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ON CONTINUITY AND COMPACTNESS OF SOME VECTOR-VALUED INTEGRALS

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ABSTRACT. Necessary and sufficient conditions in order that the indefinite integral of any Henstock-Kurzweil-Pettis (respectively, Denjoy-Pettis) integrable function with values in a fixed Banach space have a relatively compact range or be continuous, except at most on a countable set, are obtained.

1. Introduction. The *Henstock-Kurzweil-Pettis integral* and the *Denjoy-Pettis integral* [**1**] (HKP and DP integrals, in short), which are Denjoy type extensions of the Pettis integral, have received considerable study in a series of recent papers [**2–8, 12**]. In Example 2 of [**3**], the authors provide an indefinite HKP integral with values in c_0 that has a relatively *noncompact* range. The theorems in this note were suggested by this example and give answers to the following two questions.

The first question [**3**, Question 3] is: Can the indefinite integral of any HKP (respectively, DP) integrable function with values in a fixed Banach space X have a relatively compact range? For the obvious reasons, such a Banach space X can contain no isomorphic copy of c_0. It is proved that any indefinite X-valued HKP (respectively, DP) integral has a relatively compact range if and only if any indefinite X-valued HKP (respectively, DP) integral is continuous or, equivalently, X is a Schur space.

The second question is: Does there exist an indefinite HKP (respectively, DP) integral that is discontinuous at each point of an *uncountable* set? We give the following answer to this question: The indefinite integral of any HKP integrable function with values in a fixed Banach space X is continuous except at most on a countable set if and only if X contains no isomorphic copy of c_0. On the other hand, any indefinite

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X-valued DP integral is continuous except at most on a countable set if and only if X is a Schur space. In passing, we give answers to the above two questions for the *McShane-Pettis integral* (MP integral, in short).

For the most part our notation and terminology are standard, or can be found in [**9**].

Throughout this paper, $[0,1]$ will denote the unit interval of the real line and I its closed nondegenerate subinterval. $\ell(I)$ is the length of an interval I. X denotes a real Banach space and X^* its dual. Given $F : [0,1] \to X$, $D(F)$ and $\Delta F(I)$ denote the set of discontinuities of F on $[0,1]$ and the *increment* of F on an interval I. As usual, ∂E is the *boundary* of a set E. Finally, C represents the Cantor ternary set and $\{(a_k^{(j)}, b_k^{(j)})\}$, $k \in \mathbf{N}$, $j = 1, \ldots, 2^{k-1}$ is the natural enumeration of the intervals in $[0,1]$ contiguous to C.

In what follows, we will need some notions related to the integration and differentiation of vector-valued functions. They are summarized below for the reader's convenience.

We first define scalar derivatives and approximate scalar derivatives [**13**].

Definition 1. Let $F : [0,1] \to X$ and $E \subset [0,1]$. A function $f : E \to X$ is a *scalar derivative* (an *approximate scalar derivative*) of F on E if, for each x^* in X^*, the function x^*F is differentiable (approximately differentiable) almost everywhere on E and $(x^*F)' = x^*f$ ($(x^*F)'_{\mathrm{ap}} = x^*f$) almost everywhere on E (the exceptional set may vary with x^*).

Next, we define two classes, VB_* and VBG_*, of vector-valued functions of bounded variation and four classes, AC, AC_*, ACG, and ACG_*, of absolutely continuous vector-valued functions [**12**]. Let $F : [0,1] \to X$, and let E be a non-empty subset of $[0,1]$.

Definition 2. F is said to be VB_* on E if there exists a positive number M such that

$$\left\| \sum_{k=1}^{K} \Delta F(I_k) \right\| \leq M$$

for each finite collection of mutually non-overlapping intervals $\{I_k : k = 1, \dots, K\}$ with $\partial I_k \cap E \neq \varnothing$.

Definition 3. F is said to be AC (AC_*) on E if, for each positive number ε, there exists a positive number η such that

$$\left\| \sum_{k=1}^{K} \Delta F(I_k) \right\| < \varepsilon$$

for each finite collection of mutually non-overlapping intervals $\{I_k : k = 1, \dots, K\}$ with $\partial I_k \subset E$ ($\partial I_k \cap E \neq \varnothing$) and

$$\sum_{k=1}^{K} \ell(I_k) < \eta.$$

We say that F is VBG_* (ACG, ACG_*) on E if E can be written as a countable union of sets on each of which F is VB_* (AC, AC_*).

Remark 1. The reader should refer to [**11**] for other equivalent characterizations of the VB_* property of a vector-valued function. Note that the function F in Definition 2 is necessarily bounded on $[0, 1]$. On the other hand, Lemma 5.3.8 of [**15**] states that, for real-valued functions our Definition 2 is equivalent to the classical definition of a VB_* function on a set (see [**9**, Definition 6.1]) under the additional hypothesis that function F be bounded on $[0, 1]$. As a result, for bounded real-valued functions our definition of the VBG_* property is equivalent to the classical definition of a VBG_* function on a set.

