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TORSION-FREE, GENUS-ONE, CONGRUENCE SUBGROUPS OF PSL $(2, \mathbf{R})$ AND MULTIPLICATIVE η-PRODUCTS

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ABSTRACT. This paper contains a classification of the torsion-free, genus-one, congruence subgroups of PSL $(2, \mathbf{R})$. We find that there are eight classes of such subgroups which are in one-to-one correspondence with the eight weight 2 multiplicative η-products, first classified by Dummit, McKay and Kisilevsky. An intriguing connection with McKay's second monstrous E_8 corresponding is described.

1. Introduction. In this paper we will consider subgroups of PSL $(2, \mathbf{R})$ which are commensurable with PSL $(2, \mathbf{Z})$, but which are not necessarily subgroups of PSL $(2, \mathbf{Z})$. We will call such a subgroup a congruence subgroup if it contains some principal congruence subgroup. Our aim is to give a classification of the torsion-free, genus-one, congruence subgroups of PSL $(2, \mathbf{R})$. To motivate this result, we first give some background on congruence subgroups of small genus, moonshine and η-products.

Following the discovery of Moonshine [**6**], there has been renewed interest in the study and classification of congruence subgroups of low genus. Cox and Parry [**7**] classified the genus-zero, congruence subgroups of the modular group. Thompson [**39**] showed that there are finitely many congruence subgroups of SL $(2, \mathbf{R})$ of a given genus.

Thompson's result was made effective by Zograf [**41**]. Using these results, the author tabulated the genus-zero and genus-one congruence subgroups [**8, 9**]. The results are that there are 506 PSL $(2, \mathbf{R})$ conjugacy classes of congruence subgroups of genus-zero. More than half of these classes are thought to be associated with moonshine or generalized moonshine.

For the congruence subgroups of genus-one, there are 982 PSL $(2, \mathbf{R})$ conjugacy classes of groups. The connections of these groups with

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moonshine is less well understood, and one aim of this paper is to make a tentative link with moonshine in the case that the groups are torsion-free. This connection involves η-products and their moonshine properties and so we next give some background on η-products.

The use of η-products and η-quotients to construct modular functions and forms for $\Gamma_0(N)$ is well known and was first systematically investigated by Newman [**34, 35**]. This technique finds many applications; for example, it was exploited by Conway and Norton [**6**], and later by Ford, McKay and Norton [**15**], to find η-quotient expansions for many of the Hauptmoduls occurring in moonshine and generalized moonshine.

In general, the modular forms constructed in this way are not eigenforms. Dummit, Kisilevsky and McKay [**14**] were the first to classify the η-products which have multiplicative coefficients. There are just 30, and these are shown in Table 1. The more general classification of multiplicative η-quotients was considered by Martin [**28**], and Martin and Ono [**29**], and again the possibilities are somewhat limited.

The products of Table 1 have interesting moonshine properties. Each product is associated with a partition of 24, and 21 of these are cycle types of M_{24}. The resulting connection with M_{24} has been extensively studied, see [**30**] for a more detailed account. Conway and Norton [**6**] conjectured a connection with monstrous moonshine in that each such class gives rise to a class in the Monster group. This was proved by Kondo and Takashi [**21**]. Dong and Mason [**13**] have also shown how M_{24} can be used to create a "toy model" for the Moonshine module of Frenkel, Lepowsky and Meurman.

The multiplicative η-products have another moonshine connection related to McKay's second Monstrous E_8 observation (cf. [**4**, page 528]). McKay's observation is a correspondence between nine classes in the monster group and the nodes of an affine E_8 diagram. This connection is still somewhat mysterious, and understanding it is one of Borcherds' 14 problems in moonshine [**3**]. Some progress has been made by Norton and Glauberman [**16**], who have extended McKay's observation. Also, Lam, Yamada and Yamauchi [**24, 23**] and Lam and Yamauchi [**22**] have made a connection between McKay's second correspondence and certain coset subalgebras of the lattice vertex operator algebra $V_{\sqrt{2}E_8}$.

TABLE 1. Multiplicative η-products. Partitions marked with an asterisk are not cycle types of M_{24}.

Partition	Weight	Partition	Weight	Partition	Weight
24^1 *	1/2	$14^1 7^1 2^1 1^1$	2	4^6	3
8^3 *	3/2	$12^1 6^1 4^1 2^1$	2	$6^2 3^2 2^2 1^2$	4
$23^1 1^1$	1	$11^2 1^2$	2	$5^4 1^4$	4
$22^1 2^1$ *	1	$10^2 2^2$	2	$4^4 2^4$	4
$21^1 3^1$	1	$9^2 3^2$ *	2	3^8	4
$20^1 4^1$ *	1	$8^2 4^2$ *	2	$4^4 2^2 1^4$	5
$18^1 6^1$ *	1	6^4	2	$3^6 1^6$	6
$16^1 8^1$ *	1	$8^2 4^1 2^1 1^2$	3	2^{12}	6
12^2	1	$7^3 1^3$	3	$2^8 1^8$	8
$15^1 5^1 3^1 1^1$	2	$6^3 2^3$ *	3	1^{24}	12

The author and Duncan [**10**] have recently observed that there are nine multiplicative η-products of weight 4 or more and that these products correspond in a natural way to the nodes of an affine E_8 diagram. The fixing groups of these products are (up to 24th roots of unity), with one exception, the discrete groups corresponding to McKay's nine monstrous classes. A slightly more involved construction yields the groups which occur in McKay's correspondence.

Thus, the η-products of Table 1 of weight at least 4 and their fixing groups are closely tied to McKay's second correspondence. It is thus intriguing that, by the results of this paper, the torsion-free, genus-one, congruence subgroups are "up to isogeny" classified by the η-products of weight 2, the only remaining even weight forms in Table 1. This gives an interesting analogy between the two cases. While this is not a definitive moonshine connection, the situation is certainly noteworthy and gives a fuller picture of the properties of the multiplicative η-products. It suggests that further investigation is warranted.

The classification is based on a novel method introduced by Sebbar to classify the torsion-free, genus-zero, congruence subgroups of $\mathrm{PSL}(2, \mathbf{R})$ [**36**]. Sebbar's approach makes use of Larcher groups [**26, 27**]. In this paper, we refine Sebbar's method by introducing minimum Larcher subgroups. For a subgroup G of $\mathrm{PSL}(2, \mathbf{R})$, its minimum Larcher subgroup is, roughly speaking, the intersection of all congruence subgroups of G which have the same parabolic elements of G. The virtue of these groups is that they have an explicit description and their signatures are known. They are thus in some sense "nice" congruence subgroups. The idea of the classification is to deal first with these tractable groups, and then to relate them to less tractable groups.

It is perhaps worth noting that, in the genus-zero case, the torsion-free, congruence groups are precisely the torsion-free, minimum Larcher subgroups. These groups have been extensively studied by Sebbar [**37**] and Sebbar and McKay [**31–33**], and they have some interesting moonshine properties. As the results of the current paper show that the genus-one, minimum Larcher subgroups also have interesting properties, this suggests that further study of these groups will be profitable.

The structure of the paper is as follows. In Section 2 we define generalized Larcher subgroups and minimum Larcher subgroups, and find some basic properties. In Section 3 these results are applied to find the set of minimum Larcher subgroups which are (projectively) torsion-free, genus-one and regular (contain -1_2, where 1_2 is the identity of $\mathrm{SL}\,(2,\mathbf{R})$). In Section 4 it is shown that there is a certain maximum group corresponding to each such minimum Larcher group and that each such maximum group is the fixing group, up to conjugacy, of one of the weight 2, multiplicative η-products. Finally, the $\mathrm{PSL}\,(2,\mathbf{R})$ conjugacy classes of torsion-free, genus-one, congruence subgroups are found and the results are presented in a set of tables and diagrams.

