ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 2, 2013

EXISTENCE AND ITERATION OF POSITIVE SOLUTIONS TO THIRD ORDER THREE-POINT BVP WITH INCREASING HOMEOMORPHISM AND POSITIVE HOMOMORPHISM

DE-XIANG MA AND XIAO-ZHONG YANG

ABSTRACT. In this paper, we obtain the existence of positive solutions and establish two corresponding iterative schemes for the following third order three-point boundary value problem

$$\begin{cases} (\phi(u''))'(t) + q(t)f(u(t)) = 0, & 0 \le t \le 1, \\ u(0) = \beta u(\xi), & u'(1) = 0, \\ & \phi(u''(0)) = \delta\phi(u''(\xi)), \end{cases}$$

where $\phi : \mathcal{R} \to \mathcal{R}$ is an increasing homeomorphism and positive homomorphism. The main tool is the monotone iterative technique. An example is given to show our results.

1. Introduction. The purpose of this paper is to consider the existence of positive solutions and establish two corresponding iterative schemes for the following third order three-point boundary value problem (BVP for short),

(1)
$$\begin{cases} (\phi(u''))'(t) + q(t)f(u(t)) = 0, & 0 \le t \le 1, \\ u(0) = \beta u(\xi), & u'(1) = 0,\ \phi(u''(0)) = \delta\phi(u''(\xi)), \end{cases}$$

where $\phi : \mathcal{R} \to \mathcal{R}$ is an increasing homeomorphism and positive homomorphism with $\phi(0) = 0$. Here, the following conditions hold:

(H1) $0 < \xi, \beta, \delta < 1$;

(H2) $f : [0, +\infty) \to \mathcal{R}^+$ is continuous and nondecreasing;

Keywords and phrases. Iteration, positive solution, third order three-point, boundary value problem, increasing homeomorphism and positive homomorphism.
Supported by special Funds for Co-construction Project of Beijing, Doctor's Foundation of North China Electric Power University, by the Revitalization Plan in Arts and Sciences of North China Electric Power University and Fundamental Research Funds for the Central Universities (10ML38).
The first author is the corresponding author.
Received by the editors on May 3, 2010, and in revised form on July 12, 2010.
DOI:10.1216/RMJ-2013-43-2-539

(H3) $q \in L^1[0,1]$ is nonnegative on $(0,1)$, and q is not identically zero on any compact subinterval of $(0,1)$. Furthermore, q satisfies

$$0 < \int_0^1 q(t)\,dt < +\infty.$$

A projection $\phi : \mathcal{R} \to \mathcal{R}$ is called an increasing homeomorphism and positive homomorphism, if the following conditions are satisfied:

(a) if $x \leq y$, then $\phi(x) \leq \phi(y)$ for all $x, y \in \mathcal{R}$;

(b) ϕ is a continuous bijection and its inverse mapping ϕ^{-1} is also continuous;

(c) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathcal{R}$.

Remark 1.1. (i) $\phi^{-1}(xy) = \phi^{-1}(x)\phi^{-1}(y)$ can be easily obtained.

(ii) We get by $\phi(0) = 0$ that $\phi(x) \geq 0$ if $x \geq 0$ and $\phi(x) \leq 0$ if $x \leq 0$.

Remark 1.2. ϕ generates a $p-$Laplacian operator, i.e., $\phi_p(x) = |x|^{p-2}x$ for some $p > 1$.

Very recently, BVP (1) has been studied in [**9**]. As for the background for BVP (1), we refer the reader to [**9**]. For more details about the third order three-point BVP, we refer the reader to [**2–5, 7–10**] and their references.

Now, we concentrate on [**9**]. The authors in [**9**] obtained the existence of three positive solutions for BVP (1) by using the fixed point theorem due to Avery-Peterson. They proved the following lemma (a key lemma, which is critical in changing BVP (1) into an equivalent integral equation).

Lemma (see [**9**, Lemma 2.1]). *For $h \in C([0,1], R)$, the* BVP

$$\begin{cases} u''(t) + h(t) = 0, & t \in I, \\ u(0) = \beta u(\xi), & u'(1) = 0 \end{cases}$$

has a unique solution

$$u(t) = \int_0^t (1-s)h(s)\,ds + \frac{\beta}{1-\beta}\int_0^\xi (1-s)h(s)\,ds. \tag{2}$$

Unfortunately, this lemma is wrong since we get $u''(t) + h(t) \neq 0$ by (2). So the conclusions of [**9**] should be reconsidered. In fact, the same mistake has also been made in Lemma 2.11 in [**5**].