Remark 2. It is easily seen that, for real-valued functions, our definition of an ACG function on E is equivalent to the classical definition of an ACG function on E (see [**9**, Definition 6.1]) under the additional hypothesis that the function $F|_E$ be continuous.

Remark 3. Note that an ACG_* function on E is necessarily continuous on E. Moreover, in the case where X equals $\mathbf{R}$ and E is *closed*, combining Lemmas 5.3.2 and 5.3.3 of [**15**] shows that our definition of an ACG_* function on E is equivalent to the classical definition of an ACG_* function on E (see [**9**, Definition 6.1]) under the additional hypothesis that the function F be continuous on $[0,1]$.

It will be convenient to use the descriptive definition of the HKP and the DP integrals [**12**].

Definition 4. Let $f : [0,1] \to X$. The function f is HKP *integrable* (DP *integrable*) on $[0,1]$ if there exists a function $F : [0,1] \to X$ such that $F(0) = 0$, for each x^* in X^*, the function x^*F is ACG_* (ACG and continuous) on $[0,1]$, and f is a scalar derivative (an approximate scalar derivative) of F on $[0,1]$.

If, in the definition of the HKP integral, the ACG_* property is replaced with the AC property, we obtain the MP *integral* (cf. [**10**, Definition 22]). A straightforward argument can be given to show that the function F in Definition 4 is unique. Such a function will be referred to as the *indefinite* integral of function f. Given I, we write $\int_I f = \Delta F(I)$. At last,

$$\|f\|_A = \sup_{t\in(0,1]} \left\| \int_0^t f \right\|$$

is the *Alexiewicz norm* of function f.

We further define a *step function* f on I to be an X-valued function to which there corresponds a partition of I into mutually non-overlapping intervals $\{I_k : k = 1, \dots, K\}$ so that f is constant, at value x_k, say, on the interior of each I_k.

2. Indefinite HKP and DP integrals. We begin with a simple characterization of indefinite DP integrals that have a relatively compact range. The following approximation theorem is Theorem 4 of [**3**].

Theorem A. *Suppose that* $f : [0,1] \to X$ *is* DP *integrable on* $[0,1]$. *Then the set* $\{\int_I f : I \subset [0,1]\}$ *is relatively compact if and only if there* *exists a sequence of step functions* $\{f_n\}$ *on* $[0,1]$ *such that*

$$\lim_n \|f - f_n\|_A = 0. \tag{1}$$

The next theorem completes Theorem A.

Theorem 1. *Suppose that* $f : [0,1] \to X$ *is* DP *integrable on* $[0,1]$, *and let* F *be the indefinite integral of* f. *The following two statements are equivalent*:

(i) F *is continuous on* $[0,1]$.

(ii) *The set* $\{\int_I f : I \subset [0,1]\}$ *is relatively compact.*

Proof. (i) ⇒ (ii). Since F is continuous on $[0,1]$, the set $\Lambda = F([0,1])$ is compact, then so is the set $\Lambda - \Lambda \supset \{\int_I f : I \subset [0,1]\}$.

(ii) ⇒ (i). Let $\{f_n\}$ be a sequence of step functions on $[0,1]$ as in Theorem A, and let F_n be the indefinite integral of f_n. Now it follows from (1) that $\{F_n\}$ converges uniformly to F on $[0,1]$. Since each F_n is continuous on $[0,1]$, F is continuous on $[0,1]$. □

Recall that a real Banach space X is said to have the *Schur property* (or to be a *Schur space*, in short) if each weakly null sequence in X converges in norm.

Theorem 2. *The following three statements are equivalent*:

(i) X *is a Schur space.*

(ii) *If* $f : [0,1] \to X$ *is* HKP (*respectively,* DP) *integrable on* $[0,1]$, *then the indefinite integral of* f *is continuous on* $[0,1]$.

(iii) *If* $f : [0,1] \to X$ *is* HKP (*respectively,* DP) *integrable on* $[0,1]$, *then the set* $\{\int_I f : I \subset [0,1]\}$ *is relatively compact.*

Proof. (i) ⇒ (ii). Let F be the indefinite integral of f. Since function x^*F is continuous on $[0,1]$ for each x^* in X^* and X is a Schur space, F is continuous on $[0,1]$.

(ii) ⇒ (iii). This implication follows from Theorem 1.

(iii) $\Rightarrow$ (i). On the contrary, assume X fails the Schur property; then there is a sequence $\{x_n\}$ in X such that, for all x^* in X^*, $\lim_n x^*(x_n) = 0$ and $\|x_n\| \geq 1$ for all n. We make note of the fact that $\{x_n\}$ has no convergent subsequence. Let $\{[a_n, b_n]\}$ be a fixed sequence of nondegenerate intervals in $[0, 1]$ such that $a_1 > 0$, $b_n < a_{n+1}$ for each n and $\lim_n b_n = 1$. For a fixed positive integer n, we let F_n denote the real-valued function defined on $[0, 1]$ that equals 0 on the set $\{0, a_n, b_n, 1\}$, equals 1 at $(a_n + b_n)/2$, and is linear on the intervals between these points. Then, define $F(t)$ by $\sum_{n=1}^{\infty} F_n(t)x_n$ for all t in $[0, 1]$. It is obvious that x^*F is ACG_* on $[0, 1]$ for each x^* in X^*. Further, $F(0) = 0$ and F has a scalar derivative, f say, on $[0, 1]$. F therefore is the indefinite HKP integral of f. On the other hand, as the set $\{\int_I f : I \subset [0, 1]\}$ contains $\{x_n : n \in \mathbf{N}\}$, it is not relatively compact. This is the desired contradiction. □

