$ of Lemma 2:

$$\eta(a) = \left\{ \begin{array}{c} \upsilon(a) + \pi/2 \text{ if } 0 \le \upsilon(a) < \pi/2 \\ \upsilon(a) - \pi/2 \text{ if } \pi/2 \le \upsilon(a) < 3\pi/2 \\ \upsilon(a) - 3\pi/2 \text{ if } 3\pi/2 \le \upsilon(a) \le 2\pi \end{array} \right\}.$$

Clearly, $\eta(a) \in [0, \pi)$. Now let $\upsilon \in [0, \pi) \setminus \{\eta(a)\}$ and $\{w'_n\}$, $\{w''_n\}$ be as described in the hypothesis of Theorem 2. By choosing ε sufficiently small, the line L passing through a with argument υ intersects the region $\{a + \rho e^{i\theta} : \rho > 0, |\varphi - \upsilon(a)| < \pi/2 - \varepsilon\}$. Hence, one of the sequences, say $\{w'_n\}$, is such that, for N sufficiently large, $w'_n \in \{a + \rho e^{i\varphi} : \rho > 0, |\varphi - \upsilon(a)| < \pi/2 - \varepsilon\}$ when $n \ge N$. For each $n > N$, choose $k(n)$ such that $\delta''_{k(n)}$ from Lemma 2 satisfies $\delta''_{k(n)} < |w'_n - a|$. Since $|w'_n - a| \to 0$ as $n \to \infty$, $\delta''_{k(n)} \to 0$ as $n \to \infty$. Let $P = \{m > N : |w'_m - a| \le \delta'_{k(n)}\}$. P has a minimal element $m(n) = m(k(n))$. Since $|w_{m(n)'} - a| \le \delta'_{k(n)}$ and $\delta'_{k(n)} \to 0$ as $n \to \infty$, $m(n) \to \infty$ as $n \to \infty$. Since $A(a, \upsilon(a), \delta'_k, \delta''_k, \pi/2 - \varepsilon)$ is contained in D, the immediate predecessor of $w'_{m(n)}$ in $\{w'_n\}$, denoted $w'_{m(n)-1}$, satisfies the inequality $|w'_{m(n)-1} - a| \ge \delta''_{k(n)}$. Thus,

$$\frac{|w'_{m(n)} - a|}{|w'_{m(n)-1} - a|} \le \frac{\delta'_{k(n)}}{\delta''_{k(n)}} = 1 - \frac{\delta''_{k(n)} - \delta'_{k(n)}}{\delta''_{k(n)}} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Hence,

$$\liminf_{n \to \infty} \frac{|w'_{n+1} - a|}{|w'_n - a|} = 0.$$

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