In the rest of this paper we shall find it convenient to refer to genus-one, regular, congruence subgroups of $\mathrm{SL}\,(2,\mathbf{R})$ whose images in $\mathrm{PSL}\,(2,\mathbf{R})$ are torsion-free as $\mathcal{T}$-subgroups. There groups are in one-to-one correspondence with the genus-one, torsion-free, congruence subgroups of $\mathrm{PSL}\,(2,\mathbf{R})$.

2. Generalized Larcher subgroups. In the rest of this paper Γ will denote $\mathrm{SL}\,(2,\mathbf{R})$ and we will use $\mathcal{C}$ to denote the set of regular, congruence subgroups of $\mathrm{SL}\,(2,\mathbf{R})$.

Definition 2.1. For any subgroup H of $\mathcal{C}$, and for any subgroup K of H which is also in $\mathcal{C}$, we say that K is a generalized Larcher subgroup (GLS) of H if and only if H and K contain the same parabolic elements.

Next we define the set $\mathcal{LS}$ of Larcher groups. For positive integers p, q, r, χ and τ such that $p \mid qr$ and $\chi \mid \gcd\,(p, qr/p)$, let

$$H(p,q,r;\chi,\tau) = \left\{ \begin{pmatrix} 1+ap & bq \\ cr & 1+dp \end{pmatrix} \in \Gamma \mid \right.$$
$$\left. a,b,c,d \in \mathbf{Z}, \quad c \equiv \tau a \pmod{\chi} \right\}.$$

Define $\mathcal{LS}$ to be the set of all the groups of the form $\pm H(p,q,r;\chi,\tau)$, where if G is a subgroup of $\mathrm{SL}\,(2,\mathbf{R})$, we define $\pm G$ to be the group generated by G and -1_2. Larcher's original set of groups is a subset of $\mathcal{LS}$, but this larger set is easier to describe.

Larcher showed that every element of $\mathcal{C} \cap \Gamma$ contains a GLS which is, up to conjugacy in Γ, an element of $\mathcal{LS}$. So the properties of the set of cusp widths of a general congruence subgroup of Γ follow by establishing the corresponding properties for the groups in $\mathcal{LS}$. Since $\mathcal{LS}$ is given explicitly, the properties of the set of cusp widths of the groups in $\mathcal{LS}$ are easier to obtain than the properties of the set of cusp widths of a general congruence subgroup of Γ. This was Larcher's original motivation for introducing these groups. The signature and level of the groups in $\mathcal{LS}$ are known [**11, 12**].

Sebbar observed that any torsion-free, genus-zero, congruence subgroup of $\mathrm{PSL}\,(2,\mathbf{R})$ is necessarily conjugate in $\mathrm{PSL}\,(2,\mathbf{R})$ to (the image of) a Larcher group. This yields a classification of these groups. Our aim here is to extend this result to the case of genus-one subgroups. It is no longer the case that these groups are conjugate to elements of $\mathcal{LS}$. However, Larcher subgroups turn out once again to enter into the classification. In fact we will find that a more "canonical" set $\mathcal{S}$, the set of minimum Larcher groups of Γ, is the key to the classification.

Definition 2.2. Let H be a group in $\mathcal{C}$. We define the minimum Larcher subgroup $L_{\min}(H)$ of H to be the intersection of all the generalized Larcher subgroups of H.

Lemma 2.3. *Suppose H is a group in $\mathcal{C}$. Then $L_{\min}(H)$ is a* GLS *of H.*

Proof. $L_{\min}(H)$ is regular and contains the same parabolic elements as H, so we have to show that it is a congruence subgroup. If K is a

GLS of H, then $K \cap \Gamma$ is a congruence subgroup. Since K has the same parabolic elements as H, the set of cusp widths of $K \cap \Gamma$ is the same as the set of cusp widths of $H \cap \Gamma$. So, by Wolhfahrt's theorem [**40**], the level of $K \cap \Gamma$ is the same as the level of $H \cap \Gamma$. If this level is n, then $L_{\min}(H)$ contains $\Gamma(n)$, as required. $\square$

Lemma 2.4. *If G is a group in $\mathcal{C}$, then $L_{\min}(G)$ is a normal subgroup of G.*

Proof. As G is commensurable with Γ, it has the same cusps as Γ, namely, $\mathbf{Q} \cup \{\infty\}$. It follows that every element of G is a multiple of an integer matrix. So, if H is a regular, congruence subgroup of G, so is any conjugate of H in G. Moreover, if m is a parabolic element of G and g is any element of G, then gmg^{-1} is in H, since H is a GLS of G. Hence, m is in H^g and so H^g is a GLS of G. So the set of generalized Larcher subgroups of G is invariant under conjugation in G, and so $L_{\min}(G)$ is a normal subgroup, as required. $\square$

Lemma 2.5. *If K is a* GLS *of H, then $L_{\min}(K) = L_{\min}(H)$.*

Proof. As $L_{\min}(K)$ is a GLS of H, we have $L_{\min}(H) \subseteq L_{\min}(K)$. But $L_{\min}(H)$ is also a GLS of K, and so $L_{\min}(K) \subseteq L_{\min}(H)$. $\square$

Corollary 2.6. *For K in $\mathcal{C}$, K is $L_{\min}(H)$ for some H in $\mathcal{C}$ if and only if $K = L_{\min}(K)$. Moreover, if K is a* GLS *of some group H in $\mathcal{C}$ and $L_{\min}(K) = K$, then $K = L_{\min}(H)$.*

Let $\mathcal{S}$ be the set of minimum Larcher subgroups which are contained in Γ. The previous corollary shows that $\mathcal{S} = \{H \in \mathcal{C} \cap \Gamma \mid L_{\min}(H) = H\}$. We also have the following useful result:

Lemma 2.7. *Every element of $\mathcal{S}$ is contained, up to conjugacy in* $\mathrm{SL}\,(2, \mathbf{Z})$*, in the set $\mathcal{LS}$.*

Proof. Suppose H is an element of $\mathcal{S}$, then by Corollary 2.6, we have $L_{\min}(H) = H$. So H is the only generalized Larcher subgroup of H. Now Larcher's result is that every group in $\mathcal{C} \cap \Gamma$ contains, up to conjugacy in $\mathrm{SL}\,(2, \mathbf{Z})$, a subgroup L which is contained in $\mathcal{LS}$ and such that H and L have the same parabolic elements. Thus, in the terminology used in this paper, L is a GLS of H. But as H is the only GLS of H we have $H = L$ and so some conjugate of H in Γ is contained in $\mathcal{LS}$. $\square$

As we will see later, every minimum Larcher subgroup is conjugate in $\mathrm{SL}\,(2, \mathbf{R})$ to an element of $\mathcal{S}$. For this we will require the following technical lemma.

Lemma 2.8. *If G is a group in $\mathcal{C}$ and g is an element of $SL\,(2, \mathbf{R})$ such that G^g is commensurable with Γ, then G^g is also in $\mathcal{C}$ and $L_{\min}(G^g) = (L_{\min}(G))^g$.*

Proof. As G and G^g are commensurable with Γ, by considering the action of g on cusps, we deduce that g is a multiple of an integer matrix. It follows that G^g is in $\mathcal{C}$. If m is a parabolic element of G^g, then gmg^{-1} is in $L_{\min}(G)$. So $L_{\min}(G)^g$ is a GLS of G^g, and hence $L_{\min}(G^g) \subseteq L_{\min}(G)^g$. Similarly, $L_{\min}(G) \subseteq (L_{\min}(G^g))^{g^{-1}}$ and so the required equality follows. $\square$

Lemma 2.9. *The set $\mathcal{S}$ contains $\pm\Gamma(n)$, $n = 1, 2, 3, \ldots$. The set $\mathcal{S}$ also contains all of the genus-zero, regular, congruence subgroups of Γ whose images in $PSL\,(2, \mathbf{R})$ are torsion-free.*

Proof. If H is a GLS of $\pm\Gamma(n)$, then it has level n and so contains $\Gamma(n)$ and, by definition, it also contains -1_2. Hence, $H = \pm\Gamma(n)$. So $L_{\min}(\pm\Gamma(n)) = \pm\Gamma(n)$, and so $\pm\Gamma(n) \in \mathcal{S}$, $n = 1, 2, 3, \ldots$, by Corollary 2.6. The latter half of the proposition is due to Sebbar [**36**]. $\square$

3. The classification of $\mathcal{T}$-subgroups: Minimum groups. In this section our aim is to start the classification of the $\mathcal{T}$-subgroups of $\mathrm{SL}\,(2, \mathbf{R})$ which, as explained in the introduction, we have defined to be the genus-one, regular, congruence subgroups of $\mathrm{SL}\,(2, \mathbf{R})$ whose images in $\mathrm{PSL}\,(2, \mathbf{R})$ are torsion-free. We start with a definition.

