ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 4, 2013

INFINITELY MANY PERIODIC SOLUTIONS FOR ASYMPTOTICALLY LINEAR HAMILTONIAN SYSTEMS

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ABSTRACT. In this paper, we consider the asymptotically linear Hamiltonian systems with symmetry

$$u''(t) + A(t)u(t) + \nabla H(t, u(t)) = 0, \quad t \in \mathbf{R};$$

here $A(\cdot)$ is a continuous T-periodic symmetric matrix, $H : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is T-periodic in its first variable and $\nabla H(t,x)$ denotes its gradient with respect to the x variable. By using the minimax technique in critical point theory, we obtain infinitely many periodic solutions under several classes of new conditions. It turns out that our main results are sharp improvements of several known results in the literature. In particular, unlike the existing results concerning the existence of infinitely many periodic solutions, where the nonlinearity $\nabla H(t,x)$ is required to be bounded, our main results here allow $\nabla H(t,x)$ to have a sublinear growth behavior at infinity.

1. Introduction and main results. Consider the second-order Hamiltonian systems

$$u''(t) + A(t)u(t) + \nabla H(t, u(t)) = 0, \quad t \in \mathbf{R}, \tag{1.1}$$

where $A(\cdot)$ is a continuous T-periodic symmetric matrix and $H : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is T-periodic ($T > 0$) in its first variable. Moreover, we always assume that $H(t,x)$ is continuous in t for each $x \in \mathbf{R}^n$, continuously differentiable in x for each $t \in [0,T]$ and $\nabla H(t,x)$ denotes its gradient with respect to the x variable.

In this paper, we are interested in the existence and multiplicity of periodic solutions of (1.1) when $\nabla H(t,x)$ has a sublinear growth behavior at infinity, that is,

$$\lim_{|x|\to\infty} \frac{|\nabla H(t,x)|}{|x|} = 0.$$

2010 AMS *Mathematics subject classification.* Primary 34B15, 34B18, 34C25.
Keywords and phrases. Periodic solution, asymptotically linear Hamiltonian system, minimax technique.
Research supported by the Specialized Fund for the Doctoral Program of Higher Education and the National Natural Science Foundation of China.
Received by the editors on February 24, 2010, and in revised form on January 20, 2011.
DOI:10.1216/RMJ-2013-43-4-1281 Copyright ©2013 Rocky Mountain Mathematics Consortium

This problem was studied in [**19**] by the minimax methods and in [**9, 10, 12, 14, 23, 25**] by Morse theory without the evenness assumption on the potential, and obtained finitely many periodic solutions. We notice that all the results mentioned above required that $\lim_{|x|\to\infty} |\nabla H(t,x)|/|x|^{\alpha_0} = 0$ for some $\alpha_0 \in (0,1)$. However, to the best of our knowledge, little is known on the existence of infinitely many periodic solutions for system (1.1) (see [**5, 27**]). To obtain the existence of infinitely many periodic solutions, among other things, the authors in [**5, 27**] required that the nonlinearity $\nabla H(t,x)$ be bounded. We should mention that some results concerning the existence of infinitely many solutions for cooperative elliptic systems have been obtained by Ma [**11**] and Zou [**27**].

In a classical paper, Rabinowitz proposed the following subquadratic condition: there exist constants $0 < \mu < 2$ and $R > 0$ such that

$$0 < (\nabla H(t,x), x) \le \mu H(t,x), \quad \text{for } |x| \ge R.$$

This condition is now known as the Ambrosetti-Rabinowitz condition (A-R condition for short) (see [**1, 15–17**]). Following Rabinowitz, many authors studied the periodic problem under the A-R condition (see [**13, 20**] and the references therein).

In recent years, however, many authors have paid much attention to weakening the A-R condition (see [**4, 6, 7, 22, 26**]). By using a critical point theorem due to Fei [**3**], Zou and Li [**27**] obtained the following result:

Theorem A. *Assume that* $H(t,-x) = H(t,x)$ *for all* $(t,x) \in [0,T] \times \mathbf{R}^n$, *and the following conditions hold*:

(ZL1) $0 \in \sigma(-(d^2/dt^2) - A(t))$, *where* σ *denotes the spectrum with* T*-periodicity condition*;

(ZL2) $|H(t,x)| \le c|x|$ *for* $|x|$ *small*, $\liminf_{|x|\to 0} H(t,x)/|x|^2 = +\infty$ *uniformly for* $t \in [0,T]$;

(ZL3) *there exists an* $M > 0$ *such that* $|\nabla H(t,x)| \le M$ *for* $x \in \mathbf{R}^n$ *and* $t \in [0,T]$;

(ZL4) $\liminf_{|x|\to\infty}(2H(t,x) - (\nabla H(t,x),x))/|x| = h(t) \ge 0$ *uniformly for* $t \in [0,T]$ *and* $\int_0^T h(t)\,dt > 0$.

Then problem (1.1) *has infinitely many* T*-periodic solutions.*

On the other hand, it is well known [**13**] that (1.1) admits nontrivial T-periodic solutions when $|\nabla H(t,x)|$ is bounded and $\lim_{|x|\to\infty} H(t,x) = -\infty$ (or $+\infty$) as $|x|\to\infty$ uniformly for $t\in[0,T]$. This result have also been generalized by Tang and Wu [**19**]. The following theorem is a special case of a result due to [**19**] (in the original work of Tang and Wu, the authors also assume that $0\in\sigma(-(d^2/dt^2)-A(t))$. In fact, this condition can be dropped):

Theorem B. *Assume that there exists an* $\alpha\in[0,1)$ *such that the following conditions hold*:

(TW1) $\limsup_{|x|\to\infty} |\nabla H(t,x)|/|x|^\alpha \le M$ *uniformly for* $t\in[0,T]$ *and some* $M>0$;

(TW2) $H(t,x)/|x|^{2\alpha}$ *is uniformly bounded from above and* $\lim_{|x|\to\infty} H(t,x)/|x|^{2\alpha} = -\infty$ *on a positive measure subset* $E\subset[0,T]$, *or* $H(t,x)/|x|^{2\alpha}$ *is uniformly bounded from below and* $\lim_{|x|\to\infty} H(t,x)/|x|^{2\alpha} = +\infty$ *on a positive measure subset* $E\subset[0,T]$.

Then problem (1.1) *has at least one* T*-periodic solution.*

The following result was also obtained in [**27**]:

Theorem C. *Assume that* $H(t,-x)=H(t,x)$ *for all* $(t,x)\in[0,T]\times\mathbf{R}^n$, (ZL1), (ZL2) *and the following conditions hold*:

(ZL5) $\lim_{|x|\to\infty} |\nabla H(t,x)| = 0$ *uniformly for* $t\in[0,T]$;

(ZL6) *there exists a* $\beta\in(0,1)$ *such that* $\limsup_{|x|\to\infty} H(t,x)/|x|^\beta = h(t)\le 0$ *uniformly for* $t\in[0,T]$ *and* $\int_0^T h(t)\,dt<0$.

Then problem (1.1) *has infinitely many* T*-periodic solutions.*

In the present paper, we study the existence of infinitely many T-periodic solutions of (1.1) by using the minimax technique in critical point theory. Under new versions of weakened A-R conditions, we obtain the following theorems:

Theorem 1. *Assume that* $H(t,-x)=H(t,x)$ *for all* $(t,x)\in[0,T]\times\mathbf{R}^n$, *and the following conditions hold*:

(H0) $\lim_{|x|\to 0} |x|^{-2}\{H(t,x) - H(t,0)\} = +\infty$ *uniformly for* $t \in [0,T]$;

(H1) $\lim_{|x|\to\infty} |\nabla H(t,x)|/|x| = 0$ *uniformly for* $t \in [0,T]$;

(H2) $\liminf_{|x|\to\infty}[2H(t,x) - (\nabla H(t,x),x)] = h(t) > -\infty$ *uniformly for* $t \in [0,T]$ *and* $\int_0^T h(t)\,dt > 2\int_0^T H(t,0)\,dt$.

Then problem (1.1) *has infinitely many* T*-periodic solutions.*

Theorem 2. *Assume that* $H(t,-x) = H(t,x)$ *for all* $(t,x) \in [0,T] \times \mathbf{R}^n$*,* (H0), (H1) *and the following condition hold*:

(H3) *there exists a* $\beta \in (0,2]$ *such that* $\liminf_{|x|\to\infty}(2H(t,x) - (\nabla H(t,x),x))/|x|^\beta = h(t) \geq 0$ *uniformly for* $t \in [0,T]$ *and* $\int_0^T h(t)\,dt > 0$.

Then problem (1.1) *has infinitely many* T*-periodic solutions.*

Remark 1. In conditions $(H2)$ and $(H3)$, $h(t) \equiv +\infty$ on $[0,T]$ or a positive measure subset of $[0,T]$ is allowed and, in this case, $\int_0^T h(t)\,dt = +\infty$. However, if $h(t)$ is finite for every $t \in [0,T]$, then it follows that $h(t)$ is continuous on $[0,T]$.

Remark 2. Under assumption (H2), we can choose a positive number $\theta_0 \in (0,1)$ such that

$$\theta_0 \int_0^T h(t)\,dt > 2\int_0^T H(t,0)\,dt. \tag{1.2}$$

Remark 3. Clearly, Theorem 1 (in the case that $2H(t,x) - (\nabla H(t,x),x)$ is bounded from below) and Theorem 2 are sharp improvements of Theorem A. In particular, in contrast with the existing ones, our results allow $\nabla H(t,x)$ to have a sublinear growth behavior at infinity. In proving these results, a new technique is developed (see the remark at the end of Section 2), and this technique can be applied in many different contexts.

If the potential is even, we can also establish the existence of infinitely many periodic solutions for system (1.1) under (H0) and some conditions which are slightly weaker than that of Theorem B.

Theorem 3. *Assume that* $H(t,-x) = H(t,x)$ *for all* $(t,x) \in [0,T] \times \mathbf{R}^n$, (H0) *and there exist* $\alpha \in [0,1)$ *and* $M > 0$ *such that the following conditions hold*:

(H4) $\limsup_{|x|\to\infty} |\nabla H(t,x)|/|x|^\alpha \le M$ *uniformly for* $t \in [0,T]$;

(H5) (i) $\limsup_{|x|\to\infty} H(t,x)/|x|^{2\alpha} \le M$ *uniformly for* $t \in [0,T]$ *and* $\lim_{|x|\to\infty} H(t,x)/|x|^{2\alpha} = -\infty$ *on a positive measure subset* $E \subset [0,T]$, *or*

(ii) $\liminf_{|x|\to\infty} H(t,x)/|x|^{2\alpha} \ge -M$ *uniformly for* $t \in [0,T]$ *and* $\lim_{|x|\to\infty} H(t,x)/|x|^{2\alpha} = +\infty$ *on a positive measure subset* $E \subset [0,T]$.

Then problem (1.1) *has infinitely many* T*-periodic solutions.*

The following theorem and its corollary are sharp improvements of Theorem C.

Theorem 4. *Assume that* $H(t,-x) = H(t,x)$ *for all* $(t,x) \in [0,T] \times \mathbf{R}^n$, (H0) *and there exists an* $\alpha \in (0,1)$ *such that the following conditions hold*:

(H6) $\lim_{|x|\to\infty} |\nabla H(t,x)|/|x|^\alpha = 0$ *uniformly for* $t \in [0,T]$; *and*

(H7) (i) $\limsup_{|x|\to\infty} H(t,x)/|x|^{2\alpha} = h(t) \le 0$ *uniformly for* $t \in [0,T]$ *and* $\int_0^T h(t)\,dt < 0$, *or*

(ii) $\liminf_{|x|\to\infty} H(t,x)/|x|^{2\alpha} = h(t) \ge 0$ *uniformly for* $t \in [0,T]$ *and* $\int_0^T h(t)\,dt > 0$.

