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FORMAL GROUPS OF Q-CURVES WITH COMPLEX MULTIPLICATION

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ABSTRACT. In this paper, we discuss formal groups of **Q**-curves with complex multiplication by an imaginary quadratic field K. We define the Hecke character of the idele group I_K attached to a **Q**-curve, and we also define the L-function by using its twisted L-functions. We generalize the result of Honda [**2**] on formal groups of elliptic curves over **Q** to **Q**-curves. The difference from the previous generalizations [**1, 6, 7**] is that **Q**-curves are not always defined over abelian extensions of **Q**.

1. Introduction. Let E be an elliptic curve over $\mathbf{Q}$. We associate with E two formal groups. The first formal group is the formal completion $\widehat{E}(x,y)$ of the Néron model over $\mathbf{Z}$ of E along its zero section. It is defined over $\mathbf{Z}$. The second formal group is the formal group $\widehat{L}(x,y)$ of the L-function $L(E/\mathbf{Q},s)$ attached to l-adic representations on E.

Theorem 1.1 [**2**]. *$\widehat{L}(x,y)$ is defined over $\mathbf{Z}$, and it is strongly isomorphic over $\mathbf{Z}$ to $\widehat{E}(x,y)$.*

Deninger-Nart [**1**] generalized Theorem 1.1 to higher dimensional abelian varieties A of GL_2-type with real multiplication by using certain matrix L-functions attached to λ-adic representations on A. Furthermore, the author [**6**] generalized these results to certain building blocks B over finite abelian extensions of $\mathbf{Q}$, by using some twisted matrix L-functions attached to λ-adic representations on the scalar restriction of B.

In this paper, we discuss formal groups of $\mathbf{Q}$-curves with complex multiplication by an imaginary quadratic field K. Such $\mathbf{Q}$-curves are classified by Nakamura [**5**]. We define the Hecke character of the idele

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group I_K attached to a **Q**-curve, and we generalize Theorem 1.1 to **Q**-curves by using its twisted L-functions (see Theorem 3.8). The difference from the previous generalizations [**1, 6, 7**] is that such **Q**-curves are not always defined over abelian extensions of **Q**.

In Section 2, we review a basic fact of **Q**-curves with complex multiplication. In Section 3, we investigate the formal group structure.

2. Q-curves with complex multiplication.

2.1. Two-cocycles. Let K be an imaginary quadratic field with the discriminant $D_K \neq -3, -4$. The roots of unity in its maximal order $\mathcal{O}_K$ are ± 1. Let H be the Hilbert class field of K. Namely, H is the maximal unramified abelian extension of K. By class field theory, the ideal class group Cl_K of K is isomorphic to the Galois group $G_{H/K}$ of H over K.

Let E be an elliptic curve over H with complex multiplication by K.

Definition 2.1. We call E a **Q**-curve if there exists a non-zero isogeny φ_σ over H from ${}^\sigma E$ to E for each σ in $G_{H/\mathbf{Q}}$.

Theorem 2.2 [**5**]. *There exists a* **Q***-curve over* H *if and only if* D_K *is not of the form*

$$D_K = -4p_1 \cdots p_{t-1} \quad (t \geq 2),$$

where $p_1, \ldots, p_{t-1}$ *are primes satisfying* $p_1 \equiv \cdots \equiv p_{t-1} \equiv 1 \bmod 4$.

Let E be a **Q**-curve over H with complex multiplication by K. For each σ in $G_{H/\mathbf{Q}}$ we fix a non-zero isogeny φ_σ from ${}^\sigma E$ to E. We define a mapping from $G_{H/\mathbf{Q}} \times G_{H/\mathbf{Q}}$ to K^* by

$$c(\sigma, \tau) := \varphi_\sigma {}^\sigma\varphi_\tau \varphi_{\sigma\tau}^{-1}, \tag{2.1}$$

which is a two-cocycle of $G_{H/\mathbf{Q}}$. The cohomology class $[c_E]$ of c in $\mathrm{H}^2(H/\mathbf{Q}, K^*)$ is independent of the choice of φ_σ. We note that $[c_E] = [c_{E'}]$ if E and E' are isogenous over H.

