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PRIME FACTOR RINGS OF SKEW POLYNOMIAL RINGS OVER A COMMUTATIVE DEDEKIND DOMAIN

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ABSTRACT. This paper is concerned with prime factor rings of a skew polynomial ring over a commutative Dedekind domain. Let P be a non-zero prime ideal of a skew polynomial ring $R = D[x;\sigma]$, where D is a commutative Dedekind domain and σ is an automorphism of D. If P is not a minimal prime ideal of R, then R/P is a simple Artinian ring. If P is a minimal prime ideal of R, then there are two different types of P, namely, either $P = \mathfrak{p}[x;\sigma]$ or $P = P' \cap R$, where $\mathfrak{p}$ is a σ-prime ideal of D, P' is a prime ideal of $K[x;\sigma]$ and K is the quotient field of D. In the first case R/P is a hereditary prime ring and in the second case, it is shown that R/P is a hereditary prime ring if and only if $M^2 \not\supseteq P$ for any maximal ideal M of R. We give some examples of minimal prime ideals such that the factor rings are not hereditary or hereditary or Dedekind, respectively.

1. Introduction. Let D be a commutative Dedekind domain with its quotient field K, and let σ be an automorphism of D. We denote by $R = D[x;\sigma]$ the skew polynomial ring over D in an indeterminate x.

The aim of the paper is to study the structure of the prime factor ring R/P for any prime ideal P of R, which is one of the ways to investigate the structure of rings. If P is not a minimal prime ideal of R, then the Krull dimension of R/P is zero ([**15**]), that is, it is a simple Artinian ring. So we can restrict to the case P is a minimal prime ideal of R.

There are two types of minimal prime ideals P of R, that is, either $P = \mathfrak{p}[x;\sigma]$ or $P = P' \cap R$, where $\mathfrak{p}$ is a non-zero σ-prime ideal of D and P' is a non-zero prime ideal of $K[x;\sigma]$.

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In the first case R/P is always a hereditary prime ring. In the second case R/P is a hereditary prime ring if and only if $P \nsubseteq M^2$ for any maximal ideal M of R, which is motivated by [**9**] and he only considered in the case where P is principal generated by a monic polynomial and $\sigma = 1$ (note that in this case, P is a minimal prime ideal and see [**16**] and [**13**] for related papers). We give some examples of minimal prime ideals P such that R/P is not hereditary or hereditary or Dedekind, respectively, by using Gauss's integers $D = \mathbf{Z} \oplus \mathbf{Z}i$, where $\mathbf{Z}$ is the ring of integers.

We refer the readers to [**14, 15**] for some known terminologies not defined in this paper.

1. Notes on hereditary prime PI rings. Throughout this section, let R be a hereditary prime PI ring with the center C, and let Q be the quotient ring of R, which is a simple Artinian ring. It is well known that R is a classical C-order in Q and that C is a Dedekind domain (see [**15**, (13.9.16)]).

In this section, we will shortly discuss some relations between the maximal ideals of R and C, which are used in latter sections. For any R-ideal A, we use the following notation:

$$\begin{aligned}
(R:A)_l &= \{q \in Q \mid qA \subseteq R\},\ (R:A)_r = \{q \in Q \mid Aq \subseteq R\},\\
(A:A)_l &= \{q \in Q \mid qA \subseteq A\} = O_l(A),\ \text{the } \textit{left order} \text{ of } A,\\
(A:A)_r &= \{q \in Q \mid Aq \subseteq A\} = O_r(A),\ \text{the } \textit{right order} \text{ of} A,
\end{aligned}$$

and

$$A_v = \big(R:(R:A)_l\big)_r, \qquad {}_vA = \big(R:(R:A)_r\big)_l,$$

which are both R-ideals containing A. Note that $A_v = A = {}_vA$, because R is a hereditary prime ring. A finite set of distinct idempotent maximal ideals $M_1, \dots, M_m$ of R such that $O_r(M_1) = O_l(M_2), \dots, O_r(M_m) = O_l(M_1)$ is called a *cycle.* We will also consider an invertible maximal ideal to be a trivial case of a cycle.

It is well known that an ideal P is a maximal invertible ideal if and only if $P = M_1 \cap \cdots \cap M_m$, where $M_1, \dots, M_m$ is a cycle (see [**5**, (2.5) and (2.6)]). Let P be a maximal invertible ideal. Then

$C(P) = \{c \in R \mid c \text{ is regular} \bmod P\}$ is a regular Ore set, and we denote by R_P the localization of R at P (see [**11**, Proposition 2.7]). We denote by $\mathrm{Spec}\,(R)$ and $\text{Max-in}\,(R)$ the set of all prime ideals and the set of all maximal invertible ideals, respectively. For any ring S, $J(S)$ stands for Jacobson radical of S.