Then problem (1.1) *has infinitely many* T*-periodic solutions.*

Corollary. *Assume that* $H(t,-x) = H(t,x)$ *for all* $(t,x) \in [0,T] \times \mathbf{R}^n$, (H0) *and there exist two numbers* $\alpha \in [0,1)$ *and* $\beta \in (2\alpha, 1+\alpha)$ *such that* (H4) *and the following condition hold*:

(H8) (i) $\limsup_{|x|\to\infty} H(t,x)/|x|^{\beta} = h(t) \le 0$ *uniformly for* $t \in [0,T]$ *and* $\int_0^T h(t)\,dt < 0$, *or*

(ii) $\liminf_{|x|\to\infty} H(t,x)/|x|^{\beta} = h(t) \ge 0$ *uniformly for* $t \in [0,T]$ *and* $\int_0^T h(t)\,dt > 0$.

Then problem (1.1) *has infinitely many* T*-periodic solutions.*

This corollary is an immediate consequence of Theorem 4, since all the conditions of Theorem 4 with α replaced by $\beta/2$ hold true.

Remark 4. Throughout this paper, without loss of generality, we assume the following conditions also hold:

$$\int_0^T H(t,0)\,dt = 0. \tag{1.3}$$

Otherwise, $\widetilde{H}(t,x) = H(t,x) - (1/T)\int_0^T H(t,0)\,dt$ satisfies all the conditions in Theorems 1–5 and (1.3).

Remark 5. In [**24**], Wang considered the effort of concave nonlinearities for the solution structure of nonlinear boundary value problems. In particular, some results similar to ours have been obtained for elliptic equations and periodic boundary value problems of first-order Hamiltonian systems and nonlinear wave equations. It seems that a similar result is also valid for second order Hamiltonian systems. But, in [**24**], the author required that $2H(t,x) - (\nabla H(t,x), x)$ be of constant sign for all small $|x| \neq 0$ and all $t \in [0,T]$, which is not assumed in the present paper.

The remainder of this paper is organized as follows. In Section 2, we state and prove an abstract critical point theorem, which is needed in proving our main results. In Section 3, we shall establish some preliminary lemmas. Finally, we give the proofs of our main theorems in the last section.

2. An abstract critical point theorem. To prove our main results, we first state and prove an abstract critical point theorem in this section.

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Following Fei [**3**], we say that $I \in C^1(H,\mathbf{R})$ satisfies condition (B) in $(c_1,c_2) \subset [-\infty,+\infty]$ with respect to $B := \{x \in H \mid \|x\| \le r\}$ for some $r > 0$ if (i) any sequence $\{x_n\} \subset B$ such that $I(x_n) \to c \in (c_1,c_2)$ and $I'(x_n) \to 0$ is such that $\{x_n\}$ possesses a convergent subsequence; and (ii) for any $c \in (c_1,c_2)$, there exist $\sigma = \sigma(c) \in (0,\min\{c-c_1,c_2-c\})$ and $\theta < 1$ such that $\langle I'(x),x\rangle + \theta\|I'(x)\|\cdot\|x\| \ge 0$ for any $x \in I^{-1}([c-\sigma,c+\sigma]) \cap \{x \in H \mid \|x\| = r\}$.

The following deformation lemma was proved in Fei [**3**].

Lemma 2.1. *Suppose* $I \in C^1(H, \mathbf{R})$ *satisfies condition* (B) *in* (c_1, c_2) *relative to the set* B *and* $I'(u) \neq 0$ *on* ∂B. *If* $\overline{\varepsilon} > 0$, $c \in (c_1, c_2)$ *and* N *is any neighborhood of* K_c, *then there exist* $0 < \varepsilon < \overline{\varepsilon}$, $T > 0$, *and* $\eta \in C([0, T] \times B, B)$ *such that*:

(1) $\eta(0, u) = u$ *for* $u \in B$ *and* $\eta(t, u)$ *is a homeomorphism from* B *into* B;

(2) $\eta(t, u) = u$ *for* $t \in [0, T]$ *if* $u \notin I^{-1}([c - \overline{\varepsilon}, c + \overline{\varepsilon}]) \cap B$;

(3) $I(\eta(t, u)) \leq I(u)$ *for* $(t, u) \in [0, T] \times B$;

(4) $\eta(T, A_{c+\varepsilon} \setminus N) \subset A_{c-\varepsilon}$; $\eta(T, A_{c+\varepsilon}) \subset A_{c-\varepsilon}$ *if* $K_c = \varnothing$;

(5) *If* $I(u)$ *is even in* u, $\eta(t, u)$ *is odd in* u.

In proving our critical point theorem, the genus introduced by Krasnoselskii is needed [**8**] (see also [**1, 2, 15, 18**]). Let $\Sigma = \{A \subset H \mid A \text{ closed}, \ A = -A\}$. Then, for $A \in \Sigma$, the Krasnoselskii genus of A, denoted by $i(A)$, is well defined. Recall that $i(\varnothing) = 0$, for $\varnothing \neq A \in \Sigma$,

$$i(A) = \inf\{m \mid \text{there exists an } h \in C(A, \mathbf{R}^m \setminus \{0\}), \ h \text{ is odd}\}$$

and $i(A) = \infty$ if such an h does not exist, in particular, if $0 \in A$.

The Krasnoselskii genus satisfies the following properties (see [**1**, Lemma 1.2] and [**18**, Propositions 5.3 and 5.4]).

Lemma 2.2. *Let* $E, F \in \Sigma$. *Then the following hold*:

(1) $E \subset F \Rightarrow i(E) \leq i(F)$;

(2) $i(E \cup F) \leq i(E) + i(F)$;

(3) *If* $h \in C(E, F)$ *is an odd homeomorphism, then* $i(E) = i(F)$;

(4) *If* E *is compact and* $0 \notin E$, *then* $i(E) < \infty$ *and there exists a closed neighborhood* N *of* E *such that* $N \in \Sigma$ *and* $i(N) = i(E)$;

(5) *If* $i(E) > k$ *and* V *is a* k*-dimensional subspace of* H, *then* $E \cap V^{\perp} \neq \varnothing$;

(6) *If* E *is compact and* $1 \leq i(E) < \infty$, *then* $0 \notin E$ *and* E *contains at least* $i(E)$ *vectors* $x_k, 1 \leq k \leq i(E)$, *such that* $\langle x_k, x_l \rangle = 0$ $(k \neq l)$.

An abstract critical point theorem (see [**3**, Theorem 2.5]) was proved by Fei. But Fei's original proof contains a mistake. A simple counterexample is as follows: Let $I(u) = \|u\|^2 - 1$; then all the assumptions of [**3**, Theorem 2.5] are verified for any choice of a subspace H_1, of a closed subspace H_2 with finite codimension, of $\rho < 1$ and $b < -1$. But the only critical point of this functional is trivial.

Fortunately, we can state and prove the following critical point theorem which is a corrected version of [**3**, Theorem 2.5].

Theorem 2.3. *Suppose that $I \in C^1(H, \mathbf{R})$ is even. Furthermore,*

(i) *there exist two closed subspaces H_1 and H_2 with* $\operatorname{codim} H_2 < \infty$ *and two constants $\rho > 0$ and $b \in (-\infty, I(0))$ such that*

$$\sup\{I(x) \mid x \in H_1,\ \|x\| = \rho\} < I(0) \quad \textit{and} \quad \inf\{I(x) \mid x \in H_2\} > b,$$

(ii) *I satisfies condition (B) in $(b, I(0))$ with respect to $B = \{x \in H \mid \|x\| \le r,\ r > \rho\}$.*

Then if $\dim H_1 - \operatorname{codim} H_2 > 0$*, I has at least* $\dim H_1 - \operatorname{codim} H_2$ *distinct pairs of critical points, whose corresponding critical values belong to $(b, I(0))$.*

Proof. For $k \ge 0$, let $\Sigma_k = \{A \in \Sigma \mid i(A) \ge k, A \subset B\}$, $c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} I(u)$. Then our conclusions follows from Claims 1–4 stated below.

Claim 1. $c_k \le c_{k+1}$.

This claim follows from the fact that $\Sigma_{k+1} \subset \Sigma_k$ for all $k \ge 0$.

Claim 2. *If $k > \operatorname{codim} H^+$, then $c_k > b$.*

For $A \in \Sigma_k$, $i(A) \ge k > \operatorname{codim} H^+$, by Lemma 2.2 (5), it follows that $A \cap H^+ \ne \varnothing$. Therefore, $\sup_{u \in A} f(u) \ge \inf_{u \in H^+} f(u) > b$, from which Claim 2 follows.

Claim 3. *If $k \le \dim H^-$, then $c_k < I(0)$.*

Since $r > \rho$, $S_\rho \cap H^- \in \Sigma_k$, which yields $c_k \le \sup_{u \in S_\rho \cap H^-} f(u) < I(0)$.

Claim 4. *If* $c = c_{k+1} = c_{k+2} = \cdots = c_{k+j}$, $k \geq \operatorname{codim} H^+$, $k + j \leq \dim H^-$, *then* $K_c = \{u \in B \mid I'(u) = 0, I(u) = c\}$ *contains at least* j *distinct pairs of critical points.*

In fact, by Claims 2, 3 and condition B (i), $c \in (b, I(0))$, K_c is compact and $0 \notin K_c$, therefore, by Lemma 2.2 (4), $i(K_c) < \infty$ and there exists a closed neighborhood N of K_c such that $N \subset B$, $N \in \Sigma$, and $i(N) = i(K_c)$.

By Lemma 2.1, there exist $0 < \varepsilon < 1/2 \min\{I(0) - c, c - b\}$, and $\eta \in C(B, B)$ such that η is an odd homeomorphism from B into B, $\eta(\overline{A_{c+\varepsilon} \setminus N}) \subset A_{c-\varepsilon}$.

Since $c = c_{k+j} < c + \varepsilon$, there exists an $A \in \Sigma_{k+j}$ such that $\sup_{u \in A} f(u) < c + \varepsilon$. That is, $A \subset A_{c+\varepsilon}$. Therefore, by Lemma 2.2 (1), we have $k + j \leq i(A) \leq i(A_{c+\varepsilon})$.

Since $\sup_{u \in A_{c-\varepsilon}} f(u) \leq c - \varepsilon$, $A_{c-\varepsilon} \notin \Sigma_{k+1}$, which implies that $i(A_{c-\varepsilon}) \leq k$.

By Lemma 2.2 (2) and (3), we obtain

$$\begin{aligned} i(A_{c+\varepsilon}) &\leq i(\overline{A_{c+\varepsilon} \setminus N}) + i(N) \leq i(\eta(\overline{A_{c+\varepsilon} \setminus N})) + i(N) \\ &\leq i(A_{c-\varepsilon}) + i(N) \leq k + i(K_c). \end{aligned}$$

Therefore, $i(K_c) \geq j$ and, hence, by Lemma 2.2 (6), K_c contains at least j distinct pairs of critical points.

Remark. In applying Theorem 2.3, an important ingredient is checking that the functional I satisfies condition (B) in $(b, I(0))$ with respect to some ball $B = \{x \in H \mid \|x\| \leq r\}$. To do this, we split H as $H = W \oplus H_0$ and separate H into two parts S_1 and S_2 by using a supersurface

$$\{u \in H \mid u = w + u_0,\ w \in W,\ u_0 \in H_0,\ \|w\| = \delta \|u_0\|^\alpha + J\},$$

with $\delta > 0$, $\alpha \in (0, 1]$ and $J > 0$. Then we verify condition (B) in S_1 and S_2 independently. This technique seems to be new and can be applied in many different contexts.