We fix an invariant differential ω_E of E. For each σ we define α_σ in H^* by

$$\varphi_\sigma^* \omega_E = \alpha_\sigma {}^\sigma(\omega_E).$$

Then we have

$$c(\sigma,\tau) = \alpha_\sigma {}^\sigma\alpha_\tau \alpha_{\sigma\tau}^{-1} \quad ({}^\forall\sigma,\tau \in G_{H/\mathbf{Q}}).$$

Definition 2.3. We say that the two-cocycle c is symmetric on $G_{H/K}$, if it satisfies

$$c(\sigma,\tau) = c(\tau,\sigma) \quad ({}^\forall\sigma,\tau \in G_{H/K}).$$

Proposition 2.4 (cf., e.g., [**3**, Theorem 3.2]). *If the two-cocycle c is symmetric on $G_{H/K}$, then there exists a one-cocycle*

$$\beta: \ G_{H/K} \longrightarrow \overline{\mathbf{Q}}^*$$

such that

$$c(\sigma,\tau) = \beta(\sigma)\beta(\tau)\beta(\sigma\tau)^{-1} \quad ({}^\forall\sigma,\tau \in G_{H/K}). \tag{2.2}$$

By using the equivalence between (1) and (3) of Proposition 2.6, as below, we introduce Nakamura [**5**, Theorem 2] of the following form.

Proposition 2.5. *Among the cohomology classes $[c_E]$ in $H^2(H/\mathbf{Q}, K^*)$ associated to some $\mathbf{Q}$-curve E, there are 2^{t-1} classes represented by a symmetric class on $G_{H/K}$.*

Let ψ be the Hecke character from I_H to $\mathbf{C}^*$ associated to the elliptic curve E with complex multiplication (cf., e.g., [**8**, Section 5]).

Proposition 2.6. *The following conditions are equivalent.*

(1) *The two-cocycle c is symmetric on $G_{H/K}$.*

(2) $H(E_{tors}) \subset K^{ab}$.

(3) *There exists a Hecke character* ϕ *from* I_K *to* $\mathbf{C}^*$ *such that* $\psi = \phi \circ N_{H/K}$.

Proof. The equivalence between (2) and (3) is due to [**8**, Theorem 7.44]. We show the equivalence between (1) and (2).

We first show

$$\varphi_\tau {}^\tau\varphi_\sigma({}^{\tau\sigma}x) = \varphi_\sigma {}^\sigma\varphi_\tau({}^{\sigma\tau}x) \tag{2.3}$$

for all x in E_{tors} and for all σ, τ in G_K. Since E_{tors} is the compositum of all the group $E[\mathfrak{a}]$ of ideal section points, we may assume x is a basis of $E[\mathfrak{a}]$. Since $E[\mathfrak{a}]$ is a free $\mathcal{O}_K/\mathfrak{a}$-module of rank one, there exists an element a_σ in $\mathcal{O}_K/\mathfrak{a}$ such that $\varphi_\sigma({}^\sigma x) = a_\sigma x$. Thus we have $\varphi_\tau {}^\tau\varphi_\sigma({}^{\tau\sigma}x) = \varphi_\tau(a_\sigma {}^\tau x) = a_\sigma a_\tau x$. Similarly we have $\varphi_\sigma {}^\sigma\varphi_\tau({}^{\sigma\tau}x) = a_\tau a_\sigma x$. Hence, we have the assertion.

Secondly we show that (1) implies (2). Since c is symmetric and $\sigma\tau = \tau\sigma$ on H, we have $\varphi_\tau {}^\tau\varphi_\sigma = \varphi_\sigma {}^\sigma\varphi_\tau$ for all σ, τ in G_K. We take y in E_{tors} such that $ny = x$, where n is the order of $\ker \varphi_\tau {}^\tau\varphi_\sigma$. It follows from (2.3) for y that ${}^{\sigma\tau}y - {}^{\tau\sigma}y$ is in $\ker \varphi_\tau {}^\tau\varphi_\sigma$. By multiplying by n, we have ${}^{\sigma\tau}x - {}^{\tau\sigma}x = 0$. Thus, $K(x)$ is contained in K^{ab}.

Finally, we show that (2) implies (1). It follows from (2) that ${}^{\sigma\tau}x = {}^{\tau\sigma}x$ for all x in E_{tors} and for all σ, τ in G_K. Thus, it follows from (2.3) that $\varphi_\tau {}^\tau\varphi_\sigma = \varphi_\sigma {}^\sigma\varphi_\tau$. Since $\sigma\tau = \tau\sigma$ on H, the two-cocycle c is symmetric on H. □

Proposition 2.7. *Assume that the cocycle* c *is symmetric on* $G_{H/K}$. *Then the equation* $|\beta(\tau)| = \sqrt{d_\tau}$ *holds, where* d_τ *is the degree of* φ_τ.

Proof. On the one hand, by taking the degree on (2.1), we have $\mathrm{N}_{K/\mathbf{Q}}\ c(\sigma,\tau) = d_\sigma d_\tau d_{\sigma\tau}^{-1}$. On the other hand, by taking the absolute value on (2.2), we have $\mathrm{N}_{K/\mathbf{Q}}\ c(\sigma,\tau) = |\beta(\sigma)|^2|\beta(\tau)|^2|\beta(\sigma\tau)|^{-2}$. Since the mapping $\tau \mapsto d_\tau/|\beta(\tau)|^2$ is a homomorphism from the finite group $G_{H/K}$ to the group of positive real numbers, it must be the trivial homomorphism. Thus, we have $d_\tau = |\beta(\tau)|^2$. □

2.2. Hecke characters. Let E be a $\mathbf{Q}$-curve over H with complex multiplication by $\mathcal{O}_K$. As below, in this paper we always assume that the two cocycle c is symmetric on $G_{H/K}$. Furthermore, we assume that $\varphi_1 = 1$. Then, we have $\alpha_1 = 1$ and $\beta(1) = 1$.