Lemma 1.1 (1) *Let* $P \in \text{Max-in}\,(R)$*, and let* $\mathfrak{p} = P \cap C$*. Then* $\mathfrak{p} \in \mathrm{Spec}\,(C)$*.*

(2) *C is a discrete rank one valuation ring if and only if $J(R)$ of R is the intersection of a cycle.*

Proof. (1) Let $P = M_1 \cap \cdots \cap M_m \in \text{Max-in}\,(R)$. If $m = 1$, then $\mathfrak{p} = P \cap C \in \mathrm{Spec}\,(C)$. If $m \geq 2$, then M_i are all idempotents. Set $\mathfrak{p} = M_1 \cap C$, then $M_1 \supseteq \mathfrak{p}R$, an invertible ideal. So

$$(R : M_2)_l = O_l(M_2) = O_r(M_1) = (R : M_1)_r \subseteq (R : \mathfrak{p}R)_r = (R : \mathfrak{p}R)_l$$

imply

$$M_2 = (M_2)_v = (R : (R : M_2)_l)_r \supseteq (R : (R : \mathfrak{p}R)_l)_r = \mathfrak{p}R.$$

Thus $M_2 \cap C = \mathfrak{p}$ follows. Continuing this process, we have $P \cap C = \mathfrak{p}$.

(2) Suppose that C is a discrete rank one valuation ring with $J(C) = \mathfrak{p}$, the unique maximal ideal. Then $J(R) \supseteq \mathfrak{p}R$ (see [**18**, (6.15)]). So $J(R)$ is invertible by [**5**, (4.13)]. Let $J(R) = P_1 \cap \cdots \cap P_k$, where $P_i \in \text{Max-in}\,(R)$. It suffices to prove that $k = 1$. We assume that $k \geq 2$. Then $R_{P_1} \supset R$ and $\mathbf{Z}(R_{P_1}) \supseteq \mathbf{Z}(R) = C$, where $\mathbf{Z}(R_{P_1})$ is the center of R_{P_1}, so that $\mathbf{Z}(R_{P_1}) = C$. Since R_{P_1} is a finitely generated C-module (see [**15**, (13.9.16)]), there is a $c \in C(P_1)$ with $R_{P_1} = cR_{P_1} \subseteq R$, a contradiction. Hence, $k = 1$ and so $J(R)$ is the intersection of a cycle.

Suppose that $J(R)$ is the intersection of a cycle. Then $\mathfrak{p} = J(R) \cap C \in \mathrm{Spec}\,(C)$ by (1). Let $\mathfrak{p}_1 \in \mathrm{Spec}\,(C)$. Then $\mathfrak{p}_1 R = J(R)^l$ for some $l \geq 1$ by [**5**, (2.1)] and the assumption. It follows that $\mathfrak{p}_1 \subseteq J(R) \cap C = \mathfrak{p}$ and so $\mathfrak{p}_1 = \mathfrak{p}$, that is, C is a discrete rank one valuation ring.

The following proposition is just a generalization of a Dedekind C-order to a hereditary prime PI ring (see, [**18**, (22.4)]).

Proposition 1.2. *Suppose that R is a hereditary prime* PI *ring. Then there is a one-to-one correspondence between* $\text{Max-in}\,(R)$ *and* $\text{Spec}\,(C)$*, which is given by:* $P \to \mathfrak{p} = P \cap C$*, where* $P \in \text{Max-in}\,(R)$*.*

Proof. Let $P \in \text{Max-in}\,(R)$. Then $\mathfrak{p} = P \cap C \in \text{Spec}\,(C)$ by Lemma 1.1. Conversely, let $\mathfrak{p} \in \text{Spec}\,(C)$. Then there is a maximal ideal M of R containing $\mathfrak{p}R$, an invertible ideal. So there is a $P \in \text{Max-in}\,(R)$ with $P \supseteq \mathfrak{p}R$ by [**5**, (2.4)]. This shows that $P \cap C = \mathfrak{p}$ by Lemma 1.1. To prove the correspondence is one-to-one, let $P, P_1 \in \text{Max-in}\,(R)$ with $P \cap C = \mathfrak{p} = P_1 \cap C$. Then $P_{\mathfrak{p}}, P_{1\mathfrak{p}} \in \text{Max-in}\,(R_{\mathfrak{p}})$ and $\mathbf{Z}(R_P) = C_{\mathfrak{p}}$, a discrete rank one valuation ring. Thus $P_{\mathfrak{p}} = J(R_{\mathfrak{p}}) = P_{1\mathfrak{p}}$ by Lemma 1.1 and so $P = P_{\mathfrak{p}} \cap R = P_{1\mathfrak{p}} \cap R = P_1$. Hence, the correspondence is one-to-one.

2. Prime factor rings of skew polynomial rings. Throughout this section, let D be a commutative Dedekind domain with its quotient field K, and let σ be an automorphism of D. We always assume that $D \neq K$ to avoid the trivial case. Let $R = D[x;\sigma]$ be a skew polynomial ring over D.

The aim of this section is to study the structure of the factor rings of R by minimal prime ideals. It is well known that R is a Noetherian maximal order in $K(x;\sigma)$, the quotient ring of $K[x;\sigma]$ and $\text{gl.dim}\, R = 2$ (see [**2**, Proposition 3.3] and [**15**, (7.5.3)]). We denote by $\text{Spec}_0(R) = \{P \in \text{Spec}(R) \mid P \cap D = (0)\}$. It is well known that there is a one-to-one correspondence between $\text{Spec}_0(R)$ and $\text{Spec}(K[x;\sigma])$, which is given by $P \to P' = PK[x;\sigma]$ and $P' \to P' \cap R$, where $P \in \text{Spec}_0(R)$ and $P' \in \text{Spec}(K[x;\sigma])$ (see [**7**, (9.22)]).