3. Preliminary lemmas. Let H_T^1 be the Sobolev space defined by

$$H_T^1 = \{u : [0,T] \to \mathbf{R}^n \mid u \text{ is absolutely continuous}, \\ u(0) = u(T), \quad \text{and} \quad u' \in L^2(0,T;\mathbf{R}^n)\}$$

with the norm

$$\|u\| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |u'(t)|^2 dt \right)^{1/2}.$$

Let B be the linearized operator defined by $Bx(t) = -x''(t) - A(t)x(t)$ with T-periodic condition. Then B has a sequence of eigenvalues

$$\lambda_{-m} \le \lambda_{-m+1} \le \cdots \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots$$

with $\lambda_k \to +\infty$ as $k \to \infty$. Note that 0 may not be an eigenvalue in the last sequence. Let ϑ_j be the eigenvector of B corresponding to λ_j, $j = -m, -m+1, \dots, -1, 1, 2, \dots, k, \dots$, setting

$$H_0 = \ker \left(-\frac{d^2}{dt^2} - A(t) \right),$$

$$H_- = \text{the negative eigenspace of } -\frac{d^2}{dt^2} - A(t),$$
$$H_+ = \text{the positive eigenspace of } -\frac{d^2}{dt^2} - A(t),$$

then $H_T^1 = H_- \oplus H_0 \oplus H_+$. Throughout this paper, for any $u \in H_T^1$, we always denote by u^0, u^+ and u^- the vectors in H_T^1 with $u = u^0 + u^+ + u^-$, $u^0 \in H_0$ and $u^\pm \in H_\pm$.

We remark that $H_0 \ne \{0\}$ if $0 \in \sigma(B)$, and $H_0 = \{0\}$ if $0 \notin \sigma(B)$. Evidently, there exists a constant $c_0 > 0$ such that

$$\pm\langle Bu^\pm, u^\pm \rangle \ge c_0 \|u^\pm\|^2, \quad \text{for all } u^\pm \in H_\pm.$$

Therefore, we have

$$\langle B(u^+ + u^-), u^+ - u^- \rangle \ge c_0 \|u^+ + u^-\|^2.$$

Combining the fact that $\|u^+ - u^-\| = \|u^+ + u^-\|$, the last inequality implies

$$\|B(u^+ + u^-)\| \geq c_0\|u^+ + u^-\|. \tag{3.1}$$

Under the assumption (H1) ((H4) and (H6), respectively), by virtue of [**13**, Theorem 1.4], it follows that the functionals

$$\begin{aligned}\varphi(u) = \frac{1}{2}\int_0^T |u'(t)|^2 dt - \frac{1}{2}\int_0^T (A(t)u(t), u(t))\,dt \\ - \int_0^T H(t, u(t))\,dt, \quad u \in H_T^1\end{aligned} \tag{3.2}$$

and

$$\psi(u) = \int_0^T H(t, u(t))\,dt, \quad u \in H_T^1 \tag{3.3}$$

are continuously differentiable, and

$$\begin{aligned}\langle \varphi'(u), v\rangle &= \int_0^T (u', v')\,dt - \int_0^T (A(t)u(t), v(t))\,dt \\ &\quad - \int_0^T (\nabla H(t, u), v)\,dt, \quad u, v \in H_T^1, \\ \langle \psi'(u), v\rangle &= \int_0^T (\nabla H(t, u), v)\,dt, \quad u, v \in H_T^1.\end{aligned} \tag{3.5}$$

It is well known that the T-periodic solution of problem (1.1) corresponds to the critical points of the functional φ.

For $k \geq 1$, let $H_k = \operatorname{span}\{\vartheta_j \mid j = -m, \dots, -1, 1, 2, \dots, k\}$. Then we have the following:

Lemma 3.1 [**27**]. *Assume* (H0) *holds. Then there exist $\rho = \rho(k) > 0$ and $\varpi = \varpi(k) < 0$ such that $\varphi(u) \leq \varpi < 0$, for all $u \in H_k \cap \{u \in H_T^1 \mid \|u\| = \rho\}$.*

To prove our main results, we further need the following lemmas.

Lemma 3.2. *Assume* (H6) *with* $\alpha \in (0,1]$ (*this condition reduces to* (H1) *when* $\alpha = 1$). *Then, for any* $\eta > 0$ *and* $\delta > 0$, *there exists a* $J_{\eta,\delta} > 0$ *such that*

$$\langle \varphi'(u), u \rangle + \frac{1}{2}\|\varphi'(u)\| \cdot \|u\| \geq 0,$$

for all $u \in S_{\eta,\delta} := \{u \in H_T^1 \mid \varphi(u) \geq -\eta, \|u^+ + u^-\| \geq \delta\|u^0\|^\alpha + J_{\eta,\delta}\}$.

Proof. For $u \in S_{\eta,\delta}$, we have $\|u^+ + u^-\| \geq \delta\|u^0\|^\alpha$. Therefore,

$$\|u^+ + u^-\|^2 = \|u\|^2 - \|u^0\|^2 \geq \|u\|^2 - \frac{1}{\delta^{2/\alpha}}\|u^+ + u^-\|^{2/\alpha},$$

from which it is not hard to check that, for $\|u\|$ large enough,

$$\|u^+ + u^-\| \geq \frac{\delta}{2}\|u\|^\alpha. \tag{3.6}$$

For any $\varsigma \in (0, (c_0\delta)/(16)T^{(\alpha-1)/2})$, it follows from (H6) that there exists an $M > 0$ such that

$$|\nabla H(t,x)| \leq \varsigma|x|^\alpha + M, \quad \text{for all } x \in \mathbf{R}^n \text{and } t \in [0,T].$$

Therefore, we have

(3.7)

$$\begin{aligned}|H(t,x)| \leq |H(t,x) - H(t,0)| + |H(t,0)| \leq \varsigma|x|^{1+\alpha}& \\ + M|x| + |H(t,0)|.&\end{aligned}$$

$$\begin{aligned}|\psi(u)| &\leq \int_0^T |H(t,u)|\,dt \leq \varsigma\int_0^T |u|^{1+\alpha}\,dt + M\int_0^T |u|\,dt \\ &\quad + \int_0^T |H(t,0)|\,dt \\ &\leq \varsigma T^{(1-\alpha)/2}\bigg(\int_0^T |u|^2 dt\bigg)^{(1+\alpha)/2} + MT^{1/2}\bigg(\int_0^T |u|^2 dt\bigg)^{1/2} \\ &\quad + \int_0^T |H(t,0)|\,dt \\ &\leq \varsigma T^{(1-\alpha)/2}\|u\|^{1+\alpha} + MT^{1/2}\|u\| + \int_0^T |H(t,0)|\,dt,\end{aligned}$$

and

$$
\begin{aligned}
|\langle \psi'(u), v\rangle| &\le \int_0^T |\nabla H(t,u)| \cdot |v|\, dt \\
&\le \varsigma \int_0^T |u|^\alpha \cdot |v| dt + M \int_0^T |v|\, dt \\
&\le \left\{ \varsigma \left(\int_0^T |u|^{2\alpha} dt \right)^{1/2} + MT^{1/2} \right\} \left(\int_0^T |v|^2 dt \right)^{1/2} \\
&\le \left(\varsigma T^{(1-\alpha)/2} \left(\int_0^T |u|^2 dt \right)^{\beta/2} + MT^{1/2} \right) \|v\|,
\end{aligned}
$$

and hence

$$
\|\psi'(u)\| \le \varsigma T^{(1-\alpha)/2} \|u\|^\alpha + MT^{1/2}. \tag{3.8}
$$

For $u \in S_{\eta,\delta}$, $\|u\| \ge \|u^+ + u^-\| \ge J_{\eta,\delta}$; thus, it follows from (3.1) and (3.6)–(3.8) that

$$
\begin{aligned}
&\langle \varphi'(u), u\rangle + \frac{1}{2}\|\varphi'(u)\| \cdot \|u\| \\
&\quad = 2\left[\varphi(u) + \psi(u) - \frac{1}{2}\langle \psi'(u), u\rangle \right] \\
&\qquad + \frac{1}{2}\|B(u^+ + u^-) - \psi'(u)\| \cdot \|u\| \\
&\quad \ge 2\varphi(u) + 2\psi(u) - \langle \psi'(u), u\rangle \\
&\qquad + \frac{1}{2}\|B(u^+ + u^-)\| \cdot \|u\| - \frac{1}{2}\|\psi'(u)\| \cdot \|u\| \\
&\quad \ge -2\eta - \frac{7}{2}(\varsigma T^{(1-\alpha)/2}\|u\|^{1+\alpha} + MT^{1/2}\|u\|) \\
&\qquad - 2\int_0^T |H(t,0)|\, dt + \frac{c_0\delta}{4}\|u\|^{1+\alpha} \\
&\quad \ge \|u\|^{1+\alpha}\left\{ -\left(2\eta + 2\int_0^T |H(t,0)|\, dt \right) \|u\|^{-(1+\alpha)} \right. \\
&\qquad\qquad \left. - \frac{7}{2}(\varsigma T^{(1-\alpha)/2} + MT^{1/2}\|u\|^{-\alpha}) + \frac{c_0\delta}{4} \right\} \\
&\quad \ge \|u\|^{1+\alpha}\left\{ -\left(2\eta + 2\int_0^T |H(t,0)|\, dt \right) J_{\eta,\delta}^{-(1+\alpha)} \right. \\
&\qquad\qquad \left. - \frac{7}{2}(\varsigma T^{(1-\alpha)/2} + MT^{1/2} J_{\eta,\delta}^{-\alpha}) + \frac{c_0\delta}{4} \right\} > 0,
\end{aligned}
$$

provided $J_{\eta,\delta} > 0$ large enough. This completes the proof. $\square$

Lemma 3.3. *Assume* (H4) *with* $\alpha \in [0, 1)$. *Then there is a constant* $\Delta > 0$ *such that, for any* $\eta > 0$, *there exists a* $J_\eta > 0$ *such that*

$$\langle \varphi'(u), u \rangle + \frac{1}{2}\|\varphi'(u)\| \cdot \|u\| \geq 0$$

for all $u \in S_\eta := \{u \in H_T^1 \mid \varphi(u) \geq -\eta, \|u^+ + u^-\| \geq \Delta\|u^0\|^\alpha + J_\eta\}$.

