ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 2, 2013

# AN INTERESTING TOPOLOGICAL SPACE USING WEAK TOPOLOGY

S.I. NADA AND D.L. FEARNLEY

ABSTRACT. We use weak topology to construct an example of a space which is Hausdorff, not first countable and not regular, but which preserves many of the properties of a planar open set.

**1. Introduction.** Cohen [**1**] defined weak topology as follows. Let

$$\xi = \bigcup_{\alpha \in J} X_\alpha,$$

where each $X_\alpha$ is a topological space. We say that $U \subset \xi$ is open (in the weak topology induced by the $X_\alpha$ subsets) if $U \cap X_\alpha$ is open in $X_\alpha$ for all $\alpha \in J$. In general, we note that, for a topological space $X$, if $X_\alpha \subset X$ has the subspace topology for all $\alpha \in J$ and $X = \cup_{\alpha \in J} X_\alpha$, then if we define $\xi = \cup_{\alpha \in J} X_\alpha$ to be the space whose points are the points of $X$, with weak topology induced by the $X_\alpha$ subsets, then the topology on $\xi$ is at least as fine as the topology on $X$. Every open set in $X$ intersects each $X_\alpha$ in an open set in $X_\alpha$ by definition of the subspace topology and is therefore open in $\xi$. It is possible, of course, for the topologies on $X$ and $\xi$ to be the same.

We will use the convention that $(x_n)$ refers to a sequence of points $(x_1, x_2, x_3, \dots)$ and use the notation $(x_n) \to p$ to mean that $(x_n)$ converges to $p$, and we will use $\{x_n\}$ to denote $\{x_1, x_2, x_3, \dots\}$, the set of image points for $(x_n)$. Let $R^2$ denote the plane with the Euclidean metric.

**2. Properties of the space $\overline{R}^2$.**

**Example.** For every point $p \in R^2$, let $T_p$ be the union of a vertical and a horizontal line in $R^2$ which intersect at point $p$ (a translation of

2010 AMS *Mathematics subject classification.* Primary 54A10, 54H05, 54F45.
*Keywords and phrases.* Weak topology, Hausdorff, first countable, Baire category theorem, separation axioms.
The second author is the corresponding author.
Received by the editors on July 9, 2010, and in revised form on September 9, 2010.

DOI:10.1216/RMJ-2013-43-2-563 

the coordinate axes with $p$ at the intersection) with the usual subspace topology. We define $\overline{R}^2$ to be the space consisting of the points of $R^2$ with the weak topology induced by the $T_p$ subsets. In other words, a set $U \subset \overline{R}^2$ will be open if and only if every translation of the axes, $T_p$, intersects $U$ in an open subset of $T_p$.

The topology of $\overline{R}^2$ is strictly finer than the topology on $R^2$. The interior of the rose defined by the polar curve $r = \cos(2\theta)$ plus the origin is open in $\overline{R}^2$, but would not be open in $R^2$.

It follows that, if $(x_n) \to p$ in $\overline{R}^2$, then $(x_n) \to p$ in $R^2$, but the converse is false. For example, the sequence $(1/n, 1/n) \to (0, 0)$ in $R^2$, but the aforementioned rose is an example of an open set about the origin containing no members of this sequence, so this sequence does not converge in $\overline{R}^2$.

**Lemma.** *A set $U$ is open in $\overline{R}^2$ if and only if, for every $x \in U$, there is an $\varepsilon_x > 0$ and an open ball in the plane centered at $x$ with radius $\varepsilon_x$ (denoted as $B_{\varepsilon_x}(x)$) so that $B_{\varepsilon_x}(x) \cap T_x \subset U$.*

*Proof.* First assume that $U$ is open in $\overline{R}^2$. Then by the definition of the topology on $\overline{R}^2$ it follows that, for every $x \in U$, the set $T_x \cap U$ is open in $T_x$ under the subspace topology, which implies that for some $\varepsilon_x > 0$, the set $B_{\varepsilon_x}(x) \cap T_x \subset U$.