Definition 3.1.

$$\Gamma_0(f)^+ = \{e^{-1/2}\begin{pmatrix} ae & b \\ cf & de \end{pmatrix} \in \mathrm{SL}\,(2,\mathbf{R})\Big| \\ a,b,c,d,e,f \in \mathbf{Z},\ e>0,\ f>0,\ e||f\}.$$

When f is square-free, the group $\Gamma_0(f)^+$ is the normalizer of $\Gamma_0(f)$ in $\mathrm{SL}\,(2,\mathbf{R})$. The importance of these "Helling groups" is illustrated by the following theorem:

Theorem 3.2 (Helling [**5, 17, 18**]). *If G is a subgroup of $SL\,(2,\mathbf{R})$ which is commensurable with $SL\,(2,\mathbf{R})$, then G is conjugate to a subgroup of $\Gamma_0(f)^+$ for some square-free f. The groups $\Gamma_0(f)^+$ are distinct, maximal discrete subgroups of $SL\,(2,\mathbf{R})$.*

We shall also need Sebbar's lemma.

Lemma 3.3. *If t is a parabolic element of $\Gamma_0(f)^+$ with f square-free, then $t \in \Gamma$.*

Proof. If t is a parabolic matrix, then $\mathrm{tr}\,(t) = \pm 2$. So, if $t = e^{-1/2}\left(\begin{smallmatrix} ae & b \\ cf & de \end{smallmatrix}\right)$, then $ae + de = \pm 2e^{1/2}$. But $ae+de$ is an integer and e divides the square-free integer f. It follows that $e = 1$, and so $t \in \Gamma$. $\square$

Proposition 3.4. *If G is a group in $\mathcal{C}$ and G is a subgroup of $\Gamma_0(f)^+$ for some square-free f, then $L_{\min}(G) = L_{\min}(G\cap\Gamma)$.*

Proof. By Lemma 3.3, $G\cap\Gamma$ has the same parabolic elements of G and so it follows that it is a GLS of G. The result now follows from Lemma 2.5. $\square$

The following corollary will be used to establish a link between the $\mathcal{T}$-subgroups and the set $\mathcal{S}$ of minimum Larcher subgroups introduced in Section 2.

Corollary 3.5. *If G is a group in $\mathcal{C}$, then up to conjugacy in $SL(2,\mathbf{R})$, $L_{\min}(G)$ is contained in $\mathcal{S}$.*

Proof. By Theorem 3.2 and Lemma 2.8, we can reduce to the case that G is a subgroup of $\Gamma_0(f)^+$ for some square-free f. Then, by Proposition 3.4, $L_{\min}(G) = L_{\min}(G \cap \Gamma) = H$, say, with $L_{\min}(H) = H$. As noted in Section 2, this implies that H is in $\mathcal{S}$ up to conjugacy in Γ, so $L_{\min}(G)$ is contained in $\mathcal{S}$ up to conjugacy in $\mathrm{SL}(2,\mathbf{R})$, as required. □

We will also make use of the following result.

Lemma 3.6. *Let G be a subgroup of $SL(2,\mathbf{R})$ which is commensurable with $SL(2,\mathbf{Z})$. Then the normalizer of G in $SL(2,\mathbf{R})$ is, up to conjugacy in $SL(2,\mathbf{R})$, a subgroup of $\Gamma_0(f)^+$ for some square-free integer f.*

Proof. Since G is a non-cyclic, discrete subgroup of $\mathrm{SL}(2,\mathbf{R})$, it follows that N is a discrete subgroup of $\mathrm{SL}(2,\mathbf{R})$. See, for example, [**19**, Theorem 5.7.5]. Then, by Siegel's theorem (see, for example, [**2**, Theorem 10.4.5]) the index of G in N is finite. So N is commensurable with $\mathrm{SL}(2,\mathbf{Z})$. The result now follows from Theorem 3.2. □

We next show that the minimum Larcher subgroup of a $\mathcal{T}$-subgroup is also a $\mathcal{T}$-subgroup. We start with the following well-known theorem.

Theorem 3.7 (Riemann-Hurwitz formula). *Let $\phi : B' \to B$ be a surjective holomorphic mapping of degree n between two compact Riemann surfaces. Let g be the genus of B and g' the genus of B'. Then*

$$2g' - 2 = n(2g-2) + \sum_{z \in B'} (e_z - 1),$$

where e_z is the ramification index of the covering at $z \in B'$.

As explained, for example, in [**38**, Section 1.5], if G is a discrete subgroup of $\mathrm{SL}(2,\mathbf{R})$ such that $G\backslash\mathfrak{H}^*$ is compact and G' is a subgroup of G of finite index, then Theorem 3.7 may be applied to the covering map $\phi : G'\backslash\mathfrak{H}^* \to G\backslash\mathfrak{H}^*$. If $\overline{G}$ and $\overline{G}'$ are the images of G and G' in $\mathrm{PSL}\,(2, \mathbf{R})$, then the degree of this covering is $\mathrm{Index}\,(\overline{G} : \overline{G}')$. By [**38**, Proposition 1.37], the ramification index e_z, $z \in B'$, is equal to 1, except possibly at points which are images of elliptic or parabolic fixed points of G. More precisely, if $z \in B'$ is the image of $w \in \mathfrak{H}^*$, then e_z is equal to the index of $\overline{G}'_w$ in $\overline{G}_w$, where $\overline{G}'_w$ is the subgroup of $\overline{G}'$ which fixes w and $\overline{G}_w$ is the subgroup of $\overline{G}$ which fixes w.

We immediately obtain the required property of $L_{\min}(G)$, when G is a $\mathcal{T}$-subgroup, as follows.

Proposition 3.8. *Let G be a $\mathcal{T}$-subgroup. Then $L_{\min}(G)$ is also a $\mathcal{T}$-subgroup.*

Proof. The fact that G is projectively torsion-free implies that $L_{\min}(G)$ is projectively torsion-free. Consider the covers of $G\backslash\mathfrak{H}^*$ and $L_{\min}(G)\backslash\mathfrak{H}^*$. Since the groups are projectively torsion-free, there is no ramification at the images of elliptic points. Moreover, the groups contain the same parabolic elements and so there is no ramification at the images of cusps. By assumption, the genus of $G\backslash\mathfrak{H}^*$ is one. So, by Theorem 3.7, the genus of $H\backslash\mathfrak{H}^*$ is one. Since G is regular and congruence, so is $L_{\min}(G)$ and so the proposition follows. $\square$

These results provide some of the tools needed to classify the $\mathcal{T}$-subgroups up to conjugacy. If G is one such group, then, by Proposition 3.8, so is $L_{\min}(G)$, and the inclusion is normal. By Corollary 3.5, $L_{\min}(G)$ is, up to conjugacy, in $\mathcal{S}$. So we can classify $\mathcal{T}$-subgroups by first finding all the genus-one, projectively torsion-free elements of $\mathcal{S}$. By Lemma 2.7, these are, up to conjugacy in Γ, contained in $\mathcal{LS}$ (the set of Larcher groups). The genus-one elements of $\mathcal{LS}$ were listed in [**12**]. For convenience we reproduce the proof here.