We point out that the methods used in [**2–5, 7–9**] are all fixed point theorems, such as Krasnoselskii, Leggett-Williams and Avery and Peterson and so on. Completely different from the above references, in this paper, by improving the classical monotone iterative technique of Amann [**1**], we obtain not only the existence of positive solutions for (1) but also give two iterative schemes approximating the solutions.

The paper is organized as follows. After this section, some lemmas will be established in Section 2. In Section 3, we give our main result in Theorem 3.1, and an example is also given to demonstrate our results.

2. Preliminaries. We consider the Banach space $E = C[0,1]$ equipped with norm $\|w\| = \max_{0\leq t\leq 1} |w(t)|$. In this paper, a positive solution w^* of (1) means a solution w^* of (1) satisfying $w^*(t) > 0$, $0 < t < 1$. We recall that a function w is said to be concave on $[0,1]$, if

$$w(\lambda t_2 + (1-\lambda)t_1) \geq \lambda w(t_2) + (1-\lambda)w(t_1), \quad t_1, t_2, \lambda \in [0,1].$$

We denote

$$\begin{aligned} C^2[0,1] &= \{w : w''(t) \in C[0,1]\}, \\ C^+[0,1] &= \{w \in C[0,1] : w(t) \geq 0,\ t \in [0,1]\}, \\ P &= \big\{w \in C[0,1] \mid w(t) \geq 0 \text{ is concave and} \\ &\qquad\qquad \text{nondecreasing on } [0,1]\big\}. \end{aligned}$$

It is easy to see that P is a cone in $C[0,1]$. For $w \in P$, we have $\|w\| = w(1)$ and

$$\|w\| t \leq w(t) \leq \|w\|, \quad t \in [0,1]. \tag{3}$$

In the following, we always suppose (H1), (H2) and (H3) hold.

Lemma 2.1. *Suppose $y \in C^2[0,1]$ with $\phi(y'') \in L^1[0,1]$ satisfies*

$$\begin{cases} (\phi(y''))'(t) \leq 0, & 0 \leq t \leq 1, \\ y(0) = \beta y(\xi), & y'(1) = 0,\ \phi(y''(0)) = \delta\phi(y''(\xi)). \end{cases}$$

Then, $y(t) \geq 0$ is concave and nondecreasing on $[0,1]$, i.e., $y \in P$.

Proof. Since $(\phi(y''))'(t) \le 0$, we know that $\phi(y'')(t)$ is nonincreasing; thus, we have $\phi(y''(0)) = \delta\phi(y''(\xi)) \le \delta\phi(y''(0))$, which means $\phi(y''(0)) \le 0$, and then $\phi(y''(t)) \le \phi(y''(0)) \le 0$. As a result, $y''(t) \le 0$, i.e., $y(t)$ is concave. Combining $y'(1) = 0$, we have $y'(t) \ge y'(1) = 0$, i.e., $y(t)$ is nondecreasing. From $y(0) = \beta y(\xi) \ge \beta y(0)$, we get $y(0) \ge 0$, and then $y(t) \ge y(0) \ge 0$. Above all, $y \in P$.

For any $x \in C^+[0,1]$, suppose that u is a solution of the following BVP

$$\begin{cases} (\phi(u''))'(t) + q(t)f(x(t)) = 0, & 0 \le t \le 1, \\ u(0) = \beta u(\xi), & u'(1) = 0,\ \phi(u''(0)) = \delta\phi(u''(\xi)). \end{cases} \tag{4}$$

Then, integrating both sides of equation (4) on $[0,t]$, we have

$$\phi(u'')(t) = \phi(u'')(0) - \int_0^t q(s)f(x(s))\,ds,$$

by the boundary value condition $\phi(u''(0)) = \delta\phi(u''(\xi))$, we have $\phi(u''(0)) = -\delta/(1-\delta)\int_0^\xi q(r)f(x(r))\,dr$; thus, we get

$$u''(t) = \phi^{-1}\bigg(-\frac{\delta}{1-\delta}\int_0^\xi q(r)f(x(r))\,dr - \int_0^t q(r)f(x(r))\,dr\bigg). \tag{5}$$