By Proposition 2.6, the Hecke character ψ associated to E factors as $\psi = \phi \circ N_{H/K}$ for some Hecke character ϕ from I_K to $\mathbf{C}^*$. We construct ϕ explicitly.

We always fix a normalized isomorphism $[\cdot]_E$ from $\mathcal{O}_K$ to $\mathrm{End}\,(E)$, that is,

$$[\mu]_E^* \omega_E = \mu \omega_E \quad ({}^\forall \mu \in \mathcal{O}_K).$$

Then there exists an analytic isomorphism

$$\xi : \ \mathbf{C}/\mathfrak{a} \longrightarrow E$$

for an ideal $\mathfrak{a}$ in $\mathcal{O}_K$ by [**8**, Proposition 4.8].

Proposition 2.8 [**8**, Theorem 5.4]. *Let σ be an automorphism* $\mathbf{C}$ *over K, let s be an element of the idele group I_K of K such that the Artin symbol $[s, K^{ab}]$ for s in $G_{K^{ab}/K}$ is equal to σ on K^{ab}. Then there exists a unique isomorphism*

$$\xi' : \ \mathbf{C}/s^{-1}\mathfrak{a} \longrightarrow {}^\sigma E$$

satisfying the commutative diagram

$$\begin{array}{ccc} K/\mathfrak{a} & \xrightarrow{\ \xi\ } & E \\ {\scriptstyle s^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\ K/s^{-1}\mathfrak{a} & \xrightarrow{\ \xi'\ } & {}^\sigma E. \end{array} \tag{2.4}$$

We fix a given s in I_K. We put $\sigma := [s, K^{ab}]$. Then there exists a unique element γ in K satisfying the commutative diagram

$$\begin{array}{ccccc} & & K/\mathfrak{a} & \xrightarrow{\ \xi\ } & E \\ & & {\scriptstyle s^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\ K & \longrightarrow & K/s^{-1}\mathfrak{a} & \xrightarrow{\ \xi'\ } & {}^\sigma E \\ {\scriptstyle \gamma}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \varphi_\sigma} \\ K & \longrightarrow & K/\mathfrak{a} & \xrightarrow{\ \xi\ } & E. \end{array} \tag{2.5}$$

As below, we use the notation σ_s, γ_s instead of σ, γ in the case where we need to clarify s.

Proposition 2.9. *The equation $\gamma_s\gamma_t = c(\sigma_s, \sigma_t)\gamma_{st}$ holds for each s, t in I_K.*

Proof. By composing the diagram (2.5) for s and the one for t, we have

$$\xi \circ \gamma_s s^{-1}\gamma_t t^{-1} = \varphi_{\sigma_s}\sigma_s\varphi_{\sigma_t}\sigma_t \circ \xi.$$

Since $\sigma_s\varphi_{\sigma_t} = {}^{\sigma_s}\varphi_{\sigma_t}\sigma_s$ on ${}^{\sigma_t}E_{tors}$,

$$\xi \circ \gamma_s\gamma_t s^{-1}t^{-1} = c(\sigma_s, \sigma_t)\varphi_{\sigma_{st}}\sigma_{st} \circ \xi.$$

On the other hand, by using diagram (2.5) for st, we have

$$\xi \circ \gamma_{st}(st)^{-1} = \varphi_{\sigma_{st}}\sigma_{st} \circ \xi.$$

Thus, the uniqueness of γ in diagram (2.5) implies the assertion

$$\gamma_s\gamma_t = c(\sigma_s, \sigma_t)\gamma_{st}. \qquad \square$$

Since the two-cocycle c is symmetric, c splits by a one-cocycle β. By (2.2) and Proposition 2.9, we can define the homomorphism ϕ from I_K to $\mathbf{C}^*$ by

$$\phi(s) := \frac{\gamma_s s_\infty^{-1}}{\beta(\sigma_s)},$$

which is independent of choices of φ_τ. We note that $\beta(\sigma_s)\phi(s)$ is independent of choices of $\beta(\tau)$.

Proposition 2.10. *The function ϕ is a Hecke character of I_K. Furthermore, the equation $\psi = \phi \circ N_{H/K}$ holds.*

Proof. If s is in K^*, then $\sigma = 1$ and $\beta(\sigma) = 1$. We can take $\gamma = s$, and thus $\phi(K^*) = 1$.