The signature of a subgroup H of $\mathrm{SL}(2,\mathbf{Z})$ is a tuple $(\mu, \nu_2, \nu_3, \nu_\infty, \nu'_\infty)$, where μ is the index of $\pm H$ in $\mathrm{SL}\,(2,\mathbf{Z})$; ν_2 is the number of inequivalent elliptic fixed points of H of order 2; ν_3 is the number of inequivalent elliptic fixed points of H of order 3; ν_∞ is the number of regular cusps of H; and ν'_∞ is the number of irregular cusps of H. Let $H(p, N; \chi) = H(p, N, 1; \chi, 1)$; then the following lemma shows that to find the signature of $H(p,q,r;\chi,\tau)$, we can reduce to computing the signature of $H(p,N;\chi)$.

Lemma 3.9. *Suppose p,q,r,χ and τ are positive integers such that $p \mid qr$ and $\chi \mid \gcd(p, qr/p)$. Let $g = \gcd(\chi,\tau)$. Then the groups $H(p,q,r;\chi,\tau)$ and $H(p,gqr;\chi/g)$ have the same signature.*

To give the signatures of $H(p,N;\chi)$, we let $k = \operatorname{lcm}[\gcd(p^2,N)\chi, N]$ and define $c(p,N;\chi)$ by

$$c(p,N;\chi) = \frac{\chi N \phi(p)}{\phi(N)} \sum_{d \mid k/\chi} \frac{\phi(d)\phi(d')}{\operatorname{lcm}[d,d',pk/N]},$$

where $d\,d' = k/\chi$ and ϕ is Euler's function. Also define $\nu_2(N)$ and $\nu_3(N)$ to be the number of inequivalent elliptic fixed points of order 2 and 3, respectively, of $\Gamma_0(N)$. See, for example, [**38**, Proposition 1.43] for explicit formulas. Then the required signatures are given by the following theorem.

Theorem 3.10 [**11**]. *Suppose p, N and χ are positive integers such that $p \mid N$ and $\chi \mid \gcd(p, N/p)$. Let $c = c(p,N;\chi)$ and*

$$\psi(N) = N \prod_{\substack{\ell \mid N \\ \ell \text{ prime}}} \left(1 + \frac{1}{\ell}\right).$$

The signature $(\mu,\nu_2,\nu_3,\nu_\infty,\nu'_\infty)$ of $H(p,N;\chi)$ is

$$\mu = \begin{cases} \chi\phi(p)\psi(N) & \text{if } p = 2 \text{ and } \chi = 1, \\ & \text{or } p = 1, \\ \frac{1}{2}\chi\phi(p)\psi(N) & \text{otherwise.} \end{cases}$$

$$\nu_2 = \begin{cases} \nu_2(N) & \text{if } p = 1, \\ & \text{or } p = 2 \text{ and } 2 \| N, \\ 0 & \text{otherwise.} \end{cases}$$

$$\nu_3 = \begin{cases} \nu_3(N) & \text{if } p = 1, \\ & \text{or } p = 3 \text{ and } 3 \| N, \\ 0 & \text{otherwise.} \end{cases}$$

$$(\nu_\infty, \nu'_\infty) = \begin{cases} (c, 0) & \textit{if } p = 2 \textit{ and } \chi = 1, \\ & \textit{or } p = 1, \\ (\frac{2}{5}c, \frac{1}{5}c) & \textit{if } p = 2,\ \chi = 2,\ 2 \,||\, (N/p), \\ (\frac{1}{4}c, \frac{1}{2}c) & \textit{if } p = 2,\ \chi = 2,\ 2^k \,||\, (N/p),\ k \textit{ odd},\ k > 1, \\ (\frac{1}{3}c, \frac{1}{3}c) & \textit{if } p = 2,\ \chi = 2,\ 2^k \,||\, (N/p),\ k \textit{ even}, \\ (\frac{2}{5}c, \frac{1}{5}c) & \textit{if } p = 4,\ 2 \nmid (N/p),\ (\textit{so } \chi = 1), \\ (\frac{1}{2}c, 0) & \textit{otherwise.} \end{cases}$$

The signature of $\pm H$ *is* $(\mu, \nu_2, \nu_3, \nu_\infty + \nu'_\infty, 0)$.

As is well known, the formula for the genus of a subgroup of $\mathrm{SL}\,(2, \mathbf{Z})$ is given by

$$g = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2} - \frac{\nu'_\infty}{2}.$$

So Theorem 3.10 also gives an explicit formula for the genus of $\pm H(p, q, r; \chi, \tau)$. There are only finitely many groups of the form $\pm H(p, q, r; \chi, \tau)$ of any given genus, as is shown in the next theorem.

Theorem 3.11. *If the group* $\pm H(p, q, r; \chi, \tau)$ *has genus* g, *then for some* τ' *which is congruent to* τ *modulo* χ, *we have* $\pm H(p, q, r; \chi, \tau) = \pm H(p, q, r; \chi, \tau')$ *and* $1 \leq \tau' \leq \chi \leq p \leq qr < 128(g + 1)$.

Proof. Suppose H is a congruence subgroup of $\mathrm{SL}\,(2, \mathbf{Z})$. Then Zograf has shown in [**41**] that $g + 1 > (1/128)I$, where g is the genus of H and I is the index of H in $\mathrm{SL}\,(2, \mathbf{Z})$. By Theorems 3.9 and 3.10, the index of $\pm H(p, q, r; \chi, \tau)$ is $\chi\phi(p)\psi(qr)$ if $p = 1$, or $p = 2$ and $\chi = 1$, or $p = 2$, $\chi = 2$ and τ is even. For these cases we get a bound $qr < 128(g + 1)$, since qr is a lower bound for $\chi\phi(p)\psi(qr)$. Otherwise, the index is $1/2\chi\phi(p)\psi(qr)$. If $p \geq 3$, then $\phi(p) \geq 2$, and so again we get the bound $qr < 128(g + 1)$. This leave the case $p = 2$, $\chi = 2$ and τ is odd. But, since $\chi = 2$, we again get the same bound for qr. This yields the required result since p is bounded above by qr, χ is bounded by p and we can take $1 \leq \tau' \leq \chi$ where $\tau' \equiv \tau \pmod{\chi}$. □

Theorem 3.11 gives an explicit, finite list of values of the parameters p, q, r, χ and τ for which the group $\pm H(p, q, r; \chi, \tau)$ has genus g. As the signature of $\pm H(p, q, r; \chi, \tau)$ is given in Theorem 3.10, it follows that a list of all the groups in $\mathcal{LS}$ of any given genus can be computed. In particular, a list of all the genus-one, projectively torsion-free groups in $\mathcal{LS}$ can be found. In principle, this calculation can be done by hand, but the results of [**12**] were done by computer. After removing groups which do not satisfy $L_{\min}(H) = H$ and some obvious conjugations in $\mathrm{SL}(2, \mathbf{R})$, the remaining cases are listed in the first column of Table 2. There remains the possibility some of these groups are conjugate in $\mathrm{SL}(2, \mathbf{R})$. We can rule out this possibility by finding the normalizers of these groups as follows.

Definition 3.12. For positive integers p, q and r such that p divides qr, let $H(p, q, r) = H(p, q, r; 1, 1)$.

Atkin and Lehner [**1**] have found the normalizer of $\Gamma_0(N) = H(1, 1, N)$ and Lang [**25**] has found the normalizer of $\Gamma_1(N) = H(N, 1, N)$. It would be interesting to find the normalizers of all the Larcher subgroups; however, here we will need only the following special case.