From equation (5), combining $u(0) = \beta u(\xi)$ and $u'(1) = 0$, we have

$$\begin{aligned} u(t) = \int_0^t \bigg[\int_s^1 \phi^{-1}\bigg(\frac{\delta}{1-\delta}\int_0^\xi q(r)f(x(r))\,dr \\ + \int_0^\tau q(r)f(x(r))\,dr\bigg)\,d\tau\bigg]\,ds \\ + \frac{\beta}{1-\beta}\int_0^\xi \bigg[\int_s^1 \phi^{-1}\bigg(\frac{\delta}{1-\delta}\int_0^\xi q(r)f(x(r))\,dr \\ + \int_0^\tau q(r)f(x(r))\,dr\bigg)\,d\tau\bigg]\,ds. \end{aligned} \tag{6}$$

Now, for each $x \in P$, we define an operator $T : P \to E$ by

(7)
$$(Tx)(t) = \int_0^t \left[\int_s^1 \phi^{-1}\left(\frac{\delta}{1-\delta} \int_0^\xi q(r) f(x(r))\, dr + \int_0^\tau q(r) f(x(r))\, dr \right) d\tau \right] ds + \frac{\beta}{1-\beta} \int_0^\xi \left[\int_s^1 \phi^{-1}\left(\frac{\delta}{1-\delta} \int_0^\xi q(r) f(x(r))\, dr + \int_0^\tau q(r) f(x(r))\, dr \right) d\tau \right] ds.$$

It is obvious that $(Tx) \in E$ is well defined, and we may easily verify that Tx is a solution of the following BVP

(8)
$$\begin{cases} (\phi(u''))'(t) + q(t) f(x(t)) = 0, & 0 \le t \le 1, \\ u(0) = \beta u(\xi), & u'(1) = 0,\ \phi(u''(0)) = \delta \phi(u''(\xi)). \end{cases}$$

So now, to find a solution of BVP (1) in P is equal to finding a fixed point of T in P. As for T, we have the following properties.

Lemma 2.2. *$T : P \to P$ is completely continuous and nondecreasing.*

Proof. First we prove $TP \subseteq P$. For all $u \in P$, by the definition of Tu, we see $(Tu) \in C^2[0,1]$ and

(9)
$$\begin{cases} (\phi((Tu)''))'(t) = -q(t) f(u(t)) \le 0, & 0 \le t \le 1, \\ (Tu)(0) = \beta (Tu)(\xi), & (Tu)'(1) = 0, \\ & \phi((Tu)''(0)) = \delta \phi((Tu)''(\xi)). \end{cases}$$

By Lemma 2.1, we know $Tu \in$P.

Since $f : [0, +\infty) \to \mathcal{R}^+$ is continuous and nondecreasing, $q \in L^1[0,1]$ is nonnegative on $(0,1)$ and, by the expression of Tu, we may easily know $T : P \to P$ is completely continuous and nondecreasing.

3. Existence of two monotone solutions of BVP (1).

Theorem 3.1. *Assume that* (H1), (H2) *and* (H3) *hold. If there exist a $\theta \in (0,1)$ and two positive numbers $b < a$ such that*

(H4) $f(a) \le \phi(aA)$, $f(\theta b) \ge \phi(Bb)$, *where*

$$A = \frac{2(1-\beta)}{\phi^{-1}((1/1-\delta)\int_0^1 q(r)\,dr)}, \qquad B = \frac{1}{\int_\theta^1 [\int_s^1 \phi^{-1}(\int_\theta^\tau q(r)\,dr)\,d\tau]\,ds}. \tag{10}$$

Then, there are two sequences $\{w_n\}$ *and* $\{v_n\}$ *in* P *which satisfy*

$$\begin{aligned} bt = v_0(t) &\le v_1(t) \le v_2(t) \le \cdots \\ &\le v_n(t) \le w_n(t) \le \cdots \\ &\le w_2(t) \le w_1(t) \le w_0(t) = a, \end{aligned}$$

$$w_n \longrightarrow w^*, \qquad v_n \longrightarrow v^*,$$

w^* *and* v^* *are two solutions of* BVP (1) *in* P.

Proof. Denote $P[b,a] = \{w \in P : b \le \|w\| \le a\}$. We first prove that $TP[b,a] \subset P[b,a]$.