If $s_{\mathfrak{p}} = 1$ for all non-archimedian primes $\mathfrak{p}$, then $\sigma = 1$ and $\beta(\sigma) = 1$. We can take $\gamma = 1$, so that $\phi(s) = s_\infty^{-1}$. Thus, ϕ is continuous on the archimedian part of I_K. On the other hand, suppose that $s_{\mathfrak{p}}$ is very close to 1 at all $\mathfrak{p}$ dividing $\mathfrak{a}$. Then $s_{\mathfrak{p}}^{-1}$ is also close to 1, and

$$s^{-1}\mathfrak{a} = \mathfrak{a}.$$

Since we must have

$$\gamma s^{-1}\mathfrak{a} = \mathfrak{a},$$

it follows that γ is a root of unity in K. If, in addition, we select s such that σ is the identity on $E[N]$ for large N, then multiplication by s on $(K/\mathfrak{a})_N$ is the identity. Consequently, $\sigma_{|H} = 1$ and $\beta(\sigma) = 1$, and γ must also be 1. This proves that the kernel of ϕ contains an open subgroup of the finite part of I_K. Thus, $\phi(s)$ is continuous.

If $s = N_{H/K}x$ for some x in I_H, then $\sigma_{|H} = 1$, $\varphi_\sigma = 1$, $\beta(\sigma) = 1$. Thus, we have

$$\phi \circ N_{H/K}(x) = \gamma s_\infty^{-1} = \psi(x)$$

by the definition of ψ (cf., eg., [**4**, page 138]). ◻

Definition 2.11. We call ϕ the Hecke character attached to the **Q**-curve E.

Let $\mathcal{O}_{\mathfrak{P}}$ be the completion of $\mathcal{O}_H$ at a prime $\mathfrak{P}$ in H. We say ψ is unramified at $\mathfrak{P}$ if $\psi(\mathcal{O}_{\mathfrak{P}}^*) = 1$. We note that ψ is unramified at $\mathfrak{P}$ if and only if E has good reduction at $\mathfrak{P}$ by [**8**, Theorem 7.42]. If ψ is unramified at $\mathfrak{P}$, we put

$$\psi(\mathfrak{P}) := \psi(\cdots, 1, \pi_{\mathfrak{P}}, 1, \cdots),$$

where $\pi_{\mathfrak{P}}$ is a prime element of $\mathcal{O}_{\mathfrak{P}}$. We define $\phi(\mathfrak{p})$ similarly.

Proposition 2.12. *Assume that $\mathfrak{P}$ is a good prime of E. Put $\mathfrak{a} := (p)$. The equation $\phi(\mathfrak{a}) = p$ holds.*

Proof. We fix a given odd integer $\nu \geq 3$, and we put $N := p^\nu - 1$. Then $N > 2$ and $p^\nu \equiv 1 \bmod N$. Let s be an element of I_K such that $s_{\mathfrak{p}} = p^\nu$ for all $\mathfrak{p}$ dividing p, and $s_{\mathfrak{p}} = 1$ otherwise (including $\mathfrak{p} = \infty$). Since $H(E[N])$ is the ray class field of K modulo N, σ is the identity on $H(E[N])$. Then we see that $\sigma = p^\nu$ on $E[N]$, $\varphi_\sigma = 1$ and $\beta(\sigma) = 1$. Since $N > 2$, we have $\gamma = p^\nu$ in diagram (2.5). Thus, we have $\phi(\mathfrak{a}^\nu) = p^\nu$ for any odd integer $\nu \geq 3$.

Finally, since ϕ is a homomorphism and ν is odd, we see that $\phi(\mathfrak{a}) = p$. □

We restrict s to an element of I_K such that $s_\mathfrak{p} = \pi_\mathfrak{p}$ and $s_\mathfrak{q} = 1$ for all primes $\mathfrak{q} \neq \mathfrak{p}$. Then we have $s^{-1}\mathfrak{a} = \mathfrak{p}^{-1}\mathfrak{a}$ in diagram (2.5). If $\mathrm{N}\mathfrak{p} = p$, we may assume that φ_σ is a p-isogeny with the kernel ${}^\sigma E[\overline{\mathfrak{p}}]$, and thus γ must be $\pm p$. Furthermore, if $\mathfrak{P}$ is a good prime of E, we have

$$\beta(\sigma)\phi(\mathfrak{p}) = \gamma = \pm p. \tag{2.6}$$

Proposition 2.13. *Assume that $\mathfrak{P}$ is a good prime of E and p splits completely in K. Then the equation $\phi(\overline{\mathfrak{p}}) = \overline{\phi(\mathfrak{p})}$ holds.*