We start with the following easy proposition.

Proposition 2.1. (1) $\{\mathfrak{p}[x;\sigma],\ P \mid \mathfrak{p}$ *is a σ-prime ideal of D and* $P \in \text{Spec}_0(R)$ *with* $P \neq (0)\}$ *is the set of all minimal prime ideals of R.*

(2) *Let* $P \in \text{Spec}\,(R)$ *with* $P \neq (0)$*. Then P is invertible if and only if it is a minimal prime ideal of R.*

Proof. (1) Let P be a minimal prime ideal of R, and let $\mathfrak{p} = P \cap D$. If $\mathfrak{p} = (0)$, then $P \in \text{Spec}_0(R)$. If $\mathfrak{p} \neq (0)$, then there are two cases; namely, either $x \in P$ or $x \notin P$. Suppose that $x \in P$. Then $P = \mathfrak{p} + xR \supset xR$, a prime ideal, which is a contradiction. So $x \notin P$. Then $\mathfrak{p}$ is a σ-prime ideal of D and $\mathfrak{p}[x;\sigma]$ is a prime ideal of R. Hence $P = \mathfrak{p}[x;\sigma]$ follows.

Conversely, let $P \in \mathrm{Spec}_0(R)$. Then P is a minimal prime ideal of R, because $P' = PK[x;\sigma]$ is a maximal ideal as well as a minimal prime ideal of $K[x;\sigma]$. Let $P = \mathfrak{p}[x;\sigma]$, where $\mathfrak{p}$ is a σ-prime ideal. Then P is invertible, because $\mathfrak{p}$ is invertible and so P is a v-ideal. Hence P is a minimal prime ideal of R (see [**15**, (5.1.9)]).

(2) Let P be a prime and invertible ideal. Then it is a v-ideal and so it is a minimal prime ideal (see [**15**,(5.1.9)]).

Conversely, let P be a minimal prime ideal. If $P = \mathfrak{p}[x;\sigma]$, where $\mathfrak{p}$ is a σ-prime ideal of D, then P is invertible. If $P \in \mathrm{Spec}_0(R)$, with $P \neq (0)$ and $P^{\mathcal{P}} = PK[x;\sigma]$, then since any ideal of $K[x;\sigma]$ is a v-ideal and R is Noetherian, we have

$$\begin{aligned} P' = P'_v &= \big(K[x;\sigma] : (K[x;\sigma] : P')_l\big)_r = \big(K[x;\sigma] : K[x;\sigma](R:P)_l\big)_r \\ &= \big(R : (R:P)_l\big)_r K[x;\sigma] = P_v K[x;\sigma]. \end{aligned}$$

Thus, $P = P' \cap R = P_v$ follows and similarly $P = {}_vP$. Hence, P is invertible by [**4**, page 324].

Proposition 2.2. (1) *Let P be a minimal prime ideal of R with $P = \mathfrak{p}[x;\sigma]$, where $\mathfrak{p}$ is a σ-prime ideal of D. Then R/P is a hereditary prime ring. In particular, R/P is a Dedekind prime ring if and only if $\mathfrak{p} \in \mathrm{Spec}\,(D)$.*

(2) *Suppose that σ is of infinite order. Then $P = xR$ is the only minimal prime ideal of R in* $\mathrm{Spec}_0(R)$ *and R/P is a Dedekind prime ring.*

Proof. (1) The first statement follows from [**15**, (7.5.3)]. If $\mathfrak{p} \in \mathrm{Spec}\,(D)$, then $(R/P) \cong (D/\mathfrak{p})[x;\sigma]$ is a principal ideal ring so that R/P is a Dedekind prime ring. If $\mathfrak{p} \notin \mathrm{Spec}\,(D)$, then there is a maximal ideal $\mathfrak{m}$ of D with $\mathfrak{m} \supset \mathfrak{p}$ and $\mathfrak{p} = \mathfrak{m} \cap \sigma(\mathfrak{m}) \cap \cdots \cap \sigma^n(\mathfrak{m})$ for some natural number $n \geq 1$. Set $M = \mathfrak{m} + xR$, a maximal ideal of R. Then $M = M^2 + P$, because $\mathfrak{m}^2 + \mathfrak{p} = \mathfrak{m}$. Thus, M/P is idempotent and R/P is not Dedekind.

(2) Let $P = xR$. Then P is the only minimal prime ideal of R in $\mathrm{Spec}_0(R)$ by [**10**, Theorem 2] and R/P is a Dedekind prime ring because $(R/P) \cong D$.

Because of Propositions 2.1 and 2.2, we may assume that σ is of finite order to study the hereditariness of R/P. So in the remainder of this section, we may assume that σ is of finite order, say, n.