Proof. For $u \in S_\eta$, we have $\|u^+ + u^-\| \geq \Delta\|u^0\|^\alpha$. Similar to (3.6), for $\|u\|$ large enough, we have

$$\|u^+ + u^-\| \geq \frac{\Delta}{2}\|u\|^\alpha. \tag{3.9}$$

It follows from (H4) that there exists an $M > 0$ such that

$$|\nabla H(t, x)| \leq M(|x|^\alpha + 1), \quad \text{for all } x \in \mathbf{R}^n \text{and } t \in [0, T].$$

Similar to (3.7) and (3.8), we have

$$|\psi(u)| \leq MT^{(1-\alpha)/2}\|u\|^{1+\alpha} + MT^{1/2}\|u\| + \int_0^T |H(t, 0)|\, dt \tag{3.10}$$

and

$$\|\psi'(u)\| \leq MT^{(1-\alpha)/2}\|u\|^\alpha + MT^{1/2}. \tag{3.11}$$

For $u \in S_\eta$, $\|u\| \geq \|u^+ + u^-\| \geq J_\eta$, it thus follows from (3.1) and

(3.9)–(3.11) that

$$
\begin{aligned}
&\langle \varphi'(u), u\rangle + \frac{1}{2}\|\varphi'(u)\| \cdot \|u\| \\
&\quad \geq 2\varphi(u) + 2\psi(u) - \langle \psi'(u), u\rangle + \frac{1}{2}\|B(u^+ + u^-)\| \cdot \|u\| \\
&\qquad - \frac{1}{2}\|\psi'(u)\| \cdot \|u\| \\
&\quad \geq -2\eta - \frac{7}{2}(MT^{(1-\alpha)/2}\|u\|^{1+\alpha} + MT^{1/2}\|u\|) \\
&\qquad - 2\int_0^T |H(t,0)|\,dt + \frac{c_0\Delta}{4}\|u\|^{1+\alpha} \\
&\quad \geq \|u\|^{1+\alpha}\bigg\{ - \bigg(2\eta + 2\int_0^T |H(t,0)|\,dt\bigg)\|u\|^{-(1+\alpha)} \\
&\qquad\qquad - \frac{7}{2}(MT^{(1-\alpha)/2} + MT^{1/2}\|u\|^{-\alpha}) + \frac{c_0\Delta}{4}\bigg\} \\
&\quad \geq \|u\|^{1+\alpha}\bigg\{ - \bigg(2\eta + 2\int_0^T |H(t,0)|\,dt\bigg)J_\eta^{-(1+\alpha)} \\
&\qquad\qquad - \frac{7}{2}(MT^{(1-\alpha)/2} + MT^{1/2}J_\eta^{-\alpha}) + \frac{c_0\Delta}{4}\bigg\} > 0,
\end{aligned}
$$

provided that $\Delta = 15/c_0(MT^{(1-\alpha)/2} + MT^{1/2})$ and $J_\eta > 0$ large enough. This completes the proof. □

Remark. If $0 \notin \sigma(B)$, that is, $H_0 = \{0\}$, then for all $u \in H_T^1$, $u = u^+ + u^-$ with $u^\pm \in H_\pm$. In this case, Lemma 3.2 and Lemma 3.3 still hold true and can be restated as:

Lemma 3.4. *Assume that* $0 \notin \sigma(B)$. *If* (H4) *with* $\alpha \in (0, 1]$ *or* (H6) *with* $\alpha \in [0, 1)$ *holds, then for any* $\eta > 0$, *there exists a* $J_\eta > 0$ *such that*

$$\langle \varphi'(u), u\rangle + \frac{1}{2}\|\varphi'(u)\| \cdot \|u\| \geq 0$$

for all $u \in S_\eta := \{u \in H_T^1 \mid \varphi(u) \geq -\eta, \|u\| \geq J_\eta\}$.

Lemma 3.5. *Assume that* $0 \in \sigma(B)$ *and* (H2). *Then there is a number* $D > 0$ *with the following property: for any small* $\varepsilon > 0$, *there* *exists a* $\delta(\varepsilon) > 0$ *such that, for any* $J > 0$ *and* $W_{\varepsilon,J} := \{u \in H_T^1 \mid \|u^+ + u^-\| \le \delta(\varepsilon)\|u^0\| + J\}$,

$$\liminf_{u\in W_{\varepsilon,J}\,\|u\|\to\infty} \int_0^T [2H(t,u) - (\nabla H(t,u),u)]\,dt \ge \theta_0 \int_0^T h(t)\,dt - D\varepsilon,$$

where $\theta_0 \in (0,1)$ *is given in* (1.2).

Proof. For any small $\varepsilon > 0$, by using an argument as used in the proof of Lemma 3.2 in [**2**], it is not hard to show that there exist small $\gamma(\varepsilon) \in (0,1)$ and large $\Gamma(\varepsilon) > 1$ such that

$$\text{meas}\,\{t \in [0,T] \mid |u^0(t)| < \gamma(\varepsilon)\|u^0\|\} < \varepsilon, \quad \text{for all } u^0 \in H_0 \setminus \{0\}$$

and

$$\text{meas}\,\{t \in [0,T] \mid |u^+(t) + u^-(t)| > \Gamma(\varepsilon)\|u^+ + u^-\|\} < \gamma^\beta(\varepsilon)\varepsilon,$$
$$\text{for all } u^+ + u^- \in H_+ \oplus H_-,$$

where $\beta > 0$ is a constant. Set

$$E_1(u,\varepsilon) = \{t \in [0,T] \mid |u^0(t)| \ge \gamma(\varepsilon)\|u^0\|\},$$
$$E_2(u,\varepsilon) = \{t \in [0,T] \mid |u^+(t) + u^-(t)| \le \Gamma(\varepsilon)\|u^+ + u^-\|\}.$$

Then

$$\begin{aligned} &\text{meas}\,([0,T] \setminus E_1(u,\varepsilon)) < \varepsilon, \\ &\text{meas}\,([0,T] \setminus E_2(u,\varepsilon)) < \gamma^\beta(\varepsilon)\varepsilon \le \varepsilon. \end{aligned} \tag{3.12}$$

Moreover, for all $u \in H_T^1$,

$$\begin{aligned} T &\ge \text{meas}\,(E_1(u,\varepsilon) \cap E_2(u,\varepsilon)) \\ &\ge \text{meas}\,(E_1(u,\varepsilon)) - \text{meas}\,([0,T] \setminus E_2(u,\varepsilon)) \\ &\ge T - 2\varepsilon. \end{aligned}$$

Therefore, noting that $\theta_0 < 1$ and $\int_0^T h(t)\,dt > 2\int_0^T H(t,0)\,dt \ge 0$, we can choose a positive number $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0,\varepsilon_0)$, the following holds:

$$\int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} h(t)\,dt \ge \theta_0 \int_0^T h(t)\,dt \tag{3.13}$$

for all $u \in H_T^1$.

Let $\delta(\varepsilon) = (\gamma(\varepsilon))/(2\Gamma(\varepsilon))$ and $W_{\varepsilon,J} := \{u \in H_T^1 \mid \|u^+ + u^-\| \leq \delta(\varepsilon)\|u^0\| + J\}$ for $J > 0$. Then $\delta(\varepsilon) \in (0,1)$. Since $u \in W_{\varepsilon,J}$ and $\|u\| \to \infty$ imply that $\|u^0\| \to \infty$, therefore, for $u \in W_{\varepsilon,J}$ with $\|u\|$ sufficiently large, we have

$$\frac{\|u^0\|^2}{\|u\|^2} = \frac{\|u\|^2 - \|u^+ + u^-\|^2}{\|u\|^2} \geq 1 - \frac{(\delta(\varepsilon)\|u^0\| + J)^2}{\|u\|^2} \geq 1 - \frac{4\|u^0\|^2}{\|u\|^2},$$

which yields

$$\frac{\|u^0\|}{\|u\|} \geq \frac{1}{\sqrt{5}}, \quad \text{for } u \in W_{\varepsilon,J} \text{ with } \|u\| \text{ sufficiently large.}$$

For any $u \in W_{\varepsilon,J}$ and any $t \in E_1(u,\varepsilon) \cap E_2(u,\varepsilon)$, we have

$$\begin{aligned}
\frac{|u(t)|}{\|u\|} &= \frac{|u^0(t) + u^+(t) + u^-(t)|}{\|u\|} \\
&\geq \frac{|u^0(t)|}{\|u\|} - \frac{|u^+(t) + u^-(t)|}{\|u\|} \\
&\geq \gamma(\varepsilon)\frac{\|u^0\|}{\|u\|} - \Gamma(\varepsilon)\frac{\|u^+ + u^-\|}{\|u\|} \\
&\geq [\gamma(\varepsilon) - \Gamma(\varepsilon)\delta(\varepsilon)]\frac{\|u^0\|}{\|u\|} - \Gamma(\varepsilon)\frac{J}{\|u\|} \\
&= \frac{1}{2}\gamma(\varepsilon)\frac{\|u^0\|}{\|u\|} - \Gamma(\varepsilon)\frac{J}{\|u\|} \\
&\geq \frac{1}{3}\gamma(\varepsilon)\frac{\|u^0\|}{\|u\|}.
\end{aligned} \tag{3.14}$$

Therefore,

$$\liminf_{u \in W_{\varepsilon,J}, \|u\| \to \infty} \inf\left\{\frac{|u(t)|}{\|u\|} \mid t \in E_1(u,\varepsilon) \cap E_2(u,\varepsilon)\right\} \geq \frac{1}{3\sqrt{5}}\gamma(\varepsilon) > 0. \tag{3.15}$$

For any $u \in W_{\varepsilon,J}$ and $t \in E_2(u,\varepsilon) \setminus E_1(u,\varepsilon)$, we have

$$\begin{aligned}\frac{|u(t)|}{\|u\|} &\le \gamma(\varepsilon)\frac{\|u^0\|}{\|u\|} + \Gamma(\varepsilon)\frac{\|u^+ + u^-\|}{\|u\|} \\ &\le (\gamma(\varepsilon) + \Gamma(\varepsilon)\delta(\varepsilon))\frac{\|u^0\|}{\|u\|} + \Gamma(\varepsilon)\frac{J}{\|u\|} \\ &= \frac{3}{2}\gamma(\varepsilon)\frac{\|u^0\|}{\|u\|} + \Gamma(\varepsilon)\frac{J}{\|u\|} \le 2\gamma(\varepsilon)\frac{\|u^0\|}{\|u\|}.\end{aligned} \tag{3.16}$$

By virtue of (H2), there exists a large $R_\varepsilon > 0$ such that

$$\begin{gathered}2H(t,x) - (\nabla H(t,x), x) \ge h(t) - \varepsilon, \\ \text{for } t \in [0,T] \text{ and } |x| \ge R_\varepsilon.\end{gathered} \tag{3.17}$$

Set $E_3(u,\varepsilon) = \{t \in [0,T] \mid |u(t)| \ge R_\varepsilon\}$. Then, for $u \in W_{\varepsilon,J}$ with $\|u\|$ sufficiently large, (3.15) implies

$$E_1(u,\varepsilon) \cap E_2(u,\varepsilon) \subset E_3(u,\varepsilon).$$

Therefore, by (3.13) and (3.17), we obtain

$$\begin{aligned}\int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} &[2H(t,u) - (\nabla H(t,u), u)]\, dt \\ &\ge \int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} (h(t) - \varepsilon)\, dt \\ &\ge \theta_0 \int_0^T h(t)\, dt - T\varepsilon.\end{aligned} \tag{3.18}$$

Choose a large number $D_1 > 0$ such that $h(t) \ge -D_1 + 1$ for all $t \in [0,T]$. Again, by (H2), we can find a large number $R > 0$ such that

$$2H(t,x) - (\nabla H(t,x), x) \ge -D_1, \quad \text{for } t \in [0,T] \text{ and } |x| \ge R. \tag{3.19}$$

Set $E_4(u) = \{t \in [0,T] \mid |u(t)| \ge R\}$ and $D_2 = \max\{|2H(t,x) - (\nabla H(t,x), x)| \mid t \in [0,T],\ |x| \le R\}$. Then it follows from (3.12) and