Proof. We note that ϕ is independent of the choices of φ_τ. We may assume that φ_σ is a p-isogeny with the kernel ${}^\sigma E[\overline{\mathfrak{p}}]$. By Proposition 2.7, we have $|\beta(\sigma)| = \sqrt{p}$. By using (2.6), we have $|\phi(\mathfrak{p})| = \sqrt{p}$. Now the assertion follows from $\phi(\mathfrak{p})\phi(\overline{\mathfrak{p}}) = p$. □

2.3. L-functions. We define the L-function $L(\psi, s)$ by

$$L(\psi, s) := \prod_{\mathfrak{P}} (1 - \psi(\mathfrak{P})\mathrm{N}\mathfrak{P}^{-s})^{-1},$$

where the product is taken over all good primes $\mathfrak{P}$ in H. Let $L(E/H, s)$ be the L-function attached to l-adic representations on E over H. We have

$$L(E/H, s) = L(\psi, s)L(\overline{\psi}, s)$$

by [**8**, Theorem 7.42]. Furthermore, since the two-cocycle c on $G_{H/K}$ is symmetric, we have

$$L(\psi, s) = \prod_{\chi} L(\phi\chi, s),$$

where the product runs through all characters χ of the ideal class group Cl_K. Thus, we have

$$L(E/H, s) = \prod_{\chi} L(\phi\chi, s) L(\overline{\phi}\chi, s).$$

We define the L-function attached to (E, ω_E) as a **Q**-curve by

$$\xi(E, \omega_E, s) := \sum_{\tau \in G_{H/K}} \frac{\alpha_{\tau^{-1}}\beta(\tau)}{c(\tau, \tau^{-1})} \sum_{\mathfrak{a} \sim \tau} \frac{\phi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s}, \tag{2.7}$$

where $\mathfrak{a}$ is an integral ideal in K coprime to bad primes of E and $\mathfrak{a} \sim \tau$ means $[\mathfrak{a}, K^{ab}]_{|H} = \tau$. The L-function $\xi(E, \omega_E, s)$ is a linear combination of $L(\phi\chi, s)$. We note that $\xi(E, \omega_E, s)$ is independent of the choices of φ_τ and $\beta(\tau)$. Since $\beta(\tau)\phi(\mathfrak{a})$ is in K, the L-function $\xi(E, \omega_E, s)$ has coefficients in H.

3. The formal groups of Q-curves.

3.1. Notations and assumptions. Let $\mathcal{E}$ be the Néron model over $\mathcal{O}_H$ of E. We take a finite set S of primes of H satisfying the following two conditions.

(1) The ring $\mathcal{O}_{H,S}$ of S-integers is a principal integral domain.

(2) The set S is invariant by the action of $G_{H/\mathbf{Q}}$.

We fix a base ω_E of the $\mathcal{O}_{H,S}$-module of invariant differentials $\Omega^1_{\mathcal{E}_S}$ on $\mathcal{E}_S := \mathcal{E} \otimes \mathcal{O}_{H,S}$. The basis ω_E induces an isomorphism between the formal completion of $\mathcal{E}_S$ along its zero section and Spf $\mathcal{O}_{H,S}[[x]]$. This determines the formal group $\widehat{E}(x, y)$ of E over $\mathcal{O}_{H,S}$. We see that α_τ in $\mathcal{O}_{H,S}$ for each τ in $G_{H/K}$.

Proposition 3.1. *The* L*-function $\xi(E, \omega_E, s)$ is a Dirichlet series with coefficients in $\mathcal{O}_{H,S}$.*

Proof. Put $u := \alpha_{\tau^{-1}}\beta(\tau)/c(\tau, \tau^{-1})$. We note that u is independent of the choices of φ_τ. We show u is a unit of the ring of S-integers $\mathcal{O}_{H(\beta(\tau)),S}$ of $H(\beta(\tau))$. If $\tau = 1$, then $u = 1$, and thus u is a unit.

Suppose that $\tau \neq 1$. Then there exists a prime ideal $\mathfrak{q}$ of $\mathcal{O}_K$ such that $\mathfrak{q} \sim \tau$. We denote the order of $\mathfrak{q}$ in Cl_K by f. Since $\varphi_\tau {}^\tau\varphi_\tau \cdots {}^{\tau^{f-1}}\varphi_\tau$ is an endomorphism of E with the kernel $E[\overline{\mathfrak{q}}^f]$, we have

$$\overline{\mathfrak{q}}^f = (\alpha_\tau)^\tau(\alpha_\tau) \cdots {}^{\tau^{f-1}}(\alpha_\tau) = (\beta(\tau))^f.$$

It follows from $u = \beta(\tau)/{}^{\tau^{-1}}\alpha_\tau$ that $\overline{\mathfrak{q}}^f(u)$ and $\overline{\mathfrak{q}}^f(u^{-1})$ are integral ideals of $\mathcal{O}_{H(\beta(\tau)),S}$. Since there exists another prime ideal $\mathfrak{r}$ such that $\mathfrak{r} \sim \tau$, $\overline{\mathfrak{r}}^f(u)$ and $\overline{\mathfrak{r}}^f(u^{-1})$ are also integral ideals. Thus, u must be a unit of $\mathcal{O}_{H(\beta(\tau)),S}$.