Next, suppose we define $U \subset R^2$ to be any set such that, for each $x \in U$, we can choose some $\varepsilon_x > 0$ so that $B_{\varepsilon_x}(x) \cap T_x \subset U$. Then, for every $y \in R^2$ and any $x \in (T_y \cap U)$, there is some $(B_{\varepsilon_x}(x) \cap T_x) \subset U$ so that $B_{\varepsilon_x}(x) \cap T_x \cap T_y$ is an open subset of $T_y$ containing $x$. Thus, $T_y \cap U$ is open for all $y \in R^2$, and $U$ is open in $\overline{R}^2$. The result follows. □

**Theorem 1.** *Let $(x_n) \to p$ in $R^2$. If $\{x_n\} \cap T_p = \varnothing$, then $\{x_n\}$ is discrete in $\overline{R}^2$.*

*Proof.* Since the topology on $\overline{R}^2$ is finer than the topology on $R^2$, we know that $\{x_n\}$ has no limit points other than possibly $p$ in $\overline{R}^2$. Hence, we need only show that $p$ is not a limit point of $\{x_n\}$. For each point

$x \in T_p \setminus \{p\}$, since $x \notin \overline{\{x_n\}}$ in $R^2$, we can pick an open ball $B_{\varepsilon_x}(x)$ which does not intersect $\{x_n\}$. The set

$$U = \left( \bigcup_{x \in T_p \setminus \{p\}} B_{\varepsilon_x}(x) \right) \cup \{p\}$$

is open in $\overline{R}^2$ and does not intersect $\{x_n\}$. We know $U$ is open in $\overline{R}^2$ by the lemma because $U$ is the union of an open set in $R^2$ with $\{p\}$, and all of $T_p$ is contained in $U$. It follows that $\{x_n\}$ is discrete in $\overline{R}^2$. □

**Theorem 2.** *Let $(x_n) \to p$ in $R^2$. Then $(x_n) \to p$ in $\overline{R}^2$ if and only if there is a positive integer $N$ so that if $n > N$ then $x_n \in T_p$.*

*Proof.* First, assume there is a natural number $N$ so that, if $n > N$, then $x_n \in T_p$. Let $U$ be an open set in $\overline{R}^2$ containing $p$. Then, for some $\varepsilon > 0$, $B_\varepsilon(p) \cap T_p \subset U$. Since $(x_n) \to p$ in $R^2$, we know that there is an integer $N_1$ so that, if $n > N_1$, then $x_n \in B_\varepsilon(p)$. Thus, if $n > \max(N, N_1)$, then $x_n \in U$, so $(x_n) \to p$ in $\overline{R}^2$.

Next assume that, for every positive integer $N$, there is some integer $n > N$ so that $x_n \notin T_p$. Let $x_{n_k}$ be the $k$th member of $(x_n)$ which is not an element of $T_p$. Then, by Theorem 1, $\{x_{n_k}\}$ is discrete in $\overline{R}^2$ and therefore $(x_n)$ does not converge in $\overline{R}^2$ since $(x_n)$ has a subsequence $(x_{n_k})$ which does not converge. □

**Theorem 3.** *A set $K$ is compact in $\overline{R}^2$ if and only if $K$ is compact in $R^2$ and there are points $p_1, p_2, \ldots, p_n \in \overline{R}^2$ so that $K \subset \cup_{i=1}^n T_{p_i}$.*

*Proof.* First, let $K$ be compact in $\overline{R}^2$. Then, since $\overline{R}^2$ has a finer topology than $R^2$, it follows that $K$ is compact in $R^2$. Suppose that $K$ is not contained in the union of finitely many sets of the form $T_x$. Let $p_1 \in K$. Since $K \not\subset T_{p_1}$, we can find a point $p_2 \in K$ such that $p_2 \notin T_{p_1}$. If we have chosen points $p_1, p_2, \ldots, p_n$, then note that

$$K \not\subset \bigcup_{i=1}^{n} T_{p_i},$$

so we can choose a point

$$p_{n+1} \in K \setminus \left( \bigcup_{i=1}^{n} T_{p_i} \right).$$

The sequence $(p_n)$ has a subsequence $(p_{n_k})$ which converges to a point $p \in K$ in $R^2$ since $K$ is compact in $R^2$. However, $T_p$ can contain at most two elements of $\{p_n\}$ by construction, and so it follows that $(p_{n_k})$ is discrete in $\overline{R}^2$ by Theorem 1 (since a discrete set plus two more points is still discrete), and therefore $K$ is not compact in $\overline{R}^2$ because a compact set cannot contain an infinite discrete subset.