Let $w \in P[b,a]$. We have by (3),

$$0 \le w(t) \le \|w\| \le a.$$
$$\min_{t\in[\theta,1]} w(t) \ge \theta\|w\| \ge \theta b.$$

So, by assumption (H2), we have

$$0 \le f(w(t)) \le f(a) \le \phi(aA), \quad t \in [0,1], \tag{11}$$
$$f(w(t)) \ge f(\theta b) \ge \phi(bB), \quad t \in [\theta,1]. \tag{12}$$

Thus, for any $w \in P[b,a]$,

$$\begin{aligned} \|Tw\| &= (Tw)(1) \\ &= \int_0^1 \bigg[\int_s^1 \phi^{-1}\bigg(\frac{\delta}{1-\delta}\int_0^\xi q(r)f(x(r))\,dr \\ &\qquad\qquad + \int_0^\tau q(r)f(x(r))\,dr\bigg)\,d\tau\bigg]\,ds \\ &\quad + \frac{\beta}{1-\beta}\int_0^\xi \bigg[\int_s^1 \phi^{-1}\bigg(\frac{\delta}{1-\delta}\int_0^\xi q(r)f(x(r))\,dr \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad + \int_0^{\tau} q(r) f(x(r))\, dr\bigg)\, d\tau\bigg]\, ds \\
&= \int_0^1 \bigg[\int_s^1 \phi^{-1}\bigg(\frac{\delta}{1-\delta}\int_0^{\xi} q(r) f(x(r))\, dr \\
&\qquad\qquad\qquad + \int_0^{\tau} q(r) f(x(r))\, dr\bigg)\, d\tau\bigg]\, ds + (Tw)(0) \\
&\geq \int_0^1 \bigg[\int_s^1 \phi^{-1}\bigg(\frac{\delta}{1-\delta}\int_0^{\xi} q(r) f(x(r))\, dr \\
&\qquad\qquad\qquad\qquad\qquad + \int_0^{\tau} q(r) f(x(r))\, dr\bigg)\, d\tau\bigg]\, ds \\
&\geq \int_0^1 \bigg[\int_s^1 \phi^{-1}\bigg(\int_0^{\tau} q(r) f(x(r))\, dr\bigg)\, d\tau\bigg]\, ds \\
&\geq \int_{\theta}^1 \bigg[\int_s^1 \phi^{-1}\bigg(\int_0^{\tau} q(r) f(x(r))\, dr\bigg)\, d\tau\bigg]\, ds \\
&= \int_{\theta}^1 \bigg[\int_s^1 \phi^{-1}\bigg(\int_0^{\theta} q(r) f(x(r))\, dr \\
&\qquad\qquad\qquad\qquad\qquad + \int_{\theta}^{\tau} q(r) f(x(r))\, dr\bigg)\, d\tau\bigg]\, ds \\
&\geq \int_{\theta}^1 \bigg[\int_s^1 \phi^{-1}\bigg(\int_{\theta}^{\tau} q(r) f(x(r))\, dr\bigg)\, d\tau\bigg]\, ds \\
&\geq \int_{\theta}^1 \bigg[\int_s^1 \phi^{-1}\bigg(\phi(Bb)\int_{\theta}^{\tau} q(r)\, dr\bigg)\, d\tau\bigg]\, ds \\
&= Bb\int_{\theta}^1 \bigg[\int_s^1 \phi^{-1}\bigg(\int_{\theta}^{\tau} q(r)\, dr\bigg)\, d\tau\bigg]\, ds = b,
\end{aligned}
$$

$$
\begin{aligned}
\|Tw\| &= (Tw)(1) \\
&= \int_0^1 \bigg[\int_s^1 \phi^{-1}\bigg(\frac{\delta}{1-\delta}\int_0^{\xi} q(r) f(x(r))\, dr \\
&\qquad\qquad\qquad\qquad\qquad + \int_0^{\tau} q(r) f(x(r))\, dr\bigg)\, d\tau\bigg]\, ds \\
&\quad + \frac{\beta}{1-\beta}\int_0^{\xi} \bigg[\int_s^1 \phi^{-1}\bigg(\frac{\delta}{1-\delta}\int_0^{\xi} q(r) f(x(r))\, dr
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad + \int_0^\tau q(r) f(x(r))\,dr\Big)\,d\tau\Big]\,ds \\
&\le \frac{1}{1-\beta}\int_0^1 \Big[\int_s^1 \phi^{-1}\Big(\frac{\delta}{1-\delta}\int_0^\xi q(r)f(x(r))\,dr \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \int_0^\tau q(r) f(x(r))\,dr\Big)\,d\tau\Big]\,ds \\
&\le \frac{1}{1-\beta}\int_0^1 \int_s^1 \phi^{-1}\Big(\frac{1}{1-\delta}\int_0^1 q(r)f(x(r))\,dr\Big)\,d\tau\,ds \\
&\le \frac{1}{1-\beta}\int_0^1 \int_s^1 \phi^{-1}\Big(\frac{1}{1-\delta}\int_0^1 q(r)\,dr\Big)\,d\tau\,ds(aA) \\
&= \frac{1}{2(1-\beta)}\phi^{-1}\Big(\frac{1}{1-\delta}\int_0^1 q(r)\,dr\Big)(aA) = a.
\end{aligned}
$$