Theorem 3.13. *Suppose f is a square-free integer. Let $H = H(p, r, rf)$ with $p \mid rf$ and either $r \mid p$ or $r \mid 24$. Then H is normal in $\Gamma_0(f)^+$.*

Proof. Let $A = e^{-1/2} \left(\begin{smallmatrix} ae & b \\ cf & de \end{smallmatrix}\right)$ be an element of $\Gamma_0(f)^+$ and $B = \left(\begin{smallmatrix} \alpha & \beta r \\ \gamma r f & \delta \end{smallmatrix}\right)$ be an element of $H(p, r, rf)$. Then

$$ABA^{-1} = \begin{pmatrix} ade\alpha - bc(f/e)\delta + k_1 rf & ab(\delta - \alpha) + k_2 r \\ cfd(\alpha - \delta) + k_3 rf & ade\delta - bc(f/e)\alpha + k_4 rf \end{pmatrix}$$

for some integers k_1, k_2, k_3 and k_4. Now $ade - cb(f/e) = 1$ and $\alpha \equiv \delta \equiv 1 \pmod{p}$, and so the diagonal entries of ABA^{-1} are congruent to 1 modulo p since by assumption p divides rf. If r divides p, then the congruence $\alpha \equiv \delta \equiv 1 \pmod{p}$ yields $\alpha - \delta \equiv 0 \pmod{r}$. If r divides 24, then the exponent of $(\mathbf{Z}/r\mathbf{Z})^*$ is 2 and, since $\alpha\delta - \beta\gamma r^2 f = 1$, it follows that again $\alpha - \delta \equiv 0 \pmod{r}$. In either case, ABA^{-1} is in $H(p, r, rf)$ as required. $\square$

Corollary 3.14. *If $\pm H$ is one of the groups given in column $L_{\min}$ of Table 2, then the normalizer of H in* $\mathrm{SL}(2,\mathbf{R})$ *is $\Gamma_0(f)^+$, where f is given in Table 2. In particular, the groups in column $L_{\min}$ of Table 2 are in distinct* $\mathrm{SL}(2,\mathbf{R})$ *conjugacy classes.*

Proof. By Proposition 3.13, H, and hence $\pm H$, are normal in $\Gamma_0(f)^+$ where f is one of 1, 2, 3, 5, 6, 11, 14 or 15. In each case f is square-free. So, by Theorem 3.2, $\Gamma_0(f)^+$ is a maximal, discrete subgroup of $\mathrm{SL}(2,\mathbf{R})$ and so is the full normalizer of H.

If two of the groups in column $L_{\min}$ of Table 2 were conjugate in $\mathrm{SL}(2,\mathbf{R})$, then their normalizers would be conjugate in $\mathrm{SL}(2,\mathbf{R})$ also. However, the groups $\Gamma_0(f)^+$ and $\Gamma_0(f')$, with f and f' square-free integers, are not conjugate in $\mathrm{SL}(2,\mathbf{R})$ if f is not equal to f'. This is shown in [**18**, Section 3]. $\square$

Corollary 3.15. *If G is a $\mathcal{T}$-subgroup of* $\mathrm{SL}(2,\mathbf{R})$*, then, up to conjugacy in* $\mathrm{SL}(2,\mathbf{R})$*, $L_{\min}(G)$ is one of the eight groups listed in the $L_{\min}$ column of Table 2.*

To summarize, Proposition 3.8 tells us that, if G is a $\mathcal{T}$-subgroup, then it contains $L_{\min}(G)$ and $L_{\min}(G)$ is a $\mathcal{T}$-subgroup. The inclusion is normal and so every $\mathcal{T}$-subgroup of $\mathrm{SL}(2,\mathbf{R})$ is, up to conjugacy in $\mathrm{SL}(2,\mathbf{R})$, a group between one of the groups in Table 2 and its normalizer in $\mathrm{SL}(2,\mathbf{R})$ as given in Corollary 3.14.

4. The classification of $\mathcal{T}$-subgroups: Maximum groups. Since there are only finitely many groups between each minimum Larcher group and its normalizer, Corollary 3.15 gives an algorithm for computing the $\mathcal{T}$-subgroups G up to conjugacy in $\mathrm{SL}(2,\mathbf{R})$. However, there is a strong constraint on these groups, as we show in the next proposition.

Proposition 4.1. *Suppose L is a projectively torsion-free, genus-one group in $\mathcal{S}$ (and hence regular and congruence). Let $\mathbf{R}$ be the set of discrete, regular, genus-one congruence subgroups of* $\mathrm{SL}(2,\mathbf{R})$ *which have L as minimum Larcher subgroup. Then the elements of $\mathbf{R}$ are $\mathcal{T}$-subgroups, $\mathbf{R}$ has a unique maximum element $L_{\max}$ and the quotient $L_{\max}/L$ is abelian.*

Proof. Suppose G_1 and G_2 are discrete, regular, genus-one congruence subgroups of $\mathrm{SL}(2,\mathbf{R})$ such that $L = L_{\min}(G_1) = L_{\min}(G_2)$. By Theorem 3.7, G_1 and G_2 are projectively torsion-free and hence $\mathcal{T}$-subgroups. Both groups contain L normally by Lemma 2.4. Thus, if G is the group generated by G_1 and G_2, then G is contained in the normalizer of L in $\mathrm{SL}(2,\mathbf{R})$, and so, by Lemma 3.6, the index of L in G is finite. So G is discrete and regular. It is also genus-one by the following argument. The group G_1/L acts on $C = L\backslash\mathfrak{H}^*$ as a finite subgroup of the automorphism group of C and the quotient surface $(G_1/L)\backslash C$ is isomorphic to $G_1\backslash\mathfrak{H}^*$ as a Riemann surface. See, for example, [**19**, Theorem 5.9.4 and page 312]. Let t be an element of G_1. Then t induces an automorphism of C. Now C is isomorphic as a Riemann surface to $C' = \mathcal{C}/\Omega$ for some lattice Ω and the automorphisms of C' have the form $[z] \mapsto [az+b]$ where a and b are complex numbers such that $a\Omega = \Omega$ and $[z]$ is the image of $z \in \mathcal{C}$ in the quotient C'. See, for example, [**19**, Theorem 4.18.2]. So t induces an automorphism $[z] \mapsto [az+b]$ of C' for some a and b. If $a \neq 1$, then $b/(1-a)$ is a fixed point and so t has a fixed point on C. This is impossible, since the quotient surface obtained from the action of G_1 is genus 1 and by Theorem 3.7 the covering map has no ramification. Let A' be the subgroup of automorphisms of C' whose elements have the form $[z] \mapsto [z+b]$. Then A' is an abelian group which has a fixed-point-free action on C'. Let A be the corresponding abelian subgroup of automorphisms of C. Thus, the action of t on C is an element of A. By the same argument, the action of every element of G_2 on C is also an element of A. So G/L is an abelian group which has a fixed-point-free, effective action on C. By Theorem 3.7, the quotient surface $(G/L)\backslash C$, and hence $G\backslash\mathfrak{H}^*$, has genus 1.

As $\mathbf{R}$ contains only finitely many groups, we may repeat this argument finitely many times to show that the group generated by all the groups in $\mathbf{R}$ is still in $\mathbf{R}$ and so is the required unique maximum element $L_{\max}$ and that $L_{\max}/L_{\min}$ is an abelian group. □

By Proposition 4.1, the $\mathcal{T}$-subgroups containing a given minimum Larcher subgroup $L = L_{\min}$ are precisely the groups between $L_{\min}$ and $L_{\max}$. Moreover, $L_{\max}$ is the genus-one subgroup of the normalizer of L of minimum index.