It is well known that K is separable over $K_\sigma = \{k \in K \mid \sigma(k) = k\}$ and $[K : K_\sigma] = n$ (see [**1**, Theorems 14 and 15]). Furthermore, $D_\sigma = \{d \in D \mid \sigma(d) = d\}$ is also Dedekind domain by [**6**, (36.1) and (37.2)] and D is a finitely generated D_σ-module by [**20**, Corollary 1, page 265]. Since the center $\mathbf{Z}(R)$ of R is $D_\sigma[x^n]$, it follows that R is a finitely generated C-module, where $C = D_\sigma[x^n]$. Thus R is a classical C-order in $K(x;\sigma)$ and so R is a prime PI ring with $\mathcal{K}(R) = \dim(R) = 2$ (see [**15**, (6.4.8) and (6.5.4.)]), where $\mathcal{K}(R)$ is the Krull dimension of R and $\dim(R)$ is the classical Krull dimension of R.

The following lemma is due to [**19**, (1.6.27)].

Lemma 2.3. *Let σ be an automorphism of K with order n. Then*:

(1) *there is a one-to-one correspondence between* $\mathrm{Spec}\,(K[x;\sigma])$ *and* $\mathrm{Spec}\,(K_\sigma[x^n])$, *which is given by $P' \to \mathfrak{p}' = P' \cap K_\sigma[x^n]$, where $P' \in \mathrm{Spec}\,(K[x;\sigma])$.*

(2) *If $P' = xK[x;\sigma]$, then $\mathfrak{p}' = x^n K_\sigma[x^n]$ and $\mathfrak{p}'K[x;\sigma] = P'^n$. If $P' \neq xK[x;\sigma]$, then $\mathfrak{p}' = f(x^n)K_\sigma[x^n]$ for some irreducible polynomial $f(x^n)$ in $K_\sigma[x^n]$ different from x^n and $\mathfrak{p}'K[x;\sigma] = P'$.*

Lemma 2.4. *Let σ be an automorphism of D with order n. Then*:

(1) *there is a one-to-one correspondence between* $\mathrm{Spec}_0(R)$ *and* $\mathrm{Spec}_0(C)$, *which is given by $P \to \mathfrak{p} = P \cap C$, where $P \in \mathrm{Spec}_0(R)$.*

(2) *If $P = xR$, then $P^n = \mathfrak{p}R$, where $\mathfrak{p} = P \cap C$. If $P \neq xR$, then $P = \mathfrak{p}R$, where $\mathfrak{p} = P \cap C$.*

Proof. (1) Let $P \in \mathrm{Spec}_0(R)$. Then it is clear that $\mathfrak{p} = P \cap C \in \mathrm{Spec}_0(C)$. Conversely, let $\mathfrak{p} \in \mathrm{Spec}_0(C)$. If $\mathfrak{p} \neq x^n C$, then $P = \mathfrak{p}K[x;\sigma] \cap R \in \mathrm{Spec}_0(R)$ by Lemma 2.3 and [**7**, (9.22)], and so $\mathfrak{p} \subseteq \mathfrak{p}_1 = P \cap C \in \mathrm{Spec}_0(C)$. Hence $\mathfrak{p} = \mathfrak{p}_1$ by Proposition 2.1.

If $\mathfrak{p} = x^n C$, then $P = xR \in \mathrm{Spec}_0(R)$ with $\mathfrak{p} = P \cap C$. Hence the correspondence is onto.

To prove the correspondence is one to one, let P and $P_1 \in \mathrm{Spec}_0(R)$ with $P \cap C = \mathfrak{p} = P_1 \cap C$. We may assume that $P \neq xR$ and $P_1 \neq xR$. Then $PK[x;\sigma]$ and $P_1K[x;\sigma]$ both contain $\mathfrak{p}K[x;\sigma] \in \mathrm{Spec}\,(K[x;\sigma])$ and so $PK[x;\sigma] = \mathfrak{p}K[x;\sigma] = P_1K[x;\sigma]$ follows. Hence, $P = PK[x;\sigma] \cap R = P_1$.

(2) $P \in \mathrm{Spec}_0(R)$ with $\mathfrak{p} = P \cap C$. If $P = xR$, then $P^n = \mathfrak{p}R$ where $\mathfrak{p} = x^n C$. Suppose that $P \neq xR$. Let P_1 be an invertible prime ideal containing $\mathfrak{p}R$. By Proposition 2.1, P_1 is a minimal prime ideal of R. So either $P_1 = \mathfrak{p}_1[x;\sigma]$, where $\mathfrak{p}_1$ is a σ-prime ideal of D or $P_1 \in \mathrm{Spec}_0(R)$ by Proposition 2.1. If $P_1 = \mathfrak{p}_1[x;\sigma]$, then $P_1 \cap C = (\mathfrak{p}_1)_\sigma[x^n]$, a minimal prime ideal of $C[x^n]$, where $(\mathfrak{p}_1)_\sigma = \mathfrak{p}_1 \cap D_\sigma$, containing $\mathfrak{p}$ so that $\mathfrak{p} = (\mathfrak{p}_1)_\sigma[x^n]$, a contradiction, because $P \in \mathrm{Spec}_0(R)$. Hence, $P_1 \in \mathrm{Spec}_0(R)$. It follows that $\mathfrak{p}_1 = P_1 \cap C \supseteq \mathfrak{p}$ and so $\mathfrak{p}_1 = \mathfrak{p}$. Hence, $P = P_1$ by (1). Since the invertible ideal $\mathfrak{p}R$ is a finite product of invertible prime ideals (see [**4**, Theorem 1.6 and Proposition 2.3]), we have $\mathfrak{p}R = P^e$ for some $e \geq 1$. Then $\mathfrak{p}K[x;\sigma] = P^eK[x;\sigma] = P'^e$ implies $e = 1$. Hence, $P = \mathfrak{p}R$ follows.