(3.19) that

$$
\begin{aligned}
(3.20)\quad \int_{E_2(u,\varepsilon)\setminus E_1(u,\varepsilon)} & [2H(t,u) - (\nabla H(t,u),u)]\,dt \\
&\geq \int_{(E_2(u,\varepsilon)\setminus E_1(u,\varepsilon))\cap E_4(u)} [2H(t,u) - (\nabla H(t,u),u)]\,dt \\
&\quad + \int_{(E_2(u,\varepsilon)\setminus E_1(u,\varepsilon))\setminus E_4(u)} [2H(t,u) - (\nabla H(t,u),u)]\,dt \\
&\geq -D_1 \int_{(E_2(u,\varepsilon)\setminus E_1(u,\varepsilon))\cap E_4(u)} dt \\
&\quad - D_2 \int_{(E_2(u,\varepsilon)\setminus E_1(u,\varepsilon))\setminus E_4(u)} dt \\
&\geq -(D_1 + D_2)\varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
(3.21)\quad \int_{[0,T]\setminus E_2(u,\varepsilon)} & [2H(t,u) - (\nabla H(t,u),u)]\,dt \\
&= \int_{([0,T]\setminus E_2(u,\varepsilon))\cap E_4(u)} [2H(t,u) - (\nabla H(t,u),u)]\,dt \\
&\quad + \int_{([0,T]\setminus E_2(u,\varepsilon))\setminus E_4(u)} [2H(t,u) - (\nabla H(t,u),u)]\,dt \\
&\geq -D_1 \int_{([0,T]\setminus E_2(u,\varepsilon))\cap E_4(u)} dt - D_2 \int_{([0,T]\setminus E_2(u,\varepsilon))\setminus E_4(u)} dt \\
&\geq -(D_1 + D_2)\varepsilon.
\end{aligned}
$$

Therefore, by (3.18), (3.20) and (3.21), for $u \in W_{\varepsilon,J}$ with $\|u\|$ sufficiently large, we obtain

$$
\begin{aligned}
\int_0^T [2H(t,u) - (\nabla H(t,u),u)]\,dt \\
&= \int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} [2H(t,u) - (\nabla H(t,u),u)]\,dt \\
&\quad + \int_{E_2(u,\varepsilon)\setminus E_1(u,\varepsilon)} [2H(t,u) - (\nabla H(t,u),u)]\,dt
\end{aligned}
$$

$$+ \int_{[0,T]\setminus E_2(u,\varepsilon)} [2H(t,u) - (\nabla H(t,u), u)]\, dt$$

$$\geq \theta_0 \int_0^T h(t)\, dt - (T + 2D_1 + 2D_2)\varepsilon,$$

from which the conclusion follows, and the proof is complete. ◻

Lemma 3.6. *Assume that* $0 \in \sigma(B)$ *and* (H3). *Then, for any small* $\varepsilon > 0$, *there exists a* $\delta(\varepsilon) > 0$ *such that, for any* $J > 0$ *and* $W_{\varepsilon,J} := \{u \in H_T^1 \mid \|u^+ + u^-\| \leq \delta(\varepsilon)\|u^0\| + J\}$,

$$\liminf_{u \in W_{\varepsilon,J}, \|u\| \to \infty} \int_0^T \frac{2H(t,u) - (\nabla H(t,u), u)}{\|u\|^\beta} dt > 0.$$

Proof. For any small $\varepsilon > 0$, let $\gamma(\varepsilon), \Gamma(\varepsilon)$, $\delta(\varepsilon) \in (0,1)$, $E_1(u,\varepsilon)$ and $E_2(u,\varepsilon)$ with $u \in H_T^1$ be given in the proof of Lemma 3.5. Let c denote various positive constants which are independent of ε.

By virtue of (H3), there exists a large $R'_\varepsilon > 0$ such that

$$\frac{2H(t,x) - (\nabla H(t,x), x)}{|x|^\beta} \geq h(t) - \gamma^\beta(\varepsilon)\varepsilon, \tag{3.22}$$
$$\text{for } t \in [0,T] \text{ and } |x| \geq R'_\varepsilon.$$

Set $E_5(u,\varepsilon) = \{t \in [0,T] \mid |u(t)| \geq R'_\varepsilon\}$. Then, for $u \in W_{\varepsilon,J}$ with $\|u\|$ sufficiently large, it follows from (3.15) that

$$E_1(u,\varepsilon) \cap E_2(u,\varepsilon) \subset E_5(u,\varepsilon).$$

Therefore, it follows from (3.14) and (3.22) that

$$\begin{aligned}
\int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} & \frac{2H(t,u) - (\nabla H(t,u), u)}{\|u\|^\beta} dt \\
&\geq \int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} \frac{(h(t) - \gamma^\beta(\varepsilon)\varepsilon)|u|^\beta}{\|u\|^\beta} dt \\
&\geq \frac{1}{3^\beta}\gamma^\beta(\varepsilon)\left(\frac{\|u^0\|}{\|u\|}\right)^\beta \int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} h(t)\, dt \\
&\quad - c\gamma^\beta(\varepsilon)\varepsilon.
\end{aligned} \tag{3.23}$$

Again, by (H3), we can find a large number $R' > 0$ such that

$$2H(t,x) - (\nabla H(t,x),x) \geq -|x|^{\beta}, \quad \text{for } t \in [0,T] \text{ and } |x| \geq R'. \tag{3.24}$$

Set $E_6(u) = \{t \in [0,T] \mid |u(t)| \geq R'\}$ and $D_3 = \max\{|2H(t,x) - (\nabla H(t,x),x)| \mid t \in [0,T],\ |x| \leq R'\}$. Then it follows from (3.9), (3.13) and (3.21) that

$$\begin{aligned}
&\int_{E_2(u,\varepsilon)\setminus E_1(u,\varepsilon)} \frac{2H(t,u) - (\nabla H(t,u),u)}{\|u\|^{\beta}} dt \\
&\quad = \int_{(E_2(u,\varepsilon)\setminus E_1(u,\varepsilon))\cap E_6(u)} \frac{2H(t,u) - (\nabla H(t,u),u)}{\|u\|^{\beta}} dt \\
&\qquad + \int_{(E_2(u,\varepsilon)\setminus E_1(u,\varepsilon))\setminus E_6(u)} \frac{2H(t,u) - (\nabla H(t,u),u)}{\|u\|^{\beta}} dt \\
&\quad \geq - \int_{(E_2(u,\varepsilon)\setminus E_1(u,\varepsilon))\cap E_6(u)} \frac{|u(t)|^{\beta}}{\|u\|^{\beta}} dt \\
&\qquad - \int_{(E_2(u,\varepsilon)\setminus E_1(u,\varepsilon))\setminus E_6(u)} \frac{D_3}{\|u\|^{\beta}} dt \\
&\quad \geq -2^{\beta}\gamma^{\beta}(\varepsilon)\left(\frac{\|u^0\|}{\|u\|}\right)^{\beta} \varepsilon - c\gamma^{\beta}(\varepsilon)\varepsilon \geq -2^{\beta}\gamma^{\beta}(\varepsilon)\varepsilon - c\gamma^{\beta}(\varepsilon)\varepsilon
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
&\int_{[0,T]\setminus E_2(u,\varepsilon)} \frac{2H(t,u) - (\nabla H(t,u),u)}{\|u\|^{\beta}} dt \\
&\quad = \int_{([0,T]\setminus E_2(u,\varepsilon))\cap E_6(u)} \frac{2H(t,u) - (\nabla H(t,u),u)}{\|u\|^{\beta}} dt \\
&\qquad + \int_{([0,T]\setminus E_2(u,\varepsilon))\setminus E_6(u)} \frac{2H(t,u) - (\nabla H(t,u),u)}{\|u\|^{\beta}} dt \\
&\quad \geq - \int_{([0,T]\setminus E_2(u,\varepsilon))\cap E_6(u)} \frac{|u(t)|^{\beta}}{\|u\|^{\beta}} dt \\
&\qquad - \int_{([0,T]\setminus E_2(u,\varepsilon))\setminus E_6(u)} \frac{D_3}{\|u\|^{\beta}} dt \\
&\quad \geq -c\gamma^{\beta}(\varepsilon)\varepsilon.
\end{aligned} \tag{3.26}$$

Therefore, for any $u \in W_{\varepsilon,J}$ with $\|u\|$ sufficiently large, it follows from (3.23), (3.25) and (3.26) that

$$\begin{aligned}
&\int_0^T \frac{2H(t,u)-(\nabla H(t,u),u)}{\|u\|^\beta}dt \\
&\qquad = \int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} \frac{2H(t,u)-(\nabla H(t,u),u)}{\|u\|^\beta}dt \\
&\qquad\quad + \int_{E_2(u,\varepsilon)\setminus E_1(u,\varepsilon)} \frac{2H(t,u)-(\nabla H(t,u),u)}{\|u\|^\beta}dt \\
&\qquad\quad + \int_{[0,T]\setminus E_2(u,\varepsilon)} \frac{2H(t,u)-(\nabla H(t,u),u)}{\|u\|^\beta}dt \\
&\qquad \geq \frac{1}{3^\beta}\gamma^\beta(\varepsilon)\left(\frac{\|u^0\|}{\|u\|}\right)^\beta \int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} h(t)\,dt \\
&\qquad\quad - 2^\beta\gamma^\beta(\varepsilon)\varepsilon - c\gamma^\beta(\varepsilon)\varepsilon \\
&\qquad \geq \gamma^\beta(\varepsilon)\left\{\frac{1}{3^\beta 5^{\beta/2}}\int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} h(t)\,dt - 2^\beta\varepsilon - c\varepsilon\right\}.
\end{aligned}$$

Noting that $\lim_{\varepsilon\to 0}\int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} h(t)\,dt = \int_0^T h(t)\,dt > 0$, the last inequality implies that

$$\liminf_{u\in W_{\varepsilon,J},\|u\|\to\infty}\int_0^T \frac{2H(t,u)-(\nabla H(t,u),u)}{\|u\|^\beta}\,dt > 0,$$

provided $\varepsilon > 0$ is small enough. The proof is complete. $\square$

Lemma 3.7. *Assume that* $0 \in \sigma(B)$ *and* (H5). *Then, for any* $J > 0$, *and* $W_J^\alpha := \{u \in H_T^1 \mid \|u^+ + u^-\| \leq \Delta\|u^0\|^\alpha + J\}$,

$$\lim_{u\in W_J^\alpha,\|u\|\to\infty}\int_0^T \frac{H(t,u)}{\|u\|^{2\alpha}}dt = -\infty$$

in the case (H5) (i) (*or* $+\infty$, *in the case* (H5) (ii)).

Proof. We only consider the case (H5) (i); the other case can be treated similarly. Without loss of generality, we can assume that

$\lim_{|x|\to\infty} H(t,x)/|x|^{2\alpha} = -\infty$ uniformly for $t \in E \subset [0,T]$. Otherwise, we can choose a subset $E_1 \subset E$ with $\operatorname{meas}(E_1) > 0$ such that the last equality holds uniformly for $t \in E_1$ (see the proof of Lemma 2 in [**21**]).

For any small $\varepsilon > 0$, let $\gamma(\varepsilon), \Gamma(\varepsilon)$, $\delta(\varepsilon) \in (0,1)$, $E_1(u,\varepsilon)$ and $E_2(u,\varepsilon)$ with $u \in H_T^1$ be given in the proof of Lemma 3.5. Let c denote various positive constants which are independent of ε.