As the list of possible minimum Larcher subgroups and their normalizers is given in Table 2, it would be a straightforward task to find the

TABLE 2. $L_{\min}$ structure. The second column gives the structure of $L_{\min}(f)$ with notation described in the text. The column "Label" is the name of this group in [**8**]. The column "Cusps" gives the structure of the cusps of $L_{\min}(f)$, as a subgroup of $\Gamma_0(f)^+$, in partition notation. So 6^{12} means $L_{\min}(1)$ has 12 cusps of width 6 as a subgroup of $\mathrm{SL}\,(2,\mathbf{Z})$. The last two columns give the index of $L_{\min}(f)$ in $\Gamma_0(f)^+$ and $\mathrm{SL}\,(2,\mathbf{Z})$, respectively.

f	$L_{\min}$	Label	Cusps	$\lvert\Gamma_0(f)^+/L_{\min}\rvert$	I_Γ
1	$\pm H(1,6,6)$	$6F_1^1$	6^{12}	72	72
2	$\pm H(8,4,8)$	$8AH_2^1$	4^{16}	64	96
3	$\pm H(9,3,9)$	$9H_3^1$	3^{18}	54	108
5	$\pm H(5,2,10)$	$10F_5^1$	2^{12}	24	72
6	$\pm H(12,2,12)$	$12U_6^1$	2^{16}	32	96
11	$\pm H(11,1,11)$	$11A_{11}^1$	1^{10}	10	60
14	$\pm H(7,1,14)$	$7B_{14}^1$	1^{12}	12	72
15	$\pm H(15,1,15)$	$15A_{15}^1$	1^{16}	16	96

corresponding groups $L_{\max}$ by a direct computation. There is, however, a more illuminating way to characterize the groups $L_{\max}$. The idea is that each of the groups between $L_{\min}$ and $L_{\max}$ have the same space of weight two cusp forms, which is one-dimensional. Choosing a form in this space, normalized so that its first Fourier coefficient is equal to 1, allows us to characterize $L_{\max}$ as the subgroup of $\mathrm{SL}\,(2,\mathbf{R})$ which fixes this form. More interestingly, the resulting eight forms turn out to have a nice description. They are essentially the eight weight 2, multiplicative η-products [**14, 28, 29**], as we now show.

Recall that a sequence $\lambda = (\lambda_1, \dots, \lambda_k)$, $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k$, is called a partition of the number $N = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. The numbers $\lambda_1, \lambda_2, \dots, \lambda_k$ are called the parts of the partition λ. The number of parts of λ is called the length of the partition. For example, $\lambda = (4,4,8,8)$ is a partition of 24 into 4 parts. This partition may also be denoted $4^2 8^2$ in an obvious exponential notation. To each partition λ we can associate an η-product, $\prod_i \eta(\lambda_i z)$. So, for example, the η-product associated with the partition $4^2 8^2$ is $\eta(4z)^2\eta(8z)^2$.

Proposition 4.2. it Let $P_f(z)$, for $f = 1, 2, 3, 5, 6, 11, 14$ and 15, denote the following η-products:

$$\begin{aligned}
P_1(z) &= \eta(6z)^4 \\
P_2(z) &= \eta(4z)^2\eta(8z)^2 \\
P_3(z) &= \eta(3z)^2\eta(9z)^2 \\
P_5(z) &= \eta(2z)^2\eta(10z)^2 \\
P_6(z) &= \eta(2z)\eta(4z)\eta(6z)\eta(12z) \\
P_{11}(z) &= \eta(z)^2\eta(11z)^2 \\
P_{14}(z) &= \eta(z)\eta(2z)\eta(7z)\eta(14z) \\
P_{15}(z) &= \eta(z)\eta(3z)\eta(5z)\eta(15z).
\end{aligned}$$

These are the eight weight 2 *η-products with multiplicative coefficients, see* [**14, 28, 29**]. *If $\lambda(f)$ is the partition corresponding to P_f, let $r_f = \lambda(f)_1$ be the smallest part of λ, and let N_f be the product of the smallest and largest parts of $\lambda(f)$. For each value of f listed above, let $L_{\min}(f)$ and $L_{\max}(f)$ denote the corresponding groups listed in Tables* 2 *and* 3*, respectively. Then $L_{\max}(f)$ is the subgroup of* $\mathrm{SL}(2, \mathbf{R})$ *which fixes $P_f(z/r_f)$.*

Proof. By a result of Newman in [**35**], each of these η-products is a weight 2 cusp form on $\Gamma_0(N_f)$. Moreover, for each of these values of N_f, $\Gamma_0(N_f)$ is projectively torsion-free and genus-one. This follows, for example, from Propositions 1.40 and 1.43 of Shimura [**38**]. Now, the group $\Gamma_0(N_f)$ is, in the notation of this paper, the group $H(1, 1, N_f)$. So the product $P_f(z/r_f)$ is a weight 2 cusp form on the group $H(1, r_f, N_f/r_f)$. From Table 2, we see that each of these groups contains the group $L_{\min}(f)$, so that the product $P_f(z/r_f)$ is a weight 2 cusp form on $L_{\min}(f)$. By [**38**, Theorem 2.23], the spaces of weight 2 cusp forms on $L_{\min}(f)$ and $L_{\max}(f)$ are both one-dimensional, and so these two spaces coincide. In particular, $P_f(z/r_f)$ is a form on $L_{\max}(f)$. Moreover, $L_{\max}(f)$ is the subgroup of $\mathrm{SL}(2, \mathbf{R})$ which fixes $P_f(z/r_f)$ (as a weight 2 modular form). We can show this as follows. Knopp has shown that if a subgroup G of $\mathrm{SL}(2, \mathbf{R})$ has a modular form $f(z)$ which has even integral weight, has trivial multiplier system and which is not a polynomial, then G is a discrete subgroup of $\mathrm{SL}(2, \mathbf{R})$. See [**20**, Theorem 1]. Thus, if G is the fixing group of $P_f(z/r_f)$ as a weight 2 modular form, then, since $P_f(z/r_f)$ is not a polynomial, G must be a discrete subgroup of $\mathrm{SL}(2,\mathbf{R})$. By Siegel's theorem, the area of a fundamental domain of G is bounded from below and so the index of $L_{\max}(f)$ in G is finite. So by the Riemann-Hurwitz theorem the genus of G is either zero or one. So the genus of G must be one, since groups of genus-zero have no weight 2 cusp forms. The Riemann-Hurwitz formula then implies that G is projectively torsion-free and that G and $L_{\max}(f)$ have the same parabolic elements. So $L_{\min}(f)$ is the minimum Larcher subgroup of G by Lemma 2.6. This implies that $G = L_{\max}(f)$, since, by Proposition 4.1, $L_{\max}(f)$ is the maximum group with these properties. □

We can summarize these results in the following theorem.

Theorem 4.3. *There is a one-to-one correspondence between the following sets*:

- *The eight* $\mathrm{SL}(2,\mathbf{R})$ *conjugacy classes of projectively torsion-free, genus-one, minimum Larcher subgroups.*
- *The eight* $\mathrm{SL}(2,\mathbf{R})$ *conjugacy classes of projectively torsion-free, genus-one, congruence subgroups with a given minimum Larcher subgroup.*
- *The eight multiplicative weight two η-products.*
- *The eight groups $\Gamma_0(N)$ which are genus-one and projectively torsion-free.*

The problem remains of finding the $\mathrm{SL}(2,\mathbf{R})$ conjugacy classes of $\mathcal{T}$-subgroups. We now construct these groups and their conjugacy classes explicitly.

Proposition 4.4. *The generators of $L_{\max}$ over $L_{\min}$ are given in the second and third columns of Table* 4.