Since $\beta(\tau)\phi(\mathfrak{a})$ is in $\mathcal{O}_K$, the L-function $\xi(E, \omega_E, s)$ has coefficients in $\mathcal{O}_{H,S}$. □

As below, we always assume that $\mathfrak{P}$ is not in S. We denote by σ the Frobenius automorphism for $\mathfrak{P}$. We put $\mathfrak{p} := \mathfrak{P} \cap K$.

Suppose that $\mathfrak{P}$ is a good prime of E and $\mathfrak{p}$ ramifies in K. Since H is unramified over K, the order of the inertia group of $\mathfrak{P}$ is two. We denote its generator by τ. Since the inertia field $H^{\langle\tau\rangle}$ of $\mathfrak{P}$ does not contain K and there exists a κ in $\mathrm{End}\,(E)$ such that $\kappa \neq {}^\tau\kappa$. Then we have $\kappa\varphi_\tau \neq \varphi_\tau {}^\tau\kappa$.

Since E is a $\mathbf{Q}$-curve, $\mathfrak{P}$ is also a good prime of ${}^\tau E$. Since the effect of τ on the residue class field is trivial, the reduction modulo $\mathfrak{P}$ of $\kappa\varphi_\tau$ is equal to the one of $\varphi_\tau {}^\tau\kappa$. This leads to a contradiction to the injectivity of the reduction map on $\mathrm{Hom}\,({}^\tau E, E)$. Hence, $\mathfrak{p}$ is unramified if $\mathfrak{P}$ is a good prime of E.

In the case where p splits completely in K, we can take a p-isogeny with the kernel ${}^\sigma E[\overline{\mathfrak{p}}]$ as φ_σ, since the formal group $\widehat{E}(x, y)$ and $\widehat{\xi}(x, y)$ is independent of choices of φ_τ.

Here we give some properties on the numbers $\beta(\sigma)\phi(\mathfrak{p})$ and α_σ in the case where $\mathfrak{P}$ is a good prime of E.

At first, we have

$$\beta(\sigma)\phi(\mathfrak{p}) = \gamma = \pm p.$$

Indeed, if p remains prime in K, it follows from Proposition 2.12. If p splits completely in K, it follows from (2.6).

Secondly, if p splits completely in K, we have

$$\alpha_\sigma \in \mathcal{O}_{\mathfrak{P}}^*.$$

Since p splits completely in K, $\mathfrak{P}$ is an ordinary good prime. Since $\mathfrak{P}$ is a good prime, we see that α_σ is in $\mathcal{O}_{\mathfrak{P}}$. Since it follows from (2.5) that $\varphi_\sigma \sigma = \pm p$ on the l-adic Tate module $V_l(E)$, the reduction of φ_σ modulo $\mathfrak{P}$ is separable, and thus α_σ is in $\mathcal{O}_{\mathfrak{P}}^*$.

3.2. The formal group structure of $\widehat{E}$.

Proposition 3.2. *If $\mathfrak{P}$ is a good prime of E, $\widehat{E}(x,y)$ over $\mathcal{O}_{\mathfrak{P}}$ is of type $\beta(\sigma)\phi(\mathfrak{p}) - \alpha_\sigma T^f$, where $\mathrm{N}\mathfrak{p} = p^f$.*

Proof. By the commutative diagram (2.5), we have

$$f^{-1}({\alpha_\sigma}^\sigma f(x^{\mathrm{N}\mathfrak{p}})) \equiv f^{-1}(\gamma f(x)) \mod \mathfrak{P}.$$

Since $\mathfrak{P}$ is unramified, it follows from Honda [**2**, Proposition 3.3 and Lemma 4.2] that

$$\begin{aligned} \gamma f(x) - {\alpha_\sigma}^\sigma f(x^{\mathrm{N}\mathfrak{p}}) &\equiv 0 \mod \mathfrak{P}, \\ \beta(\sigma)\phi(\mathfrak{p}) f(x) - {\alpha_\sigma}^\sigma f(x^{\mathrm{N}\mathfrak{p}}) &\equiv 0 \mod \mathfrak{P}. \end{aligned}$$

That is, $\widehat{E}(x,y)$ over $\mathcal{O}_{\mathfrak{P}}$ is of type $\beta(\sigma)\phi(\mathfrak{p}) - \alpha_\sigma T^f$. □

Proposition 3.3. *If $\mathfrak{P}$ is a bad prime of E, $\widehat{E}(x,y)$ over $\mathcal{O}_{\mathfrak{P}}$ is type $\pi_{\mathfrak{P}}$.*