Next, assume that $K$ is both compact in $R^2$ and $K \subset \cup_{i=1}^{n} T_{p_i}$ for some points $p_1, p_2, \dots, p_n$. Then $K_i = K \cap T_{p_i}$ is compact in $R^2$ for every $i$. Let $C$ be an open cover of $K_i$ in $\overline{R}^2$. For each $U \in C$ we construct a set $U'$ which is open in $R^2$ as follows. For each $x \in (U \cap K_i)$, choose an open ball $B_{\varepsilon_x}(x)$ so that $(B_{\varepsilon_x}(x) \cap T_{p_i}) \subset U$. Then let

$$U' = \bigcup_{x \in K_i} B_{\varepsilon_x}(x),$$

and note that $U'$ is an open set in $R^2$ such that $(U' \cap K_i) = (U \cap K_i)$. We then define $C' = \{U' \mid U \in C\}$. Since $K_i$ is compact in $R^2$, it follows that $C'$ has a finite subcover $\{U'_1, U'_2, \dots, U'_m\}$. Thus, $\{U_1, U_2, \dots, U_m\}$ is a finite subset of $C$ which covers $K_i$. Hence, each $K_i$ is compact in $\overline{R}^2$, and so $K = \cup_{i=1}^{m} K_i$ is a finite union of compact sets, and therefore $K$ is compact in $\overline{R}^2$. □

**Theorem 4.** *The space $\overline{R}^2$ is locally path-wise connected.*

*Proof.* Let $p \in U$ be an open set in $\overline{R}^2$. Choose $\varepsilon_p > 0$ so that $L_1 = (B_{\varepsilon_p}(p) \cap T_p) \subset U$. Inductively, if some $L_i$ has been defined, then for every point $x \in L_i$ we choose $\varepsilon_x > 0$ so that $(B_{\varepsilon_x}(x) \cap T_x) \subset U$, and we let

$$L_{i+1} = \bigcup_{x \in L_i} (B_{\varepsilon_x}(x) \cap T_x).$$

Then the set

$$L = \bigcup_{i=1}^{\infty} L_i$$

is path-wise connected, open and contained in $U$. □

**Corollary 5.** $\overline{R}^2$ *has a basis of connected open sets.*

**Theorem 6.** $\overline{R}^2$ *is not first countable.*

*Proof.* Let $(U_n)$ be a sequence of open sets so that $p \in U_n$ for every positive integer $n$. For each $n$, we choose a point $x_n \in (U_n \cap (T_p \setminus \{p\}) \cap B_{1/n}(p))$. Then, for each $x_n$, we can find an $\varepsilon_{x_n} > 0$ so that $(B_{\varepsilon_{x_n}}(x_n) \cap T_{x_n}) \subset U_n$. Thus, we may also choose a point $y_n \in (B_{\varepsilon_{x_n}}(x_n) \cap T_{x_n}) \setminus T_p$. The sequence $(y_n)$ thus defined has the property that $y_n \in U_n$ for every integer $n$, $\{y_n\} \cap T_p = \varnothing$, and $(y_n) \to p$ in $R^2$. Therefore, by Theorem 1, $\{y_n\}$ is discrete in $\overline{R}^2$. But then $\overline{R}^2 \setminus \{y_n\}$ is an open set containing $p$ which does not contain any of the open sets $U_n$. It follows that $\overline{R}^2$ is not first countable. □

**Corollary 7.** $\overline{R}^2$ *is neither second countable nor metrizable.*

**Theorem 8.** $\overline{R}^2$ *is not regular.*

*Proof.* We will show that there is a compact set and a closed set which cannot be placed into disjoint open sets. Let $K = \{ (x, 0) \in R^2 \mid 0 \leq x \leq 1\}$. Note that $K$ is compact in $\overline{R}^2$. Let $C_n$ consist of the set of points $(x, y)$ whose $x$-coordinates are the rational numbers whose denominator in reduced terms is $n$, with $y$-coordinates $1/n$. Let $C = \cup_{i=1}^{\infty} C_i$.