Proof. Using the standard transformation properties of $\eta(z)$, the matrices listed in Table 4 fix $P_f(z/r_f)$ for each value of f. Moreover, these matrices commute modulo $L_{\min}(f)$ and their images modulo $L_{\min}(f)$ generate an abelian group with the abelian invariants listed

TABLE 3. $L_{\max}$ structure. The first column gives the standard name of $L_{\max}(f)$ if it is known. See [**6**] for an explanation of the notation. If the structure of the group is not known, it is denoted M_f. The column "Rank" gives the rank of $L_{\max}$ as a free group. The column "Label" gives the name of $L_{\max}(f)$ in [**8**]. The column "Cusps" gives the structure of the cusps of $L_{\max}(f)$, as a subgroup of $\Gamma_0(f)^+$, in the same notation as Table 2. The last two columns give the index of $L_{\max}(f)$ in $\Gamma_0(f)^+$ and the index of $L_{\min}(f)$ in $L_{\max}(f)$, respectively.

f	$L_{\max}$	Rank	Label	Cusps	$\lvert\Gamma_0(f)^+/L_{\max}\rvert$	$\lvert L_{\max}/L_{\min}\rvert$
1	Γ'	2	$6A_1^1$	6^1	6	12
2	M_2	2	$4A_2^1$	4^1	4	16
3	M_3	3	$3A_3^1$	3^2	6	9
5	M_5	2	$2A_5^1$	2^1	2	12
6	M_6	2	$2A_6^1$	2^1	2	16
11	$11-$	3	$1A_{11}^1$	1^2	2	5
14	$14+2$	3	$1A_{14}^1$	1^2	2	6
15	$15+3$	3	$1A_{15}^1$	1^2	2	8

in the table. This gives an upper bound on the index of $L_{\max}(f)$ in $\Gamma_0(f)^+$ given by the sixth column of Table 3. But, as the groups are projectively torsion-free, a lower bound for the index is the LCM of the projective orders of the torsion elements of $\Gamma_0(f)^+$. These orders are given in Table 5. The LCMs of these projective orders are again given by the numbers in the sixth column of Table 3. Thus, the groups generated by the matrices in Table 4 over $L_{\min}(f)$ are precisely the groups $L_{\max}(f)$. $\square$

If G is a $\mathcal{T}$-subgroup, then by Corollary 3.15 it is conjugate in $\mathrm{SL}(2,\mathbf{R})$ to a group containing one of the eight groups $L_{\min}$ of Table 2. So, by Proposition 4.1, G is conjugate to a group between $L_{\min}$ and $L_{\max}$. Moreover, $L_{\max}/L_{\min}$ is abelian and, by Proposition 4.4, is generated by the images of the elements listed in Table 4. Thus, G corresponds to some subgroup of $L_{\max}/L_{\min}$. These subgroups and their generators over $L_{\min}$ can be explicitly computed which gives a list which contains all the $\mathcal{T}$-subgroups up to conjugacy. However, there remains the question of which of these groups are conjugate in $\mathrm{SL}(2,\mathbf{R})$.

By Lemma 2.8 and Corollary 3.14, if G_1 and G_2 contain, up to conjugacy, different minimum Larcher subgroups, then G_1 and G_2 are not

TABLE 4. Generators of $L_{\max}$ over $L_{\min}$. The orders of the images of the elements in $L_{\max}/L_{\min}$ are indicated by the subscripts. For the case of two generators, their images in $L_{\max}/L_{\min}$ generate cyclic subgroups with trivial intersection. The column labeled "Ab" gives the structure of $L_{\max}/L_{\min}$ as an abelian group. The column labeled "Auto" gives the induced action of the normalizer of $L_{\max}$ and $L_{\min}$ on the generators of $L_{\max}/L_{\min}$. The order of this automorphism group is given in the next column. The column labeled "N_1" gives the numbers of subgroups between $L_{\min}$ and $L_{\max}$, and the column labeled "N_2" gives the number up to conjugacy in $SL(2, \mathbf{R})$, or equivalently up to equivalence modulo the group generated by the transformations of column "Auto."

f	Generators over $L_{\min}$		Ab	Auto	$\|$Auto$\|$	N_1	N_2
1	$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}_2$	$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}_6$	$2^2 3^1$	$x \mapsto x, y \mapsto y^{-1}$ $x \mapsto xy^3, y \mapsto xy^4$	6	10	6
2	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}_4$	$\begin{pmatrix} 0 & 1 \\ -2 & 4 \end{pmatrix}_4$	4^2	$x \mapsto x^{-1}, y \mapsto y^{-1}$ $x \mapsto y^{-1}, y \mapsto x$	4	15	10
3	$\begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}_3$	$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}_3$	3^2	$x \mapsto x^{-1}, y \mapsto y^{-1}$ $x \mapsto xy^{-1}, y \mapsto x$	6	6	4
5	$\begin{pmatrix} 5 & 2 \\ 10 & 5 \end{pmatrix}_2$	$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}_6$	$2^2 3^1$	$x \mapsto x, y \mapsto y^{-1}$	2	10	10
6	$\begin{pmatrix} 15 & -2 \\ 24 & -3 \end{pmatrix}_2$	$\begin{pmatrix} 4 & 1 \\ 6 & 2 \end{pmatrix}_8$	$2^1 8^1$	$x \mapsto x, y \mapsto y^{-1}$	2	11	11
11	$\begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}_5$		5^1	$x \mapsto x^{-1}$	2	2	2
14	$\begin{pmatrix} 4 & 1 \\ 14 & 4 \end{pmatrix}_2$	$\begin{pmatrix} 5 & 1 \\ 14 & 3 \end{pmatrix}_3$	$2^1 3^1$	$x \mapsto x, y \mapsto y^{-1}$	2	4	4
15	$\begin{pmatrix} 6 & 1 \\ 15 & 3 \end{pmatrix}_8$		8^1	$x \mapsto x^{-1}$	2	4	4

conjugate in $\mathrm{SL}(2, \mathbf{R})$. Thus, it is only necessary to consider the case where G_1 and G_2 contains the same $L_{\min}$ as minimum Larcher subgroup. Suppose $g^{-1} G_1 g = G_2$ for some $g \in \mathrm{SL}(2, \mathbf{R})$. Then, by Lemma 2.8, $L_{\min} = L_{\min}(G_2) = L_{\min}(G_1)^g = L_{\min}^g$. Thus, g is an element of the normalizer of $L_{\min}$ in $\mathrm{SL}(2, \mathbf{R})$. Since $L_{\max}$ is a normal subgroup of this normalizer, it follows that g induces an automorphism of $L_{\max}/L_{\min}$. Hence, we have shown the following.

Theorem 4.5. *The $\mathcal{T}$-subgroups are classified up to conjugacy in $\mathrm{SL}(2, \mathbf{R})$ by the subgroup lattices of $L_{\max}/L_{\min}$ up to automorphisms induced by the normalizer of $L_{\min}$.*

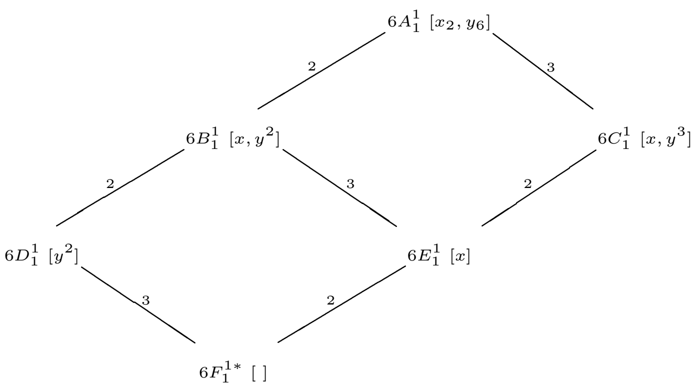
TABLE 5. Generators of $\Gamma_0(f)^+$. The last column gives the structure, in partition notation, of the conjugacy classes of cyclic subgroups in the corresponding projective groups. So $3^1 2^1$ means the projective group contains one conjugacy class of cyclic subgroups of order 3 and one of order 2.