Lemma 2.5. *Let $P \in \mathrm{Spec}_0(R)$ with $P \neq xR$. Then $P_\mathfrak{n}$ is principal generated by a central polynomial in $C_\mathfrak{n}$ for any $\mathfrak{n} \in \mathrm{Spec}\,(D_\sigma)$.*

Proof. Let $\mathfrak{p} = P \cap C$. Then $\mathfrak{p}_\mathfrak{n}$ is principal by [**12**, (3.1)], because $C_\mathfrak{n} = (D_\sigma)_\mathfrak{n}[x^n]$ and $(D_\sigma)_\mathfrak{n}$ is a discrete rank one valuation ring. Hence, $P_\mathfrak{n}$ is principal generated by a central element in $C_\mathfrak{n}$ by Lemma 2.4.

Lemma 2.6. *Let $P \in \mathrm{Spec}_0(R)$ with $P \neq xR$. Then the following are equivalent*:

(1) *$P \nsubseteq M^2$ for any maximal ideal M of R.*

(2) *$P_\mathfrak{n} \nsubseteq (M_\mathfrak{n})^2$ for any $\mathfrak{n} \in \mathrm{Spec}\,(D_\sigma)$ and for any maximal ideal M of R with $M \cap (D_\sigma \setminus \mathfrak{n}) = \varnothing$.*

Proof. (1) $\Rightarrow$ (2). Suppose that there is an $\mathfrak{n} \in \mathrm{Spec}\,(D_\sigma)$ and a maximal ideal M of R with $M \cap (D_\sigma \setminus \mathfrak{n}) = \varnothing$ satisfying $P_\mathfrak{n} \subseteq (M_\mathfrak{n})^2$. Then there is a $c \in D_\sigma \setminus \mathfrak{n}$ with $cP \subseteq M^2 \subseteq M$, which implies $P \subseteq M$

and $cR + M = R$. Hence, $P = (cR + M)P \subseteq M^2$, a contradiction. Hence, for any $\mathfrak{n} \in \mathrm{Spec}\,(D_\sigma)$ and any maximal ideal M of R with $M \cap (D_\sigma \setminus \mathfrak{n}) = \varnothing$, $P_\mathfrak{n} \nsubseteq (M_\mathfrak{n})^2$.

(2) $\Rightarrow$ (1). Suppose that there is a maximal ideal M of R with $P \subseteq M^2$. Then $M \cap D \neq (0)$ by Proposition 2.1 and so $\mathfrak{n} = M \cap D_\sigma \neq (0)$, which is a prime ideal of D_σ with $M \cap (D_\sigma \setminus \mathfrak{n}) = \varnothing$. By the assumption, $P_\mathfrak{n} \nsubseteq (M^2)_\mathfrak{n} = M_\mathfrak{n}^2$, a contradiction. Hence, $P \nsubseteq M^2$ for any maximal ideal M of R.

Lemma 2.7. *Let $P \in \mathrm{Spec}_0(R)$ with $P \neq xR$ and $\mathfrak{p} = P \cap C$. Then $\mathbf{Z}(R/P) = (C/\mathfrak{p})$.*

Proof. Since $\mathbf{Z}(R/P) = \mathbf{Z}(K[x;\sigma]/P') \cap (R/P)$, it suffices to prove that $\mathbf{Z}(K[x;\sigma]/P') = (K_\sigma[x^n]/\mathfrak{p}')$, where $\mathfrak{p}' = K_\sigma[x^n] \cap P'$. We set $\overline{K[x;\sigma]} = K[x;\sigma]/P'$. It is clear that $\mathbf{Z}(\overline{K[x;\sigma]}) \supseteq (K_\sigma[x^n]/\mathfrak{p}')$. To prove the converse inclusion, let $f(x^n) \in K_\sigma[x^n]$ be a monic polynomial with $P' = f(x^n)K[x;\sigma]$ and $\deg f(x^n) = nl$. Write

$$f(x^n) = x^{nl} + a_{l-1}x^{n(l-1)} + \cdots + a_1x^n + a_0, \quad \text{where } a_i \in K_\sigma.$$