By virtue of (H5) (i), we can choose a large number $M' > 0$ such that

$$H(t,x) \le M'(|x|^{2\alpha} + 1), \quad \text{for } t \in [0,T] \text{ and } x \in \mathbf{R}^n, \tag{3.27}$$

and, for any $K > 0$, we can choose a large $R'_K > 0$ such that

$$H(t,x) \le -K|x|^{2\alpha}, \quad \text{for } t \in E \text{ and } |x| \ge R'_K. \tag{3.28}$$

Set $E_7(u,K) = \{t \in [0,T] \mid |u(t)| \ge R'_K\}$. Since $\alpha < 1$, it follows that, for $\|u\|$ sufficiently large,

$$\Delta\|u^0\|^\alpha = \frac{\Delta}{\|u^0\|^{1-\alpha}}\|u^0\| \le \delta(\varepsilon)\|u^0\|.$$

Therefore, $W_J^\alpha \subset W_{\varepsilon,J} := \{u \in H_T^1 \mid \|u^+ + u^-\| \le \delta(\varepsilon)\|u^0\| + J\}$ and, hence, for $u \in W_J^\alpha$ with $\|u\|$ sufficiently large, $\|u^0\|/\|u\| \ge 1/\sqrt{5}$ and (3.15) imply that

$$E_1(u,\varepsilon) \cap E_2(u,\varepsilon) \subset E_7(u,K).$$

Noting that $T \ge \operatorname{meas}(E_1(u,\varepsilon) \cap E_2(u,\varepsilon)) \ge T - 2\varepsilon$ for all $u \in H_T^1$, it follows that

$$\operatorname{meas}(E_1(u,\varepsilon) \cap E_2(u,\varepsilon) \cap E) \ge \operatorname{meas}(E) - 2\varepsilon > 0,$$

provided $\varepsilon > 0$ is small enough. Therefore, by (3.14), (3.27) and (3.28), we obtain that, for any $u \in W_J^\alpha$ with $\|u\|$ sufficiently large,

$$\begin{aligned} &\int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} \frac{H(t,u)}{\|u\|^{2\alpha}}dt \\ &\qquad \le \int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)\cap E} \frac{-K|u|^{2\alpha}}{\|u\|^{2\alpha}}dt \\ &\qquad\quad + \int_{(E_1(u,\varepsilon)\cap E_2(u,\varepsilon))\setminus E} \frac{M'(|u|^{2\alpha}+1)}{\|u\|^{2\alpha}}dt \\ &\qquad \le -\frac{K}{3^{2\alpha}}\gamma^{2\alpha}(\varepsilon)\left(\frac{\|u^0\|}{\|u\|}\right)^{2\alpha}(\operatorname{meas}(E) - 2\varepsilon) + c \\ &\qquad \le -\frac{K}{45^\alpha}\gamma^{2\alpha}(\varepsilon)(\operatorname{meas}(E) - 2\varepsilon) + c. \end{aligned} \tag{3.29}$$

Moreover, it follows from (3.27) that

$$\begin{aligned} (3.30) \quad & \int_{[0,T]\setminus(E_1(u,\varepsilon)\cap E_2(u,\varepsilon))} \frac{H(t,u)}{\|u\|^{2\alpha}}dt \\ & \qquad \leq \int_{[0,T]\setminus(E_1(u,\varepsilon)\cup E_2(u,\varepsilon))} \frac{M'(|u|^{2\alpha}+1)}{\|u\|^{2\alpha}}dt \leq c. \end{aligned}$$

Therefore, by (3.29) and (3.30), for $u \in W_J^\alpha$ with $\|u\|$ sufficiently large, we obtain

$$\begin{aligned} \int_0^T \frac{H(t,u)}{\|u\|^{2\alpha}}dt &= \int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} \frac{H(t,u)}{\|u\|^{2\alpha}}dt \\ &\quad + \int_{[0,T]\setminus(E_1(u,\varepsilon)\cap E_2(u,\varepsilon))} \frac{H(t,u)}{\|u\|^{2\alpha}}dt \\ &\leq -\frac{K}{45^\alpha}\gamma^{2\alpha}(\varepsilon)(\mathrm{meas}\,(E)-2\varepsilon)+c. \end{aligned}$$

Since the constant c is independent of $K>0$, the arbitrariness of $K>0$ yields the conclusion, and the proof is complete.

Lemma 3.8. *Assume* $0 \in \sigma(B)$ *and* (H8) *with* $\beta \in (0,2)$. *Then, for any small* $\varepsilon > 0$, *there exists a* $\delta(\varepsilon) > 0$ *such that, for any* $J > 0$ *and* $W_{\varepsilon,J} := \{u \in H_T^1 \mid \|u^+ + u^-\| \leq \delta(\varepsilon)\|u^0\| + J\}$,

$$\limsup_{u\in W_{\varepsilon,J},\|u\|\to\infty} \int_0^T \frac{H(t,u)}{\|u\|^\beta}dt < 0, \quad \textit{in the case } \text{(H8) (i)}$$

or

$$\liminf_{u\in W_{\varepsilon,J},\|u\|\to\infty} \int_0^T \frac{H(t,u)}{\|u\|^\beta}dt > 0, \quad \textit{in the case } \text{(H8) (ii)}.$$

Proof. Without loss of generality, we assume (H8) (i); the other case can be treated similarly. For any small $\varepsilon > 0$, let $\gamma(\varepsilon), \Gamma(\varepsilon)$, $\delta(\varepsilon) \in (0,1)$, $E_1(u,\varepsilon)$ and $E_2(u,\varepsilon)$ with $u \in H_T^1$ be given as in the proof of Lemma 3.5. Let c denote various positive constants which are independent of ε.

By virtue of (H8) (i), there exists a large $R''_\varepsilon > 0$ such that

$$H(t,x) \le (h(t) + \gamma^\beta(\varepsilon)\varepsilon)|x|^\beta, \quad \text{for } t \in [0,T] \text{ and } |x| \ge R''_\varepsilon. \tag{3.31}$$

Set $E_8(u,\varepsilon) = \{t \in [0,T] \mid |u(t)| \ge R''_\varepsilon\}$. Then, for $u \in W_{\varepsilon,J}$ with $\|u\|$ sufficiently large, (3.15) implies

$$E_1(u,\varepsilon) \cap E_2(u,\varepsilon) \subset E_8(u,\varepsilon).$$

Therefore, by (3.14) and (3.31), we obtain

$$\begin{aligned} \int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} \frac{H(t,u)}{\|u\|^\beta} dt &\le \int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} \frac{(h(t) + \gamma^\beta(\varepsilon)\varepsilon)|u|^\beta}{\|u\|^\beta} dt \\ &\le \frac{1}{3^\beta}\gamma^\beta(\varepsilon)\left(\frac{\|u^0\|}{\|u\|}\right)^\beta \int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} h(t)\, dt \\ &\quad + c\gamma^\beta(\varepsilon)\varepsilon. \end{aligned} \tag{3.32}$$

Again, by (H8) (i), we can find a large number $R'' > 0$ such that

$$H(t,x) \le |x|^\beta, \quad \text{for } t \in [0,T] \text{ and } |x| \ge R''. \tag{3.33}$$

Set $E_9(u) = \{t \in [0,T] \mid |u(t)| \ge R''\}$ and $D_4 = \max\{|H(t,x)| \mid t \in [0,T],\ |x| \le R''\}$. Then it follows from (3.12), (3.16) and (3.33) that

$$\begin{aligned} \int_{E_2(u,\varepsilon)\setminus E_1(u,\varepsilon)} \frac{H(t,u)}{\|u\|^\beta} dt &= \int_{(E_2(u,\varepsilon)\setminus E_1(u,\varepsilon))\cap E_9(u)} \frac{H(t,u)}{\|u\|^\beta} dt \\ &\quad + \int_{(E_2(u,\varepsilon)\setminus E_1(u,\varepsilon))\setminus E_9(u)} \frac{H(t,u)}{\|u\|^\beta} dt \\ &\le \int_{(E_2(u,\varepsilon)\setminus E_1(u,\varepsilon))\cap E_9(u)} \frac{|u|^\beta}{\|u\|^\beta} dt \\ &\quad + \int_{(E_2(u,\varepsilon)\setminus E_1(u,\varepsilon))\setminus E_9(u)} \frac{D_4}{\|u\|^\beta} dt \\ &\le 2^\beta\gamma^\beta(\varepsilon)\left(\frac{\|u^0\|}{\|u\|}\right)^\beta \varepsilon + c\gamma^\beta(\varepsilon)\varepsilon, \end{aligned} \tag{3.34}$$

and

$$
\begin{aligned}
\int_{[0,T]\setminus E_2(u,\varepsilon)} \frac{H(t,u)}{\|u\|^\beta} dt &= \int_{([0,T]\setminus E_2(u,\varepsilon))\cap E_9(u)} \frac{H(t,u)}{\|u\|^\beta} dt \\
&\quad + \int_{([0,T]\setminus E_2(u,\varepsilon))\setminus E_9(u)} \frac{H(t,u)}{\|u\|^\beta} dt \\
&\le \int_{([0,T]\setminus E_2(u,\varepsilon))\cap E_9(u)} \frac{|u|^\beta}{\|u\|^\beta} dt \\
&\quad + \int_{([0,T]\setminus E_2(u,\varepsilon))\setminus E_9(u)} \frac{D_4}{\|u\|^\beta} dt \\
&\le c\gamma^\beta(\varepsilon)\varepsilon.
\end{aligned} \tag{3.35}
$$

Therefore, by (3.32), (3.34) and (3.35), for $u \in W_{\varepsilon,J}$ with $\|u\|$ sufficiently large, we obtain

$$
\begin{aligned}
\int_0^T \frac{H(t,u)}{\|u\|^\beta} dt &= \int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} \frac{H(t,u)}{\|u\|^\beta}\, dt \\
&\quad + \int_{E_2(u,\varepsilon)\setminus E_1(u,\varepsilon)} \frac{H(t,u)}{\|u\|^\beta} dt \\
&\quad + \int_{[0,T]\setminus E_2(u,\varepsilon)} \frac{H(t,u)}{\|u\|^\beta} dt \\
&\le \gamma^\beta(\varepsilon)\bigg\{ \frac{1}{3^\beta}\bigg(\frac{\|u^0\|}{\|u\|}\bigg)^\beta \int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} h(t)\, dt \\
&\quad + 2^\beta \bigg(\frac{\|u^0\|}{\|u\|}\bigg)^\beta \varepsilon + c\varepsilon \bigg\} \\
&\le \gamma^\beta(\varepsilon)\bigg\{ \frac{1}{3^\beta 5^{\beta/2}} \int_{E_1(u,\varepsilon)\cap E_2(u,\varepsilon)} h(t)\, dt + 2^\beta\varepsilon + c\varepsilon \bigg\},
\end{aligned}
$$

from which and the arbitrariness of $\varepsilon > 0$, the conclusion follows. The proof is complete. $\square$

4. Proofs of main results. Now, we are in a position to prove our main results.

Proof of Theorem 1. Firstly, we assume $0 \in \sigma(B)$. Let $D > 0$ be the constant given in Lemma 3.5. Let $\varepsilon > 0$ be so small that

$$(2+D)\varepsilon \leq \frac{1}{2}\theta_0 \int_0^T h(t)\,dt. \tag{4.1}$$

By virtue of Lemma 3.5, we can choose a $\delta(\varepsilon) > 0$ such that, for any $J > 0$,

$$\liminf_{\|u\|\to\infty} \int_0^T [2H(t,u) - (\nabla H(t,u), u)]\,dt \geq \theta_0 \int_0^T h(t)\,dt - D\varepsilon \tag{4.2}$$

for $u \in H_T^1$ with $\|u^+ + u^-\| \leq \delta(\varepsilon)\|u^0\| + J$.

Let $H_2 = \operatorname{span}\{\vartheta_\ell, \vartheta_{\ell+1}, \dots\}$. Then $\operatorname{codim} H_2 < \infty$. Recall that $\int_0^T H(t,0)\,dt = 0$; for any $u \in H_2$, we have

$$\begin{aligned}
\psi(u) = \int_0^T H(t,u)\,dt &\leq \int_0^T |H(t,u) - H(t,0)|\,dt \\
&\leq \int_0^T |u|^2 dt + M \int_0^T |u|\,dt \\
&\leq \int_0^T |u|^2 dt + MT^{\frac{1}{2}} \left(\int_0^T |u|^2 dt \right)^{1/2} \\
&\leq \|u\|^2 + MT^{1/2}\|u\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\langle Bu, u\rangle - \psi(u) \\
&\geq \frac{1}{2}\lambda_\ell\|u\|^2 - \|u\|^2 - MT^{\frac{1}{2}}\|u\| \\
&\geq \min_{t\geq 0} \left\{ \left(\frac{1}{2}\lambda_\ell - 1\right) t^2 - MT^{1/2} t \right\} \\
&\geq -\varepsilon,
\end{aligned}$$

provided ℓ is large enough. Therefore, $\inf_{u\in H_2} \varphi(u) \geq -\varepsilon$.