Altogether, we get $b \le \|Tw\| \le a$ for $w \in P[b,a]$, which means that $TP[b,a] \subset P[b,a]$.

Let $w_0(t) = a$, $t \in [0,1]$; then $w_0(t) \in P[b,a]$. Let $w_1 = Tw_0$. Then $w_1 \in P[b,a]$ and we denote

$$w_{n+1} = Tw_n, \quad (n = 0,1,2,\dots). \tag{13}$$

Since $TP[b,a] \subset P[b,a]$, we have $w_n \in P[b,a]$, $(n = 0,1,2,\dots)$. From Lemma 2.2, T is compact, so $\{w_n\}_{n=1}^\infty$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^\infty$, and there exists a $w^* \in P[b,a]$ such that $w_{n_k} \to w^*$.

Now, since $w_1 \in P[b,a] \subset P$, we have

$$0 \le w_1(t) \le \|w\| \le a = w_0(t).$$

By Lemma 2.2, T is nondecreasing, and we know that $Tw_1 \le Tw_0$ which means that $w_2(t) \le w_1(t)$, $0 \le t \le 1$. By induction,

$$\cdots \le w_n(t) \le \cdots \le w_2(t) \le w_1(t) \le w_0(t) = a.$$

Hence, we assert that $w_n \to w^*$. Let $n \to \infty$ in (13). By the continuity of T, we obtain $Tw^* = w^*$, which means that w^* is the solution of BVP (1). Since $\|w^*\| \ge b > 0$ and w^* is a nonnegative concave function on $[0,1]$, we conclude that $w^*(t) > 0$, $t \in (0,1)$.

Let $v_0(t) = bt$, $t \in [0,1]$. Then $v_0(t) \in P[b,a]$. Let $v_1 = Tv_0$. Then $v_1 \in P[b,a]$, and we denote

$$v_{n+1} = Tv_n, \quad (n = 0,1,2,\dots). \tag{14}$$

Since $TP[b,a] \subset P[b,a]$, we have $v_n \in P[b,a]$, $(n = 0,1,2,\dots)$. From Lemma 2.2, T is compact, so $\{v_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$, and there exists a $v^* \in P[b,a]$ such that $v_{n_k} \to w^*$.

Now, since $v_1 \in P[b,a] \subset P$, we have

$$v_1(t) \geq \|v\|t \geq bt = v_0(t).$$

By Lemma 2.2, T is nondecreasing, and we know that $Tv_1 \geq Tv_0$ which means that $v_2(t) \geq v_1(t)$, $0 \leq t \leq 1$. By induction,

$$bt = v_0(t) \leq v_1(t) \leq v_2(t) \leq \cdots \leq v_n(t) \leq \cdots.$$

Hence, we assert that $v_n \to v^*$. Let $n \to \infty$ in (14). By the continuity of T, we obtain $Tv^* = v^*$ which means that v^* is the solution of BVP (1). Since $\|v^*\| \geq b > 0$ and v^* is a nonnegative concave function on $[0,1]$, we conclude that $v^*(t) > 0$, $t \in (0,1)$.

$v_n(t) \leq w_n(t)$ can be easily obtained due to $v_0(t) \leq w_0(t)$ and the fact that T is nondecreasing.

Corollary 3.1. *Assume that* (H1), (H2) *and* (H3) *hold. If there exists a* $\theta \in (0,1)$ *such that*

$$\lim_{u\to 0+} \frac{f(u)}{\phi(u)} \geq \phi\Big(\frac{B}{\theta}\Big), \qquad \lim_{u\to +\infty} \frac{f(u)}{\phi(u)} \leq \phi(A), \tag{15}$$

where A, B *are defined as in* (10)*, then, all the conclusions of Theorem* 3.1 *hold.*

Proof. It is very easy to verify that condition (H4) of Theorem 3.1 can be obtained from condition (15) in Corollary 3.1. We omit the proof.