We claim that $C$ is discrete in $\overline{R}^2$. To see this, let $p \in \overline{R}^2$. First, assume that $p$ is not on the $x$-axis. Then ,if the $y$-coordinate of $p$ has absolute value $\varepsilon$, then there are only finitely many points of $C$ within $\varepsilon/2$ of $p$, so $p$ is not a limit point of $C$. Next, assume that $p = (x_p, 0)$ is on the $x$-axis. Then $T_p$ contains at most one point of $C$. Thus, we can choose $\varepsilon_p > 0$ so that $B_{\varepsilon_p}(p) \cap T_p \cap C = \varnothing$. Let $L_p = \{(x_p, y)\, od|y| < \varepsilon_p\}$. For each $z \in L_p$, we can choose $\varepsilon_z > 0$ so that $B_{\varepsilon_z}(z) \cap C = \varnothing$. Define

$$W_p = (B_{\varepsilon_p}(p) \cap T_p) \cup \bigcup_{z \in L_p} B_{\varepsilon_z}(z).$$

Notice that $W_p$ is the union of an open set in $R^2$ and an open interval on the $x$-axis containing $p$. For every point $t = (x, 0)$ on the $x$-axis, we choose $W_t$ similarly, and let $W = \cup_{x\in R} W_{(x,0)}$. Using the lemma we can see that $W$ is open in $\overline{R}^2$ because $W$ is the union of an open set in $R^2$ with the $x$-axis and for every point $t$ on the $x$-axis there is an $\varepsilon_t > 0$ so that $(B_{\varepsilon_t}(t) \cap T_t) \subset W_t \subset W$ by construction. Hence, $p \in W$ and $W \cap C = \varnothing$, so $C$ is discrete and therefore closed in $\overline{R}^2$.

Let $U$ and $V$ be open sets in $\overline{R}^2$ containing $K$ and $C$, respectively. If we let $E_n = \{ (x,0) \in K \mid B_{1/n}((x,0)) \cap T_{(x,0)} \subset U\}$, then since $\cup_{n=1}^{\infty} E_n = K$ and, by the Baire category theorem, it follows that there is some $E_m$ which is dense in a sub-interval $I$ of $K$. But then we can choose an integer $t > m$ so that $C_t$ contains a point $p$ with $x$-coordinate within $I$. Hence, for any $\varepsilon > 0$ it follows that $(B_\varepsilon(p) \cap T_p) \cap U \neq \varnothing$. Thus, $U \cap V \neq \varnothing$, and so $\overline{R}^2$ is not regular. □

**Corollary 9.** *The space $\overline{R}^2$ is not locally compact.*

*Proof.* Since $\overline{R}^2$ is Hausdorff and not completely regular, it follows that $\overline{R}^2$ is not locally compact. □

**Notation.** Let $K_\varepsilon(p)$ denote the set of points in $T_p$ whose distance from $p$ is less than or equal to $\varepsilon$. Let $L_\varepsilon(p), R_\varepsilon(p), U_\varepsilon(p), D_\varepsilon(p)$ denote the line segments consisting of the point $p$ and all points of $T_p$ which are left of, right of, above or below, respectively, the point $p$ and have distance less than or equal to $\varepsilon$ from $p$. Let $X$ and $Y$ denote the $x$ and $y$ coordinate axes in the remaining theorems.

**Theorem 10.** *Let $h : \overline{R}^2 \to \overline{R}^2$ be a homeomorphism. Then, for any $p \in R^2$, there are $\varepsilon, \delta > 0$ so that $h(K_\delta(p)) \subset T_{h(p)}$ and $K_\varepsilon(h(p)) \subset h(K_\delta(p))$.*