Now we check that φ satisfies condition (B) (ii) in $(-\varepsilon, 0)$ with respect to $B := \{u \in H_T^1 \mid \|u\| \leq r\}$ with $r > 0$ being specified below. For any

$\epsilon \in (-\varepsilon, 0)$, let $\sigma = 1/4 \min\{\epsilon + \varepsilon, -\epsilon\}$. Then $[\epsilon - \sigma, \epsilon + \sigma] \subset (-\varepsilon, 0)$. If $u \in \varphi^{-1}([\epsilon - \sigma, \epsilon + \sigma])$, by Lemma 3.2, there exists a $J_\varepsilon > 0$ such that

$$\langle \varphi'(u), u \rangle + \frac{1}{2}\|\varphi'(u)\| \cdot \|u\| \geq 0, \tag{4.3}$$
$$\text{for } u \in H_T^1 \text{ with } \|u^+ + u^-\| \geq \delta(\varepsilon)\|u^0\| + J_\varepsilon.$$

Denote $W_\epsilon := \{u \in H_T^1 \mid \|u^+ + u^-\| \leq \delta(\varepsilon)\|u^0\| + J_\varepsilon\} \cap \varphi^{-1}([\epsilon - \sigma, \epsilon + \sigma])$. Then it follows from (4.1) and (4.2) that

$$\begin{aligned}
&\liminf_{u \in W_\epsilon, \|u\| \to \infty} \left\{ \langle \varphi'(u), u \rangle + \frac{1}{2}\|\varphi'(u)\| \cdot \|u\| \right\} \\
&\quad = \liminf_{u \in W_\epsilon, \|u\| \to \infty} \left\{ 2\left[\varphi(u) + \psi(u) - \frac{1}{2}\langle \psi'(u), u \rangle \right] \right. \\
&\qquad \left. + \frac{1}{2}\|B(u^+ + u^-) - \psi'(u)\| \cdot \|u\| \right\} \\
&\quad \geq \liminf_{u \in W_\epsilon, \|u\| \to \infty} \{2\varphi(u) + 2\psi(u) - \langle \psi'(u), u \rangle\} \\
&\quad \geq \liminf_{u \in W_\epsilon, \|u\| \to \infty} \left\{ -2\varepsilon + \int_0^T [2H(t, u) - (\nabla H(t, u), u)]\, dt \right\} \\
&\quad \geq -2\varepsilon + \liminf_{u \in W_\epsilon, \|u\| \to \infty} \int_0^T [2H(t, u) - (\nabla H(t, u), u)]\, dt \\
&\quad \geq -2\varepsilon + \theta_0 \int_0^T h(t)\, dt - D\varepsilon \\
&\quad \geq \frac{1}{2}\theta_0 \int_0^T h(t)\, dt > 0.
\end{aligned}$$

Consequently, $\langle \varphi'(u), u \rangle + 1/2\|\varphi'(u)\| \cdot \|u\| \geq 0$ for any $u \in \varphi^{-1}([\epsilon - \sigma, \epsilon + \sigma])$ with $\|u\| = r$ sufficiently large. That is, φ satisfies condition (B) (ii) in $(-\varepsilon, 0)$ with respect to $B := \{u \in H_T^1 \mid \|u\| \leq r\}$. Since condition (B) (i) holds naturally, by Theorem 2.3 and Lemma 3.1, the number of the critical points of φ is more than or equal to $\dim H_1 - \operatorname{codim} H_2$, where $H_1 = H_k$ comes from Lemma 3.1. Passing to the limit as $k \to \infty$, we obtain infinitely many critical points of φ, the critical values of which belong to $(-\varepsilon, 0)$.

Finally, if $0 \notin \sigma(B)$, then by Lemma 3.4, we also have that $\langle \varphi'(u), u \rangle + 1/2\|\varphi'(u)\| \cdot \|u\| \geq 0$ for any $u \in \varphi^{-1}([\epsilon - \sigma, \epsilon + \sigma])$ with $\|u\| = r$

sufficiently large. Therefore, φ also satisfies condition (B) (ii) in $(-\varepsilon, 0)$ with respect to $B := \{u \in H_T^1 \mid \|u\| \leq r\}$. As before, we can conclude that φ has infinitely many critical points. This completes the proof. $\square$

Proof of Theorem 2. Let $\varepsilon, \delta(\varepsilon)$, J_ϵ and $W_\epsilon = \{u \in H_T^1 \mid \|u^+ + u^-\| \leq \delta(\varepsilon)\|u^0\| + J_\epsilon\} \cap \varphi^{-1}([\epsilon - \sigma, \epsilon + \sigma])$ be the same as in the proof of Theorem 1. Then it follows that

$$\begin{aligned}
&\liminf_{u \in W_\epsilon, \|u\| \to \infty} \frac{1}{\|u\|^\beta} \left\{ \langle \varphi'(u), u \rangle + \frac{1}{2} \|\varphi'(u)\| \cdot \|u\| \right\} \\
&\quad \geq \liminf_{u \in W_\epsilon, \|u\| \to \infty} \frac{1}{\|u\|^\beta} \left\{ -2\varepsilon + \int_0^T [2H(t,u) - (\nabla H(t,u), u)]\, dt \right\} \\
&\quad \geq \liminf_{u \in W_\epsilon, \|u\| \to \infty} \int_0^T \frac{2H(t,u) - (\nabla H(t,u), u)}{\|u\|^\beta} dt > 0,
\end{aligned}$$

which, together with (4.3), implies that $\langle \varphi'(u), u \rangle + 1/2 \|\varphi'(u)\| \cdot \|u\| \geq 0$ for any $u \in \varphi^{-1}([\epsilon - \sigma, \epsilon + \sigma])$ with $\|u\| = r$ sufficiently large. The rest of the proof is the same as that of Theorem 1, and the proof is complete. $\square$

Proof of Theorem 3. Without loss of generality, we assume that there exists a constant $M > 0$ such that $\limsup_{|x| \to \infty} H(t,x)/|x|^{2\alpha} \leq M$ uniformly for $t \in [0, T]$, and $\lim_{|x| \to \infty} H(t,x)/|x|^{2\alpha} = -\infty$ on a positive measure subset $E \subset [0, 1]$; the other case can be treated similarly. Let $H_2 = \operatorname{span}\{\vartheta_\ell, \vartheta_{\ell+1}, \dots\}$. Then $\operatorname{codim} H_2 < \infty$. Let $\varepsilon > 0$ be a small number, for any $u \in H_2$. We then have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2} \langle Bu, u \rangle - \psi(u) \\
&\geq \frac{1}{2} \lambda_\ell \|u\|^2 - MT^{(1-\alpha)/2} \|u\|^{1+\alpha} - MT^{1/2} \|u\| \\
&\geq \min_{t \geq 0} \left\{ \frac{1}{2} \lambda_\ell t^2 - MT^{(1-\alpha)/2} t^{1+\alpha} - MT^{1/2} t \right\} \geq -\varepsilon,
\end{aligned}$$

provided ℓ is large enough. Therefore, $\inf_{u \in H_2} \varphi(u) \geq -\varepsilon$.

Now we check that φ satisfies condition (B) (ii) in $(-\varepsilon, 0)$ with respect to $B := \{u \in H_T^1 \mid \|u\| \leq r\}$ with $r > 0$ being specified below.

Firstly, we assume $0 \in \sigma(B)$. Let $\Delta > 0$ be the constant given in Lemma 3.3. By virtue of Lemma 3.7, for any $J > 0$, we have

$$\limsup_{\|u\|\to\infty} \int_0^T \frac{H(t,u)}{\|u\|^{2\alpha}} dt = -\infty. \tag{4.4}$$

for $u \in H_T^1$ with $\|u^+ + u^-\| \le \Delta\|u^0\|^\alpha + J$.

Let $\eta = \varepsilon > 0$, and let $J_\varepsilon := J_\eta > 0$ be given as in Lemma 3.3. Then, for any large $K > 0$, it follows that

$$\psi(u) = \int_0^T H(t,u)\,dt \le -K\|u\|^{2\alpha} < 0$$

for $u \in W_\varepsilon^\alpha := \{u \in H_T^1 \mid \|u^+ + u^-\| \le \Delta\|u^0\|^\alpha + J_\varepsilon\}$ with $\|u\|$ sufficiently large.

Therefore, for any $u \in \varphi^{-1}([-\varepsilon, 0]) \cap W_\varepsilon^\alpha$ with $\|u\|$ sufficiently large.

$$\langle B(u^+ + u^-), u^+ + u^-\rangle = \langle Bu, u\rangle = 2\varphi(u) + 2\psi(u) \le -K\|u\|^{2\alpha}. \tag{4.5}$$

On the other hand, for $\|u\|$ large enough, we have

$$\begin{aligned} |\langle B(u^+ + u^-), u^+ + u^-\rangle| &\le \|B(u^+ + u^-)\| \cdot \|u^+ + u^-\| \\ &\le (\Delta\|u^0\|^\alpha + J_\varepsilon)\|B(u^+ + u^-)\| \\ &\le 2\Delta\|u\|^\alpha\|B(u^+ + u^-)\|. \end{aligned} \tag{4.6}$$

It then follows from (4.5) and (4.6) that

$$\|B(u^+ + u^-)\| \ge \frac{K}{2\Delta}\|u\|^\alpha, \tag{4.7}$$

for $u \in \varphi^{-1}([-\varepsilon, 0]) \cap W_\varepsilon^\alpha$ with $\|u\|$ sufficiently large.

For any $\epsilon \in (-\varepsilon, 0)$, let $\sigma = 1/4 \min\{\epsilon + \varepsilon, -\epsilon\}$. Then $[\epsilon - \sigma, \epsilon + \sigma] \subset (-\varepsilon, 0)$. If $u \in \varphi^{-1}([\epsilon - \sigma, \epsilon + \sigma])$, by Lemma 3.3, we have

$$\begin{gathered} \langle \varphi'(u), u\rangle + \frac{1}{2}\|\varphi'(u)\| \cdot \|u\| \ge 0, \\ \text{for } u \in H_T^1 \text{ with } \|u^+ + u^-\| \ge \Delta\|u^0\|^\alpha + J_\varepsilon. \end{gathered} \tag{4.8}$$

Denote $W_\epsilon := W_\varepsilon^\alpha \cap \varphi^{-1}([\epsilon-\sigma, \epsilon+\sigma])$. Then it follows from (3.7), (3.8) and (4.7) that

$$\begin{aligned}\langle \varphi'(u), u\rangle + \frac{1}{2}\|\varphi'(u)\| \cdot \|u\| &= 2\left[\varphi(u) + \psi(u) - \frac{1}{2}\langle \psi'(u), u\rangle\right] \\ &\quad + \frac{1}{2}\|B(u^+ + u^-) - \psi'(u)\| \cdot \|u\| \\ &\geq 2\varphi(u) + 2\psi(u) + \frac{1}{2}\|B(u^+ + u^-)\| \cdot \|u\| \\ &\quad - \frac{3}{2}\|\psi'(u)\| \cdot \|u\| \\ &\geq -2\varepsilon - 2MT^{(1-\alpha)/2}\|u\|^{1+\alpha} \\ &\quad - 2MT^{1/2}\|u\| - 2\int_0^T |H(t,0)|\, dt \\ &\quad + \frac{K}{4\Delta}\|u\|^{1+\alpha} - \frac{3}{2}MT^{(1-\alpha)/2}\|u\|^{1+\alpha} \\ &\quad - \frac{3}{2}MT^{1/2}\|u\| \\ &> 0,\end{aligned}$$

provided $K > 0$ and $\|u\|$ are sufficiently large.