Theorem 3. *The following four statements are equivalent*:

(i) *X contains no isomorphic copy of c_0.*

(ii) *If $f : [0, 1] \to X$ is* HKP *integrable on* $[0, 1]$ *and F its indefinite integral, then the set $D(F)$ is at most countable.*

(iii) *If $f : [0, 1] \to X$ is* MP *integrable on* $[0, 1]$ *and F its indefinite integral, then the set $D(F)$ is at most countable.*

(iv) *If $f : [0, 1] \to X$ is* MP *integrable on* $[0, 1]$, *then the indefinite integral of f is continuous on* $[0, 1]$.

Proof. (i) $\Rightarrow$ (ii). Since function x^*F is VBG_* on $[0, 1]$ for each x^* in X^* (see [**9**, Theorem 6.2 (b)]), it follows from Theorem 3.3 of [**12**] that F is VBG_* on $[0, 1]$. Consequently, by [**11**, Theorem 3.1], the set $D(F)$ is at most countable.

(ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (iii). These implications are obvious.

(iii) $\Rightarrow$ (i). Evidently, it suffices to construct an indefinite MP integral $F_0 : [0, 1] \to c_0$ such that $D(F_0)$ is uncountable. Let $\{e_n\}_{n=1}^{\infty}$ be the standard unit vector basis of c_0. For a fixed positive integer n, there is a unique pair (k, j) such that $k \in \mathbf{N}$, $j \in \{1, \ldots, 2^{k-1}\}$ and $n = 2^{k-1} + j - 1$. We let F_n denote the real-valued function defined on $[0, 1]$ that equals 0 on the set $\{0, a_k^{(j)}, b_k^{(j)}, 1\}$, equals 1 at $(a_k^{(j)} + b_k^{(j)})/2$ and is linear on the intervals between these points. Then, define $F_0(t)$

by $\sum_{n=1}^{\infty} F_n(t)e_n$ for all t in $[0,1]$. Note that F_0 has a scalar derivative, f_0 say, on $[0,1]$. Choose $x^* = (\alpha_1, \alpha_2, \dots) \in \ell^1 = c_0^*$ arbitrarily. It follows from the definition of function F_0 that

$$\int_0^1 |x^* f_0| = \sum_{n=1}^{\infty} 2|\alpha_n| < \infty.$$

Thus, function F_0 is both an indefinite MP integral and discontinuous on C.

(i) ⇒ (iv). This implication follows from [**12**, Corollary 4.1] (cf. [**10**, Theorem 26]). □

Theorem 4. *The following two statements are equivalent*:

(i) X *is a Schur space.*

(ii) *If* $f : [0,1] \to X$ *is* DP *integrable on* $[0,1]$ *and* F *its indefinite integral, then the set* $D(F)$ *is at most countable.*

Proof. (i) ⇒ (ii). This implication follows from Theorem 2.

(ii) ⇒ (i). The relevant example is in fact provided in the proof of Theorem 5.4 of [**12**]. However, we include this example for the sake of completeness.

On the contrary, assume X fails the Schur property. Then there is a sequence $\{x_n\}$ in X such that, for all x^* in X^*, $\lim_n x^*(x_n) = 0$ and $\|x_n\| \geq 1$ for all n. For a fixed positive integer k, we let F_k denote the real-valued function defined on $[0,1]$ that equals 0 on the set $\{0, 1, a_k^{(1)}, b_k^{(1)}, \dots, a_k^{(2^{k-1})}, b_k^{(2^{k-1})}\}$, equals 1 on the set $\{(a_k^{(1)} + b_k^{(1)})/2, \dots, (a_k^{(2^{k-1})} + b_k^{(2^{k-1})})/2\}$ and is linear on the intervals between these points. Then define $F(t)$ by $\sum_{k=1}^{\infty} F_k(t)x_k$ for all t in $[0,1]$. We claim that F is weakly continuous on $[0,1]$. Fix an arbitrary balanced weak neighborhood $\mathscr{O}$ of 0. Then there exists a positive integer K such that $x_k \in \mathscr{O}$ for each $k > K$. Since $0 \leq F_k(t) \leq 1$ for all t in $[0,1]$ and for all k, it follows that $\sum_{k=K+1}^{\infty} F_k(t)x_k \in \mathscr{O}$ for all t in $[0,1]$. By [**14**, Lemma 1], F is weakly continuous on $[0,1]$. Now it is clear that, for each x^* in X^*, function x^*F is ACG and continuous on $[0,1]$ and F has a scalar derivative, f say, on $[0,1]$. Thus, function F is both the indefinite DP integral of f and discontinuous on C, a contradiction. The proof is complete. □

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