Example 3.1. Consider the following BVP,

$$\begin{cases} (\phi(u''))'(t) + \frac{1}{\sqrt{t}}(\sqrt{u} + \frac{1}{5}u) = 0, & 0 \leq t \leq 1, \\ u(0) = \frac{1}{2}u(\frac{1}{4}), & u'(1) = 0,\ \phi(u''(0)) = \frac{1}{2}\phi(u''(\frac{1}{4})), \end{cases}$$

where $\phi(u) = u$.

We choose $\theta = 1/4$, and then by (10), we have $A = 1/4$, $B = 97/480$. Let $a = 20^2$ and $b = (120/73)^2$. It is easily verified that all the conditions of Theorem 3.1 are satisfied; thus, by Theorem 3.1, we can get two sequences

$$\begin{aligned} w_n(t) = \int_0^t \bigg[\int_s^1 \bigg(\int_0^{1/4} &\frac{\sqrt{w_{n-1}(r)} + (1/5)w_{n-1}(r)}{\sqrt{r}}\, dr \\ &+ \int_0^\tau \frac{\sqrt{w_{n-1}(r)} + (1/5)w_{n-1}(r)}{\sqrt{r}}\, dr\bigg)\, d\tau\bigg]\, ds \\ + \int_0^{1/4} \bigg[\int_s^1 \bigg(\int_0^{1/4} &\frac{\sqrt{w_{n-1}(r)} + (1/5)w_{n-1}(r)}{\sqrt{r}}\, dr \\ &+ \int_0^\tau \frac{\sqrt{w_{n-1}(r)} + (1/5)w_{n-1}(r)}{\sqrt{r}}\, dr\bigg)\, d\tau\bigg]\, ds, \\ v_n(t) = \int_0^t \bigg[\int_s^1 \bigg(\int_0^{1/4} &\frac{\sqrt{v_{n-1}(r)} + (1/5)v_{n-1}(r)}{\sqrt{r}}\, dr \\ &+ \int_0^\tau \frac{\sqrt{v_{n-1}(r)} + (1/5)v_{n-1}(r)}{\sqrt{r}}\, dr\bigg)\, d\tau\bigg]\, ds \\ + \int_0^{1/4} \bigg[\int_s^1 \bigg(\int_0^{1/4} &\frac{\sqrt{v_{n-1}(r)} + (1/5)v_{n-1}(r)}{\sqrt{r}}\, dr \\ &+ \int_0^\tau \frac{\sqrt{v_{n-1}(r)} + (1/5)v_{n-1}(r)}{\sqrt{r}}\, dr\bigg)\, d\tau\bigg]\, ds, \end{aligned}$$

where $w_0(t) = a$, $v_0(t) = bt$. Both w_n and v_n converge to the solution of BVP (16). Figures 1 and 2 show the characteristic of the curves of the successive iteration.

Remark 3.1. We recall that we have already used the technique of iteration to study BVP with a $p-$Laplacian operator, for example [**6, 10**]. The following similar problem to BVP (1) has been studied in [**10**],

(17)
$$\begin{cases} (\phi_p(u''))'(t) = q(t)f(t,u(t)), & 0 \le t \le 1, \\ u(0) = \sum_{i=1}^m \alpha_i u(\xi_i), & u'(\eta) = 0,\ u''(1) = \sum_{i=1}^n \beta_i u''(\theta_i). \end{cases}$$

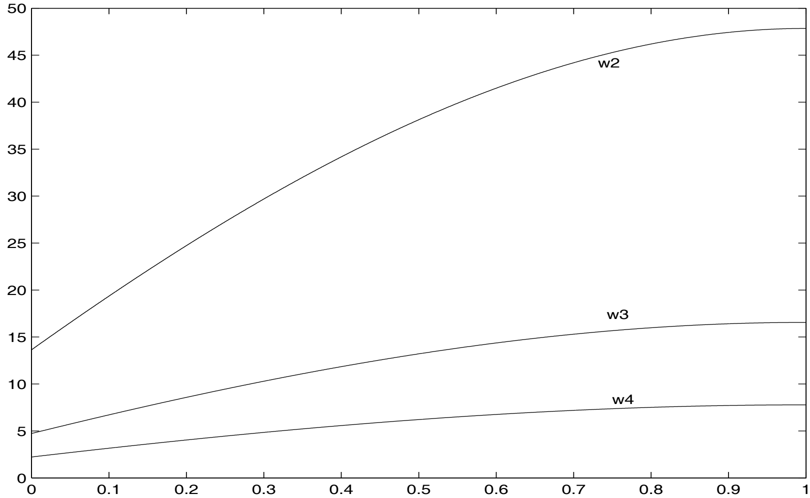

FIGURE 1. Curves of successive iteration.