Consequently, $\langle \varphi'(u), u\rangle + 1/2\|\varphi'(u)\| \cdot \|u\| \geq 0$ for any $u \in \varphi^{-1}([\epsilon - \sigma, \epsilon + \sigma])$ with $\|u\| = r$ sufficiently large. That is, φ satisfies condition (B) (ii) in $(-\varepsilon, 0)$ with respect to $B := \{u \in H_T^1 \mid \|u\| \leq r\}$. The remainder of the proof is similar to that of Theorem 1 and is omitted. The proof is complete. □

Proof of Theorem 4. We only consider the case (H7) (i), the other case can be treated similarly. Let $H_2 = \operatorname{span}\{\vartheta_\ell, \vartheta_{\ell+1}, \dots\}$. Then $\operatorname{codim} H_2 < \infty$. Let $\varepsilon > 0$ be a small number. For any $u \in H_2$, we have $\inf_{u \in H_2} \varphi(u) \geq -\varepsilon$ provided ℓ is large enough.

Now we check that φ satisfies condition (B) (ii) in $(-\varepsilon, 0)$ with respect to $B := \{u \in H_T^1 \mid \|u\| \leq r\}$ with $r > 0$ being specified below.

Firstly, we assume $0 \in \sigma(B)$. For $\varepsilon > 0$ given above, by virtue of Lemma 3.8, we can choose a $\delta(\varepsilon) > 0$ such that, for any $J > 0$,

$$\limsup_{\|u\| \to \infty} \int_0^T \frac{H(t,u)}{\|u\|^{2\alpha}} dt < 0. \tag{4.9}$$

for $u \in H_T^1$ with $\|u^+ + u^-\| \le \delta(\varepsilon)\|u^0\| + J$.

For $\eta = \varepsilon > 0$ and $\delta = \delta(\varepsilon)$, let $J_\varepsilon := J_{\eta,\delta} > 0$ be given as in Lemma 3.2. Then we can take a positive number $\varepsilon_1 > 0$ such that

$$\limsup_{u \in W_\varepsilon, \|u\| \to \infty} \int_0^T \frac{H(t,u)}{\|u\|^{2\alpha}} dt \le -2\varepsilon_1 < 0,$$

where $W_\varepsilon := \{u \in H_T^1 \mid \|u^+ + u^-\| \le \delta(\varepsilon)\|u^0\| + J_\varepsilon\}$. s

Since $\alpha < 1$, $W_\varepsilon^\alpha := \{u \in H_T^1 \mid \|u^+ + u^-\| \le \delta(\varepsilon)\|u^0\|^\alpha + J_\varepsilon\} \subset W_\varepsilon$ for $\|u\|$ large enough. Therefore, for $u \in W_\varepsilon^\alpha$ with $\|u\|$ sufficiently large, we have

$$\psi(u) = \int_0^T H(t,u)\,dt \le -\varepsilon_1\|u\|^{2\alpha} < 0.$$

Therefore, for any $u \in \varphi^{-1}([-\varepsilon, 0]) \cap W_\varepsilon^\alpha$ with $\|u\|$ sufficiently large, we have

$$\langle B(u^+ + u^-), u^+ + u^- \rangle = \langle Bu, u \rangle = 2\varphi(u) + 2\psi(u) \le -\varepsilon_1\|u\|^{2\alpha}. \tag{4.10}$$

On the other hand, for $\|u\|$ large enough, we have

$$\begin{aligned} |\langle B(u^+ + u^-), u^+ + u^- \rangle| &\le \|B(u^+ + u^-)\| \cdot \|u^+ + u^-\| \\ &\le (\delta(\varepsilon)\|u^0\|^\alpha + J_\varepsilon)\|B(u^+ + u^-)\| \\ &\le 2\delta(\varepsilon)\|u\|^\alpha\|B(u^+ + u^-)\|. \end{aligned} \tag{4.11}$$

It follows from (4.10) and (4.11) that

$$\|B(u^+ + u^-)\| \ge \frac{\varepsilon_1}{2\delta(\varepsilon)}\|u\|^\alpha, \tag{4.12}$$

for $u \in \varphi^{-1}([-\varepsilon, 0]) \cap W_\varepsilon^\alpha$ with $\|u\|$ sufficiently large.

For any $\epsilon \in (-\varepsilon, 0)$, let $\sigma = 1/4 \min\{\epsilon + \varepsilon, -\epsilon\}$. Then $[\epsilon - \sigma, \epsilon + \sigma] \subset (-\varepsilon, 0)$. If $u \in \varphi^{-1}([\epsilon - \sigma, \epsilon + \sigma])$, by Lemma 3.2, we have

$$\begin{gathered} \langle \varphi'(u), u \rangle + \frac{1}{2}\|\varphi'(u)\| \cdot \|u\| \ge 0, \\ \text{for } u \in H_T^1 \text{ with } \|u^+ + u^-\| \ge \delta(\varepsilon)\|u^0\|^\alpha + J_\varepsilon. \end{gathered} \tag{4.13}$$

Denote $W_\epsilon := W_\varepsilon^\alpha \cap \varphi^{-1}([\epsilon-\sigma, \epsilon+\sigma])$. It then follows from (3.7), (3.8) and (4.12) that

$$\begin{aligned}
&\langle \varphi'(u), u\rangle + \frac{1}{2}\|\varphi'(u)\| \cdot \|u\| \\
&\quad = 2\left[\varphi(u) + \psi(u) - \frac{1}{2}\langle \psi'(u), u\rangle\right] \\
&\qquad + \frac{1}{2}\|B(u^+ + u^-) - \psi'(u)\| \cdot \|u\| \\
&\quad \geq 2\varphi(u) + 2\psi(u) + \frac{1}{2}\|B(u^+ + u^-)\| \cdot \|u\| - \frac{3}{2}\|\psi'(u)\| \cdot \|u\| \\
&\quad \geq -2\varepsilon - 2\varsigma T^{(1-\alpha)/2}\|u\|^{1+\alpha} - 2MT^{1/2}\|u\| - 2\int_0^T |H(t,0)|\, dt \\
&\qquad + \frac{\varepsilon_1}{4\delta(\varepsilon)}\|u\|^{1+\alpha} \\
&\qquad - \frac{3}{2}\varsigma T^{(1-\alpha)/2}\|u\|^{1+\alpha} - \frac{3}{2}MT^{1/2}\|u\| \\
&\quad > 0,
\end{aligned}$$

provided $\varsigma > 0$ is sufficiently small and $\|u\|$ is sufficiently large.

Consequently, $\langle \varphi'(u), u\rangle + 1/2\|\varphi'(u)\| \cdot \|u\| \geq 0$ for any $u \in \varphi^{-1}([\epsilon - \sigma, \epsilon + \sigma])$ with $\|u\| = r$ sufficiently large. That is, φ satisfies condition (B) (ii) in $(-\varepsilon, 0)$ with respect to $B := \{u \in H_T^1 \mid \|u\| \leq r\}$. The remainder of the proof is similar to that of Theorem 1 and is omitted. The proof is complete. □

Acknowledgments. I would like to thank the referee for making me aware of the work [**24**].

REFERENCES

1. A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

2. P. Bartolo, V. Benci and D. Fortunato, *Abstract critical point theorems and applications to some problems with strong resonance at infinity*, Nonlin. Anal. **7** (1983), 241–273.

3. G. Fei, *Multiple solutions of some nonlinear strongly resonant elliptic equations without the* (PS) *condition*, J. Math. Anal. Appl. **193** (1995), 659–670.

4. ———, *On periodic solutions of superquadratic Hamiltonian systems*, Electron. J. Diff. Equat. **08** (2002), 1–12.

5. G. Fei, S. Kim and T. Wang, *Periodic solutions of classical Hamiltonian systems without Palais-Smale condition*, J. Math. Anal. Appl. **267** (2002), 665–678.

6. H. He and X. Wu, *Periodic solutions for a class of nonautonomous second order Hamiltonian systems*, J. Math. Anal. Appl. **341** (2008), 1354–1364.

7. Q. Jiang and C.-L. Tang, *Periodic and subharmonic solutions of a class of subquadratic second-order Hamiltonian systems*, J. Math. Anal. Appl. **328** (2007), 380–389.

8. M.A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, MacMillan Co., New York, 1964.

9. M. Lazzo, *Nonlinear differential problems and Morse theory*, Nonlin. Anal. **30** (1997), 169–176.

10. S. Li and J.Q. Liu, *Computations of critical groups at degenerate critical point and applications to nonlinear differential equations with resonance*, Houston J. Math. **25** (1999), 563–582.

11. S. Ma, *Infinitely many solutions for cooperative elliptic systems with odd nonlinearities*, Nonlin. Anal. **71** (2009), 1445–1461.

12. ———, *Computations of critical groups and periodic solutions for asymptotically linear Hamiltonian systems*, J. Differ. Equat. **248** (2010), 2435–2457.

13. J. Mawhin and M. Willem, *Critical point theory and Hamiltonian system*, Appl. Math. Sci. **74**, Springer-Verlag, New York, 1989.

14. N. Papageorgiou and V. Staicu, *Multiple solutions strongly resonance periodic systems*, Nonlinear Anal. **67** (2007), 1895–1907.

15. P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, in CBMS Reg. Conf. Ser. Math. **65**, American Mathematical Society, Providence, RI, 1986.

16. ———, *Periodic solutions of Hamiltonian systems*, Comm. Pure Appl. Math. **31** (1978), 157–184.

17. ———, *On subharmonic solutions of Hamiltonian systems*, Comm. Pure Appl. Math. **33** (1980), 609–633.

18. M. Struwe, *Variational methods*: *Applications to nonlinear partial differential equations and Hamiltonian systems*, Springer-Verlag, Berlin, 1990.

19. C.-L. Tang and X.-P. Wu, *Periodic solutions for a class of nonautonomous subquadratic second order Hamiltonian systems*, J. Math. Anal. Appl. **275** (2002), 870–882.

20. ———, *Notes on periodic solutions of subquadratic second order systems*, J. Math. Anal. Appl. **285** (2003), 8–16.

21. ———, *Periodic solutions for second order systems with not uniformly coercive potential*, J. Math. Anal. Appl. **259** (2001), 386–397.

22. Z.-L. Tao and C.-L. Tang, *Periodic solutions of second-order Hamiltonian systems*, J. Math. Anal. Appl. **293** (2004), 435–445.

23. Z.-Q. Wang, *Multiple solutions for indefinite functional with applications to asymptotically linear problems*, Acta Math. Sin. New Ser. **5** (1989), 101–113.

24. ———, *Nonlinear boundary value problems with concave nonlinearities near the origin*, NoDEA: Nonlin. Diff. Equat. Appl. **8** (2001), 15–33.

25. W. Zou, *Multiple solutions for second-order Hamiltonian systems via computation of the critical groups*, Nonlin. Anal. **44** (2001), 975–989.

26. ———, *Solutions of resonant elliptic systems with nonodd or odd nonlinearities*, J. Math. Anal. Appl. **223** (1998), 397–417.

27. W. Zou and S. Li, *Infinitely many solutions for Hamiltonian systems*, J. Differ. Equat. **186** (2002), 141–164.

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