*Proof.* First note that, since the topology on $\overline{R}^2$ is finer than the topology on $R^2$, $h$ is also a homeomorphism on $R^2$. Since $K_1(p)$ is compact, we know by Theorem 3 that $h(K_1(p))$ is contained in a finite number of vertical and horizontal line segments. Thus, there is some line segment $L_{\gamma_1}(p) \subset L_1(p)$ so that $h(L_{\gamma_1}(p)) = l_{h(p)}$ is a horizontal or vertical line segment containing $h(p)$ as one of its end points. We can similarly choose segments $R_{\gamma_2}(p)$, $U_{\gamma_3}(p)$, $D_{\gamma_4}(p)$ with corresponding horizontal or vertical line segment images $r_{h(p)} = h(R_{\gamma_2}(p))$, $u_{h(p)} = h(U_{\gamma_3}(p))$, $d_{h(p)} = h(D_{\gamma_4}(p))$. Since $h$ is bijective, $h(p)$ is the only point common to any two of these line segments. If we set $\delta$ equal to the minimum of $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_4$ and set $\varepsilon$ equal to the minimum of the lengths of $l_{h(p)}$, $r_{h(p)}$, $u_{h(p)}$, $d_{h(p)}$, then the desired result follows. □

**Theorem 11.** *Let $h : \overline{R}^2 \to \overline{R}^2$ be a homeomorphism. Then the image of every horizontal or vertical line under $h$ is a horizontal or vertical line. Furthermore, if the image of one horizontal line under $h$ is horizontal or vertical, respectively, then the images of all other horizontal lines are horizontal or vertical, respectively, and the images of all vertical lines are vertical or horizontal, respectively.*

*Proof.* Let $p = (0, 0)$, and let $h(p) = q$. By Theorem 10, we can find $\delta$, $\varepsilon > 0$, so that $h(K_\delta(p)) \subset T_q$ and $K_\varepsilon(q) \subset h(K_\delta(p))$. Let $l_q = h(L_\delta(p))$, $r_q = h(R_\delta(p))$, $d_q = h(D_\delta(p))$, $u_q = h(U_\delta(p))$. We wish to show first that $l_q \cup r_q$ is either a vertical line segment or a horizontal line segment, with $q$ in its interior. We will first assume (without loss of generality, since all cases are similar) that $l_q = L_{\gamma_1}(q)$ for some $\gamma_1 > \varepsilon$. Suppose that $r_q = D_{\gamma_2}(q)$ for some $\gamma_2 > \varepsilon$. Since $h$ is a homeomorphism on $B_\delta(p)$ and $L_\delta(p) \cup R_\delta(p)$ separates $B_\delta(p)$, it follows that $h(L_\delta(p) \cup R_\delta(p))$ separates $h(B_\delta(p))$. Thus, all points above $X$ in $B_\delta(p)$ must map to one component of $h(B_\delta(p)) \setminus (l_q \cup r_q)$, and all points below $X$ in $B_\delta(p)$ must map to the other component of $h(B_\delta(p)) \setminus (l_q \cup r_q)$. Without loss of generality, we may assume that points below $X$ map below $l_q$. But then we can find a horizontal line segment $H \subset B_\delta(p)$ above $L_\delta(p) \cup R_\delta(p)$ so that the end points $a$ and $b$ of $H$ are near enough to the end points of $L_\delta(p) \cup R_\delta(p)$ that $h(a)$ is directly above a point of $l_q$ and $h(b)$ is directly to the right of a point of $r_q$. We know that $h(L_\delta(p) \cup D_\delta(p))$ separates $h(a)$ and $h(b)$ in $h(B_\delta(p))$, and $h(H)$ is connected. Therefore, $h(H)$ intersects $h(D_\delta(p))$. But this is impossible because $H$ does not intersect $D_\delta(p)$. A similar contradiction arises if $l_q$ and $r_q$ form any corner at $h(p)$ or if $p$ is any point of the plane. Thus, it follows that every point of every horizontal (or vertical) line is contained in an interval along that line which is mapped into a vertical or horizontal line under $h$.