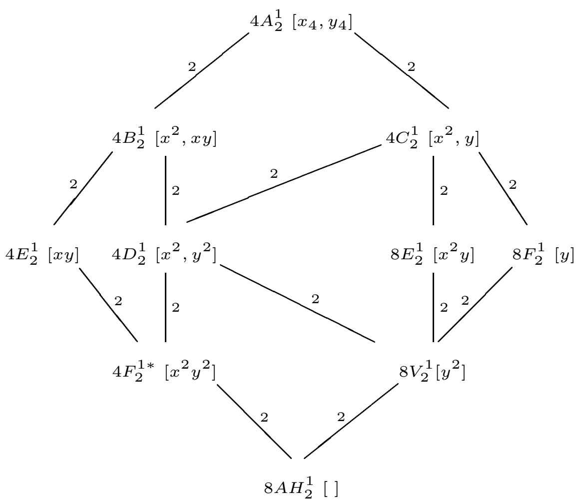
f	Generators				Torsion
1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$			$3^1 2^1$
2	$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix}$			$4^1 2^1$
3	$\begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix}$			$6^1 2^1$
5	$\begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}$	$\begin{pmatrix} 5 & -3 \\ 10 & -5 \end{pmatrix}$		2^3
6	$\begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix}$	$\begin{pmatrix} 3 & -2 \\ 6 & -3 \end{pmatrix}$		2^3
11	$\begin{pmatrix} 0 & -1 \\ 11 & 0 \end{pmatrix}$	$\begin{pmatrix} 11 & -4 \\ 33 & -11 \end{pmatrix}$	$\begin{pmatrix} 11 & -6 \\ 22 & -11 \end{pmatrix}$	$\begin{pmatrix} 22 & -15 \\ 33 & -22 \end{pmatrix}$	2^4
14	$\begin{pmatrix} 0 & -1 \\ 14 & 0 \end{pmatrix}$	$\begin{pmatrix} 7 & -2 \\ 28 & -7 \end{pmatrix}$	$\begin{pmatrix} 14 & -5 \\ 42 & -14 \end{pmatrix}$	$\begin{pmatrix} 7 & -4 \\ 14 & -7 \end{pmatrix}$	2^4
15	$\begin{pmatrix} 0 & -1 \\ 15 & 0 \end{pmatrix}$	$\begin{pmatrix} 5 & -2 \\ 15 & -5 \end{pmatrix}$	$\begin{pmatrix} 15 & -8 \\ 30 & -15 \end{pmatrix}$	$\begin{pmatrix} 10 & -7 \\ 15 & -10 \end{pmatrix}$	2^4

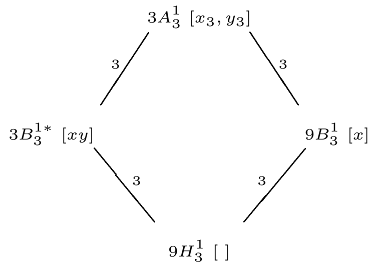
The generators of the normalizers are listed in Table 5. The induced action of these generators is given in Table 4 in the column "Auto." For $f = 1, 2$ and 3, there are two automorphisms corresponding to the two generators of $\Gamma_0(f)^+$. For the other values of f there is only one non-trivial induced transformation. The induced action on the generators of $L_{\max}/L_{\min}$ gives rise to an action on the subgroup lattices. This action is non-trivial only for $f = 1, 2$ and 3. The number of subgroups is given by the column "N_1" of Table 4 and the number modulo the induced action of conjugation in $\mathrm{SL}\,(2, \mathbf{R})$ is given in column "N_2" of Table 4.

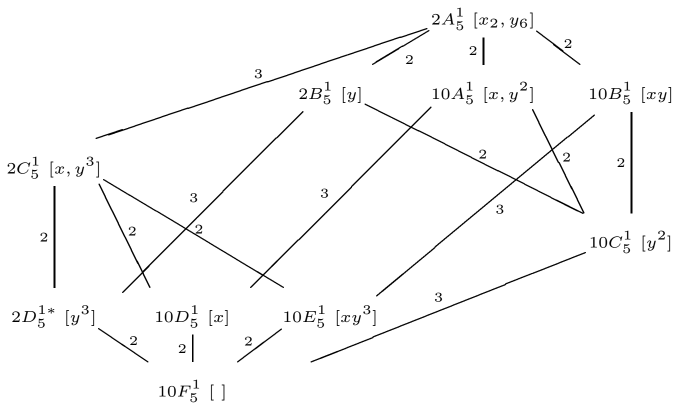
The explicit classification is given in the form of eight diagrams corresponding to the eight possible groups $L_{\min}$. Each diagram gives the structure of the subgroup lattice $L_{\max}/L_{\min}$ up to equivalence by conjugation by the normalizer of $L_{\min}$. The generators over $L_{\min}$ are listed as words in x and y (or just x) where x and y are the two generators in Table 4. For convenience, the order of each generator modulo $L_{\min}$ is indicated by subscripts. The corresponding group label from the tables of [**8**] is also listed. As the computations of [**8**] are independent of the results of this paper, the fact that the results are in agreement provides a useful check.

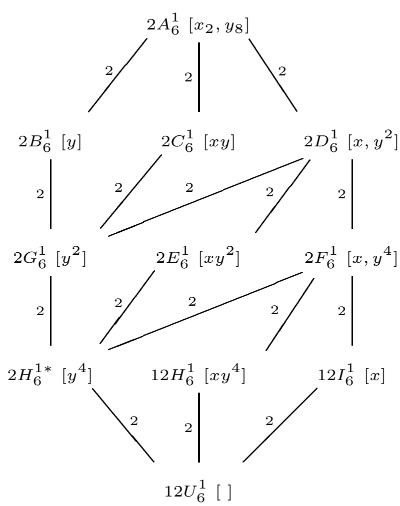
As noted above, for each minimum Larcher group $\pm H(p,q,r)$ in Table 2, there is a corresponding group $\Gamma_0(qr)$ which is genus-one and projectively torsion-free and which, up to conjugacy, has $\pm H(p,q,r)$ as its minimum Larcher subgroup. In each diagram, the group which is conjugate to $\Gamma_0(qr)$ is marked with a superscript asterisk.






$3A_3^1$ $[x_3, y_3]$
3
3
$3B_3^{1*}$ $[xy]$
$9B_3^1$ $[x]$
3
3
$9H_3^1$ [ ]


$2A_5^1$ $[x_2, y_6]$
3
2
2
2
$2B_5^1$ $[y]$
$10A_5^1$ $[x, y^2]$
$10B_5^1$ $[xy]$
$2C_5^1$ $[x, y^3]$
2
2
2
3
3
3
2
2
$10C_5^1$ $[y^2]$
$2D_5^{1*}$ $[y^3]$
$10D_5^1$ $[x]$
$10E_5^1$ $[xy^3]$
3
2
2
2
$10F_5^1$ [ ]


$2A_6^1$ $[x_2, y_8]$
2
2
2
$2B_6^1$ $[y]$
$2C_6^1$ $[xy]$
$2D_6^1$ $[x, y^2]$
2
2
2
2
2
$2G_6^1$ $[y^2]$
$2E_6^1$ $[xy^2]$
$2F_6^1$ $[x, y^4]$
2
2
2
2
2
$2H_6^{1*}$ $[y^4]$
$12H_6^1$ $[xy^4]$
$12I_6^1$ $[x]$
2
2
2
$12U_6^1$ [ ]

$$
\begin{tikzcd}
1A_{11}^{1*}\ [x_5] \arrow[d, "5"'] \\
11A_{11}^{1}\ [\ ]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 1A_{14}^{1}\ [x_2, y_3] \arrow[dl, dash, "2"'] \arrow[dr, dash, "3"] & \\
1B_{14}^{1*}\ [y] \arrow[dr, dash, "3"'] & & 7A_{14}^{1}\ [x] \arrow[dl, dash, "2"] \\
 & 7B_{14}^{1}\ [\ ] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
1A_{15}^{1}\ [x_8] \arrow[dr, dash, "2"] & \\
 & 1B_{15}^{1*}\ [x^2] \arrow[dl, dash, "2"'] \\
5A_{15}^{1}\ [x^4] \arrow[dr, dash, "2"] & \\
 & 15A_{15}^{1}\ [\ ]
\end{tikzcd}
$$

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