Suppose that $a_0 = 0$. Then $f(x^n) = h(x^n)x^n$, where $h(x^n) = x^{n(l-1)} + \cdots + a_1$, shows that $P' \subseteq xK[x;\sigma]$ and so $P' = xK[x;\sigma]$, a contradiction. So we may assume that $a_0 \neq 0$. Note that

$$\overline{K[x;\sigma]} \cong K \oplus K\overline{x} \oplus \cdots \oplus K\overline{x}^{nl-1},$$

as a ring and that

$$\overline{x}^{nl} = -\big(a_{l-1}\overline{x}^{n(l-1)} + \cdots + a_1\overline{x}^n + a_0\big).$$

Let $\overline{g(x)} = b_{nl-1}\overline{x}^{nl-1} + \cdots + b_1\overline{x} + b_0$ be any element in $\mathbf{Z}(\overline{K[x;\sigma]})$, where $b_i \in K$. Then, for any $k \in K$, $k\overline{g(x)} = \overline{g(x)}k$ implies $b_i\sigma^i(k) = b_ik$ for any i, $0 \leq i \leq nl-1$. Suppose that there is an i with $b_i \neq 0$ and $i = nj + s$ $(1 \leq s < n)$. Then $b_i\sigma^s(k) = b_ik$ and so $\sigma^s(k) = k$ for all $k \in K$, a contradiction. Thus, if $b_i \neq 0$, then $i = nj$, $0 \leq j \leq l-1$. Next,

$$\begin{aligned}\overline{g(x)}\overline{x} = b_0\overline{x} + b_1\overline{x}^2 + \cdots + b_{nl-2}\overline{x}^{nl-1} \\ + b_{nl-1}\big(-a_{l-1}\overline{x}^{n(l-1)} - \cdots - a_1\overline{x}^n - a_0\big)\end{aligned}$$

and

$$\overline{x}\overline{g(x)} = \sigma(b_0)\overline{x} + \sigma(b_1)\overline{x}^2 + \cdots + \sigma(b_{nl-2})\overline{x}^{nl-1} \\ + \sigma(b_{nl-1})\big(- a_{l-1}\overline{x}^{n(l-1)} - \cdots - a_1\overline{x}^n - a_0\big).$$

Since $\overline{x}\overline{g(x)} = \overline{g(x)}\overline{x}$, comparing the coefficients, we have $\sigma(b_{nl-1}) = b_{nl-1}$, that is, $b_{nl-1} \in K_\sigma$ and so $\sigma(b_i) = b_i$ for all $0 \leq i \leq nl-2$. Thus, we have

$$\overline{g(x)} = b_0 + b_n\overline{x}^n + \cdots + b_{n(l-1)}\overline{x}^{n(l-1)} \quad \text{and} \quad b_i \in K_\sigma.$$

Hence, $\overline{g(x)} \in (K_\sigma[x^n]/\mathfrak{p}')$.

Let $P \in \mathrm{Spec}_0(R)$ with $P \neq xR$. Since $\mathbf{Z}(R/P) = (C/\mathfrak{p}) \supseteq D_\sigma$ naturally, it follows from [**18**, (3.24)] that R/P is a hereditary prime ring if and only if $(R/P)_\mathfrak{n}(\cong R_\mathfrak{n}/P_\mathfrak{n})$ is a hereditary prime ring for any $\mathfrak{n} \in \mathrm{Spec}\,(D_\sigma)$.

Let $\mathfrak{m}$ be any maximal ideal of C with $\mathfrak{m} \supset \mathfrak{p}$. By lying over and going up theorems (see [**15**, (10.2.9) and (10.2.10)]), there is a maximal ideal M of R with $M \cap C = \mathfrak{m}$ and $M \supset P$. Set $J = \cap\{M \mid M \text{ is a maximal ideal of } R \text{ with } \mathfrak{m} = M \cap C\}$. Since $\dim(R/J) = \mathcal{K}(R/J) < \mathcal{K}(R) = 2$, M/J is a minimal prime ideal of R/J and J is a finite intersection of those M's, that is, $J = M_1 \cap \cdots \cap M_k$ (see [**15**, (3.2.2)]). Thus, we have the following lemma.

Lemma 2.8. *With the notation above, the following hold*:

(1) $P \not\subseteq M_i^2$ *if and only if* $P_\mathfrak{m} \not\subseteq M_{i\mathfrak{m}}^2$.

(2) $M_i \supset M_i^2$ *for any* i $(1 \leq i \leq k)$.

(3) $\mathrm{gl.dim}\, R_\mathfrak{m} = 2$ *and* $J(R_\mathfrak{m}) = M_{1\mathfrak{m}} \cap \cdots \cap M_{k\mathfrak{m}}$.

Proof. (1) This is proved in the same way as in [**13**, Lemma 2].