Assume that $l_q \cup r_q$ is part of the horizontal line $P$. If there are points of $X$ that do not map to $P$, let $t$ be the least upper bound of $S = \{x \in R \mid h(z,0) \in P \text{ for all } z \in [0,x]\}$. Then, for some $\alpha > 0$, $h(B_\alpha((t,0)) \cap X$ is entirely contained in a single horizontal or vertical line. Since we know that $h(t-(\alpha/2),0) \in P$, it then follows that $(h(B_\alpha((t,0)) \cap X) \subset P$, contradicting $t$ being the supremum of $S$. Thus, $h(X) = P$. Similarly, if $l_q \cup r_q$ is a vertical line segment, then $h(X)$ is a vertical line. By a similar argument, we see that every horizontal or vertical line maps to a horizontal or vertical line.

Since horizonal lines are disjoint, their images are disjoint under $h$, so if one horizontal line maps to a horizontal line then no horizontal line can map to a vertical line. Similarly, since every vertical and horizontal line intersect, their images under $h$ must intersect. Thus, if any horizontal line maps to a horizontal (respectively, vertical) line, then all horizontal lines map to horizontal (respectively, vertical) lines and all vertical lines map to vertical (respectively, horizontal) lines. □

**Theorem 12.** *A function $h : \overline{R}^2 \to \overline{R}^2$ is a homeomorphism if and only if there are homeomorphisms $h_1, h_2 : R \to R$ such that $h(x,y) = (h_1(x), h_2(y))$ or $h(x,y) = (h_1(y), h_2(x))$ for all $x, y \in \overline{R}^2$.*

*Proof.* First, assume that $h(x,y) = (h_1(x), h_2(y))$ for homeomorphisms $h_1, h_2 : R \to R$. That $h$ is a bijection follows from the fact that its coordinate functions are bijective. Let $U$ be open in $\overline{R}^2$, and let $(x_1, y_1) \in U$, where $h_1(x_0) = x_1$ and $h_2(y_0) = y_1$. Then there are open intervals $(a,b)$ and $(c,d)$ so that $\{(x_1,x) \in \overline{R}^2 \mid c < x < d\} \subset U$ and $\{(x,y_1) \in \overline{R}^2 \mid a < x < b\} \subset U$. Thus, $(\{(x_0,x) \in \overline{R}^2 \mid x \in h_1^{-1}(c,d)\} \cup \{(x,y_0) \in \overline{R}^2 \mid x \in h_1^{-1}(a,b)\}) \subset h^{-1}(U)$. Hence, $h$ is continuous and, by a similar argument, $h^{-1}$ is continuous. If $h(x,y) = (h_1(y), h_2(x))$, then the argument is similar.

Next, assume that $h : \overline{R}^2 \to \overline{R}^2$ is a homeomorphism. Then $h(X)$ is either a vertical or a horizontal line. Without loss of generality, we may assume that $h(X)$ is a horizontal line, and therefore $h(Y)$ is a vertical line. Let $\pi_X, \pi_Y : \overline{R}^2 \to R$ denote $x$-coordinate and $y$-coordinate projections, respectively. Let $h_1, h_2 : R \to R$ be defined by letting $h_1(x) = \pi_X(h(x,0))$, and let $h_2(x) = \pi_Y(h(0,x))$ for all $x \in R$. The topology of a horizontal or vertical line in $\overline{R}^2$ is the same as the topology on $R$, so $\pi_X$ is a homeomorphism on any horizontal line and $\pi_Y$ is a homeomorphism on any vertical line. Hence, $h_1$ and $h_2$ are compositions of homeomorphisms and are therefore homeomorphisms. Given any point $(x_0, y_0) \in \overline{R}^2$, we know that the line $x = x_0$ maps to a vertical line under $h$, and therefore the $x$-coordinate of $h(x_0, y_0)$ is $h_1(x_0)$, and similarly the $y$-coordinate of $h(x_0, y_0)$ is $h_2(y_0)$. Thus, $h(x, y) = (h_1(x), h_2(y))$ for all $(x, y) \in \overline{R}^2$. The proof is similar if $h(X)$ is vertical, yielding a homeomorphism of the form $h(x, y) = (h_1(y), h_2(x))$. $\square$

## REFERENCES

**1.** D.E. Cohen, *Spaces with weak topology*, Quart. J. Math. **5**, Oxford, (1954), 77–80.

Faculty of Science, Menoufia University, Egypt
**Email address: shokrynada@yahoo.com**

Utah Valley University, Orem, UT
**Email address: axiom10@gmail.com**