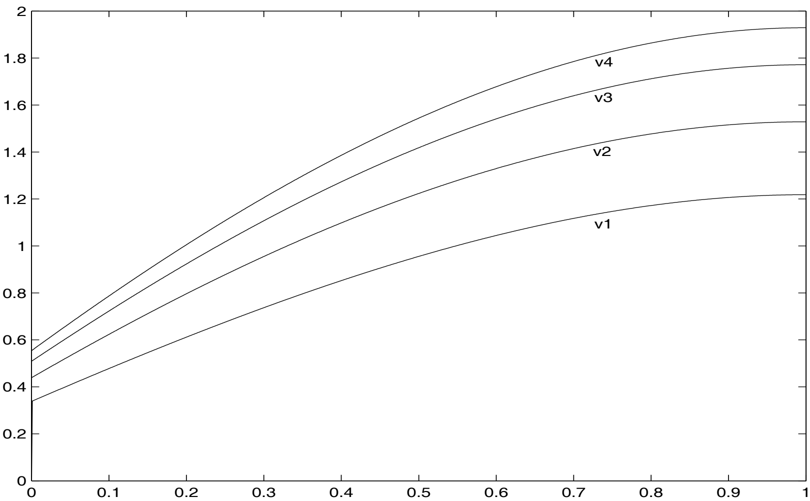

FIGURE 2. Curves of successive iteration.

But, in [**10**], when we rewrite the BVP (17) into an equivalent integral equation, an unknown parameter must be included in the integral equation. As a result, we can also give two sequences, which converge to the solution of BVP (17) theoretically, but we cannot give the curves of successive iteration due to the unknown parameter.

Acknowledgments. The authors thank the referee for his (or her) careful advices.

REFERENCES

1. H. Amann, Fixed point equations and nonlinear eigenvalue problems in order Banach spaces, SIAM Rev. **18** (1976), 620–709.

2. D.R. Anderson, *Multiple positive solutions for a three point boundary value problem*, Math. Comput. Model. **27** (1998), 49–57.

3. ———, *Green's function for a third-order generalized right focal problem*, J. Math. Anal. Appl. **288** (2003), 1–14.

4. D.R. Anderson and J.M. Davis, *Multiple solutions and eigenvalues for third order right focal boundary value problems*, J. Math. Anal. Appl. **267** (2002), 135–157.

5. J. Liu and F.Y. Xu, *Triple positive solutions for third-order m-point boundary value problems on time scales*, Discr. Dyn. Nature Soc., doi:10.1155/2009/123283.

6. D.X. Ma, W.G. Ge and Z.J. Gui, *Existence and iteration of positive solutions for a singular two-point boundary value problem with a p-Laplacian operator*, Czech. Math. J. **57** (2007), 135–152.

7. Y.B. Sang and H. Su, *Positive solutions of nonlinear third-order m-point BVP for an increasing homeomorphism and homomorphism with sign-changing nonlinearity*, J. Comput. Appl. Math. **225** (2009), 288–300.

8. Y.P. Sun, *Existence of triple positive solutions for a third-order three-point boundary value problem*, J. Comput. Appl. Math. **221** (2008), 194–201.

9. Yuming Wei, *Triple positive solutions to third order three-point BVP with increasing homeomorphism and positive homomorphism*, J. Comput. Appl. Math. **231** (2009), 134–142.

10. C.L. Zhou and D.X. Ma, *Existence and iteration of positive solutions for a generalized right-focal boundary value problem with p-Laplacian operator*, J. Math. Anal. Appl. **324** (2006), 409–424.

Department of Mathematics, North China Electric Power University, Beijing 102206, China
Email address: mdxcxg@163.com

Department of Mathematics, North China Electric Power University, Beijing 102206, China
Email address: yxiaozh@ncepu.edu.cn