(2) Set $M = M_i$ and $\mathfrak{m}_0 = M \cap D \neq (0)$, because $M \supset P$. If $x \in M$, then $M = \mathfrak{m}_0 + xR$ and $\mathfrak{m}_0$ is a maximal ideal of D with $\mathfrak{m}_0 \supset \mathfrak{m}_0^2$. Thus, $M^2 \subseteq \mathfrak{m}_0^2 + xR \subset \mathfrak{m}_0 + xR = M$. If $x \notin M$, then $\mathfrak{m}_0$ is a σ-prime ideal and $D/\mathfrak{m}_0$ is a semi-simple Artinian ring. Since $M \supseteq \mathfrak{m}_0[x;\sigma]$, we have

$$\widetilde{M} = \big(M/\mathfrak{m}_0[x;\sigma]\big) \subset \widetilde{R} = \big(R/\mathfrak{m}_0[x;\sigma]\big) \cong (D/\mathfrak{m}_0)[x;\widetilde{\sigma}],$$

which is hereditary by [**15**, (7.5.3)]. Since $\widetilde{x} \notin \widetilde{M}$, $\widetilde{M}$ is principal by [**3**, Lemma 2.6]. So $(\widetilde{M})^2 \subset \widetilde{M}$, and thus $M^2 \subset M$ follows.

(3) It follows that $2 = \text{gl.dim}\, R \geq \text{gl.dim}\, R_{\mathfrak{m}}$. If $\text{gl.dim}\, R_{\mathfrak{m}} \leq 1$, then $R_{\mathfrak{m}}$ is hereditary, which is implies $M_{\mathfrak{m}} = P_{\mathfrak{m}}$. Hence, $M = M_{\mathfrak{m}} \cap R = P_{\mathfrak{m}} \cap R = P$, a contradiction. Hence, $\text{gl.dim}\, R_{\mathfrak{m}} = 2$. Since $R_{\mathfrak{m}}$ is a PI ring with the maximal ideals $M_{1\mathfrak{m}}, \ldots, M_{k\mathfrak{m}}$, it is clear that $J(R_{\mathfrak{m}}) = M_{1\mathfrak{m}} \cap \cdots \cap M_{k\mathfrak{m}}$.

Proposition 2.9. *Let σ be an automorphism of D with order n, and let $P \in \text{Spec}_0(R)$ with $P \neq xR$. Then, $\overline{R} = R/P$ is a hereditary prime ring if and only if $P \nsubseteq M^2$ for any maximal ideal M of R.*

Proof. First note that $\mathbf{Z}(\overline{R}) = \overline{C} = (C/\mathfrak{p})$ by Lemma 2.7, where $\mathfrak{p} = P \cap C$. Suppose that $\overline{R}$ is a hereditary prime ring. Then $\overline{C}$ is a Dedekind domain (see [**15**, (13.9.16)]. Let M be a maximal ideal of R. If $P \nsubseteq M$, then $P \nsubseteq M^2$. So we may assume that $P \subseteq M$. In order to prove $P \nsubseteq M^2$, we may assume that P is a principal generated by a central element by Lemmas 2.5 and 2.6, and let $\mathfrak{m} = M \cap C$, a maximal ideal of C properly containing $\mathfrak{p}$. Then there are a finite number of maximal ideals $M_1, \ldots, M_k$ of R lying over $\mathfrak{m}$ such that $J(\overline{R}_{\overline{\mathfrak{m}}}) = (\overline{M}_1)_{\overline{\mathfrak{m}}} \cap \cdots \cap (\overline{M}_k)_{\overline{\mathfrak{m}}}$ and $\overline{C}_{\overline{\mathfrak{m}}}$ is a discrete rank one valuation ring, where $M = M_1$, $\overline{M_i} = M_i/P$ and $\overline{\mathfrak{m}} = (\mathfrak{m}/\mathfrak{p})$. If $k = 1$, then $\overline{R}_{\overline{\mathfrak{m}}}$ is a local Dedekind prime ring so that it is a principal ideal ring. So $\overline{M}_{\overline{\mathfrak{m}}} = \overline{a}\overline{R}_{\overline{\mathfrak{m}}}$ for some $a \in M_{\mathfrak{m}}$ and $M_{\mathfrak{m}} = aR_{\mathfrak{m}} + P_{\mathfrak{m}}$. Suppose that $P \subseteq M^2$. Then $M_{\mathfrak{m}} = aR_{\mathfrak{m}} + P_{\mathfrak{m}} \subseteq aR_{\mathfrak{m}} + M_{\mathfrak{m}}J(R_{\mathfrak{m}}) \subseteq M_{\mathfrak{m}}$. Hence $M_{\mathfrak{m}} = aR_{\mathfrak{m}}$ by Nakayama's lemma, which is invertible. It follows from [**8**, Proposition 1.3] that $R_{\mathfrak{m}}$ is a principal ideal ring. So $\text{gl.dim}\, R_{\mathfrak{m}} \leq 1$, which contradicts Lemma 2.8. Hence $P \nsubseteq M^2$. If $k \geq 2$, then $\overline{M}_{1\overline{\mathfrak{m}}}, \ldots, \overline{M}_{k\overline{\mathfrak{m}}}$ is a cycle by Lemma 1.1, because $\overline{C}_{\overline{\mathfrak{m}}}$ is a discrete rank one valuation ring. Suppose that $P \subseteq M^2$. Then $\overline{M}_{\overline{\mathfrak{m}}} = \overline{M}^2_{\overline{\mathfrak{m}}}$ implies

$$M_{\mathfrak{m}} = (M_{\mathfrak{m}})^2 + P_{\mathfrak{m}} = (M_{\mathfrak{m}})^2 = M^2_{\mathfrak{m}}.$$

Let $\mathfrak{m}_i$ be another maximal ideal of C. Then $M_{\mathfrak{m}_i} = R_{\mathfrak{m}_i}$ and so $R_{\mathfrak{m}_i} = (M_{\mathfrak{m}_i})^2 = (M^2)_{\mathfrak{m}_i}$. Hence, $M = \cap M_{\mathfrak{m}_j} = \cap (M^2)_{\mathfrak{m}_j} = M^2$, which contradicts Lemma 2.8, where $\mathfrak{m}_j$ runs over all maximal ideals of C. Hence, $P \nsubseteq M^2$.