Proof. Since E has complex multiplication, E has additive reduction at any bad prime. Thus, $\widehat{E}(x,y)$ over $\mathcal{O}_{\mathfrak{P}}$ is strongly isomorphic over $\mathcal{O}_{\mathfrak{P}}$ to $\widehat{\mathbf{G}}_a(x,y)$. □

3.3. The formal group structure of $\widehat{\xi}$.

Let $\widehat{\xi}(x,y)$ be the formal group of $\xi(E, \omega_E, s)$. For each τ in $G_{H/K}$, we define

$$h_\tau(x) := \beta(\tau) \sum_{\mathfrak{a} \sim \tau} \frac{\phi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}} x^{\mathrm{N}\mathfrak{a}}.$$

$h_\tau(x)$ is in $K[[x]]$ and

$$h(x) := \sum_{\tau \in G_{H/K}} \frac{\alpha_{\tau^{-1}}}{c(\tau, \tau^{-1})} h_\tau(x)$$

is the transformer of $\widehat{\xi}(x,y)$.

Proposition 3.4. *Assume that $\mathfrak{P}$ is a bad prime of E. Then $\widehat{\xi}(x,y)$ over $\mathcal{O}_{\mathfrak{P}}$ is of type $\pi_{\mathfrak{P}}$.*

Proof. Since $\mathfrak{P}$ is a bad prime, the $\mathfrak{P}$-factor of $L(\phi,s)$ is equal to one. The denominator $\mathrm{N}\mathfrak{a}$ of each coefficient of $h_\tau(x)$ is prime to $\mathfrak{P}$. Thus, we have $\pi_{\mathfrak{P}} h(x) \equiv 0 \bmod \mathfrak{P}$. □

Proposition 3.5. *Assume that $\mathfrak{P}$ is a good prime of E. If p remains prime in K, then*

$$\beta(\sigma)\phi(\mathfrak{p})h_\tau(x) \equiv h_\tau(x^{p^2}) \bmod \mathfrak{P}.$$

In particular, $\widehat{\xi}(x,y)$ over $\mathcal{O}_{\mathfrak{P}}$ is of type $\beta(\sigma)\phi(\mathfrak{p}) - T^2$.

Proof. Since $\mathfrak{p}$ is principal, $\beta(\sigma)=1$ and the left hand side is

$$\phi(\mathfrak{p})\beta(\tau)\sum_{\mathfrak{a}\sim\tau}\frac{\phi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}}x^{\mathrm{N}\mathfrak{a}} = \phi(\mathfrak{p})\beta(\tau)\sum_{\substack{\mathfrak{a}\sim\tau\\ \mathfrak{p}\nmid\mathfrak{a}}}\frac{\phi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}}x^{\mathrm{N}\mathfrak{a}} + \beta(\tau)\sum_{\mathfrak{a}\sim\tau}\frac{\phi(\mathfrak{a}\mathfrak{p}^2)}{\mathrm{N}\mathfrak{a}\mathfrak{p}}x^{\mathrm{N}\mathfrak{a}\mathfrak{p}}.$$

Since $\phi(\mathfrak{p}^2) = \mathrm{N}\mathfrak{p} = p^2$, the right hand side is equal to

$$h_\tau(x^{p^2}) = \beta(\tau)\sum_{\mathfrak{a}\sim\tau}\frac{\phi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}}x^{p^2\mathrm{N}\mathfrak{a}} \bmod \mathfrak{P}.$$

The last assertion follows from the first assertion. □

Proposition 3.6. *Assume that $\mathfrak{P}$ is a good prime of E. If $\mathfrak{p}$ splits completely in K, then*

$$\beta(\sigma)\phi(\mathfrak{p})h_\tau(x) \equiv c(\sigma,\tau)h_{\sigma\tau}(x^p) \bmod \mathfrak{P}.$$

Proof. The left hand side is

$$\beta(\sigma)\phi(\mathfrak{p})\beta(\tau)\sum_{\mathfrak{a}\sim\tau}\frac{\phi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}}x^{\mathrm{N}\mathfrak{a}}$$
$$= \phi(\mathfrak{p})\beta(\sigma)\beta(\tau)\sum_{\substack{\mathfrak{a}\sim\tau\\ \overline{\mathfrak{P}}\nmid\mathfrak{a}}}\frac{\phi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}}x^{\mathrm{N}\mathfrak{a}} + \beta(\sigma)\beta(\tau)\sum_{\mathfrak{a}\sim\sigma\tau}\frac{\phi(\mathfrak{a}\mathfrak{p}\overline{\mathfrak{P}})}{\mathrm{N}\mathfrak{a}\overline{\mathfrak{p}}}x^{\mathrm{N}\mathfrak{a}\overline{\mathfrak{P}}}.$$

Since $\overline{\mathfrak{p}}$ does not divide $\mathfrak{a}$ in the first term, the order at $\mathfrak{p}$ of $\phi(\mathfrak{a})$ is equal to the one of $\mathrm{N}\mathfrak{a}$. Thus, the first term is congruent to zero modulo $\mathfrak{P}$. It follows from Proposition 2.12 that the second term is congruent to

$$c(\sigma,\tau)h_{\sigma\tau}(x^p) = c(\sigma,\tau)\beta(\sigma\tau)\sum_{\mathfrak{a}\sim\sigma\tau}\frac{\phi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}}x^{p\mathrm{N}\mathfrak{a}}. \qquad \square$$

Proposition 3.7. *Assume that $\mathfrak{P}$ is a good prime of E. If $\mathfrak{p}$ splits completely in K, then $\widehat{\xi}(x,y)$ over $\mathcal{O}_{\mathfrak{P}}$ is of type $\phi(\mathfrak{p})\beta(\sigma)-\alpha_\sigma T$.*

Proof.

$$\begin{aligned}
&(\beta(\sigma)\phi(\mathfrak{p})-\alpha_\sigma T) * h(x)\\
&\equiv \sum_{\tau\in G_{H/K}} \frac{\alpha_{\tau^{-1}}c(\sigma,\tau)}{c(\tau,\tau^{-1})}h_{\sigma\tau}(x^p) - \frac{{\alpha_\sigma}^\sigma\alpha_{\tau^{-1}}}{c(\tau,\tau^{-1})}h_\tau(x^p) \bmod \mathfrak{P}\\
&= \sum_{\tau\in G_{H/K}} \frac{\alpha_{\tau^{-1}}c(\sigma,\tau)}{c(\tau,\tau^{-1})}h_{\sigma\tau}(x^p) - \frac{c(\sigma,\tau^{-1})\alpha_{\sigma\tau^{-1}}}{c(\tau,\tau^{-1})}h_\tau(x^p) \bmod \mathfrak{P}\\
&= \sum_{\tau\in G_{H/K}} \frac{\alpha_{\tau^{-1}}c(\sigma,\tau)}{c(\tau,\tau^{-1})}h_{\sigma\tau}(x^p) - \frac{c(\sigma,\sigma^{-1}\tau^{-1})\alpha_{\tau^{-1}}}{c(\sigma\tau,\sigma^{-1}\tau^{-1})}h_{\sigma\tau}(x^p) \bmod \mathfrak{P}\\
&= \sum_{\tau\in G_{H/K}} \left[\frac{c(\sigma,\tau)}{c(\tau,\tau^{-1})} - \frac{c(\sigma,\sigma^{-1}\tau^{-1})}{c(\sigma\tau,\sigma^{-1}\tau^{-1})}\right]\alpha_{\tau^{-1}}h_{\sigma\tau}(x^p) \bmod \mathfrak{P}.
\end{aligned}$$

It follows from the two-cocycle condition (cf., e.g., [**3**, page 6]) that

$$c(\sigma,\tau)c(\sigma\tau,\sigma^{-1}\tau^{-1}) - c(\sigma,\sigma^{-1}\tau^{-1})c(\tau,\tau^{-1}) = 0.$$

Thus, we have the assertion. $\square$

3.4. Main theorem. From Propositions 3.2–3.7, we have the following.

Theorem 3.8. *$\widehat{\xi}(x,y)$ is defined over $\mathcal{O}_{H,S}$, and it is strongly isomorphic over $\mathcal{O}_{H,S}$ to $\widehat{E}(x,y)$.*

We define b_n in $\mathcal{O}_{H,S}$ by

$$\omega_E = \sum_{n\geq 1} b_n x^{n-1} dx,$$

where x is the local parameter at the zero. We note that the right hand side is the invariant differential of $\widehat{E}(x, y)$.

Proposition 3.9. *Assume $\mathfrak{P}$ is not in S. If $\mathfrak{p}$ splits completely in K, the $\mathfrak{P}$-adic limit*

$$\lim_{\nu\to\infty} \frac{b_{rp^{\nu+1}}}{{}^{\sigma}b_{rp^{\nu}}} = \frac{\phi(\overline{\mathfrak{p}})\alpha_\sigma}{\beta(\sigma)}$$

for any natural number r coprime to p.

Proof. Since $\widehat{\xi}(x, y)$ over $\mathcal{O}_{\mathfrak{P}}$ is of type $\beta(\sigma)\phi(\mathfrak{p}) - \alpha_\sigma T$, we have

$$\beta(\sigma)\phi(\mathfrak{p})p^{-1}b_{np} - \alpha_\sigma {}^{\sigma}b_n \equiv 0 \mod \mathfrak{P}^{\nu+1},$$

where $n = rp^\nu$. The assertion follows from Proposition 2.12. □

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