Conversely, suppose that $P \nsubseteq M^2$ for any maximal ideal M of R. Let $\mathfrak{m}$ be a maximal ideal of C with $\mathfrak{m} \supset \mathfrak{p}$ and $\mathfrak{n} = \mathfrak{m} \cap D_{\sigma}$, a maximal ideal of D_σ. Since $(R_{\mathfrak{n}})_{\mathfrak{m}_{\mathfrak{n}}} = R_{\mathfrak{m}}$ and $(P_{\mathfrak{n}})_{\mathfrak{m}_{\mathfrak{n}}} = P_{\mathfrak{m}}$, we may suppose that P is principal by Lemmas 2.5 and 2.6. It follows from Lemma 2.8 and [**13**, Lemma 3] that $\overline{R}_{\overline{\mathfrak{m}}} = R_{\mathfrak{m}}/P_{\mathfrak{m}}$ is a hereditary prime ring. Hence $\overline{R}$ is a hereditary prime ring by [**18**, (3.24)].

Summarizing Propositions 2.1, 2.2, and 2.9, we have the following theorem:

Theorem 2.10. *Let $R = D[x;\sigma]$ be a skew polynomial ring over a commutative Dedekind domain, where σ is an automorphism of D, and let P be a prime ideal of R. Then*:

(1) *P is a minimal prime ideal of R if and only if either $P = \mathfrak{p}[x;\sigma]$, where $\mathfrak{p}$ is either a non-zero σ-prime ideal of D or $P \in \mathrm{Spec}_0(R)$ with $P \neq (0)$.*

(2) *If $P = \mathfrak{p}[x;\sigma]$, where $\mathfrak{p}$ is a non-zero σ-prime ideal of D, then R/P is a hereditary prime ring. In particular, R/P is a Dedekind prime ring if and only if $\mathfrak{p} \in \mathrm{Spec}\,(D)$.*

(3) *If $P \in \mathrm{Spec}_0(R)$ with $P = xR$, then R/P is a Dedekind prime ring. In particular, if the order of σ is infinite, then $P = xR$ is the only minimal prime ideal belonging to $\mathrm{Spec}_0(R)$.*

(4) *If $P \in \mathrm{Spec}_0(R)$ with $P \neq xR$ and $P \neq (0)$, then R/P is a hereditary prime ring if and only if $P \nsubseteq M^2$ for any maximal ideal M of R.*

3. Examples. Let $D = \mathbf{Z} \oplus \mathbf{Z}i$ be the Gauss integers, where $i^2 = -1$, and let σ be the automorphism of D with $\sigma(a + bi) = a - bi$, where $a, b \in \mathbf{Z}$, the ring of integers.

In this section, we will give some examples of minimal prime ideals of a skew polynomial ring over D, in order to display some of the various phenomena in Section 2.

Let p be a prime number. Then the following properties are well known in the elementary number theory:

(1) If $p = 2$, then $2D = (1 + i)^2 D$ and $(1 + i)D$ is a prime ideal.

(2) If $p = 4n + 1$, then $pD = \pi\sigma(\pi)D$ for some prime element π with $\pi D + \sigma(\pi)D = D$.

(3) If $p = 4n + 3$, then pD is a prime ideal of R.

We let $R = D[x;\sigma]$ be the skew polynomial ring, $P = (x^2+p)R \in \mathrm{Spec}_0(R)$ and $\overline{R} = R/P$.

Lemma 3.1. *If $p = 2$, then $\overline{R}$ is not a hereditary prime ring.*

Proof. Let $M = (1+i)D + xR$ be a maximal ideal of R. Then $M^2 = 2D \oplus (1+i)Dx \oplus x^2R$ and so $M^2 \ni x^2 + 2$. Hence $\overline{R}$ is not a hereditary prime ring by Theorem 2.10.

In what follows, we suppose that $p \neq 2$ unless otherwise stated. Let M be maximal ideal containing $x^2 + p$. First we will study in the case where $M \ni x$. Then $M = \pi D + xR$ for some prime element π of D with either $pD = \pi\sigma(\pi)D$ and $\pi D + \sigma(\pi)D = D$ if $p = 4n+1$ or $pD = \pi D$ if $p = 4n+3$.

Lemma 3.2. *Let $M = \pi D + xR$ be a maximal ideal of R with $M \supset P$. Then*:

(1) *If $p = 4n+1$, then $M^2 \not\ni x^2 + p$ and $M = M^2 + P$, that is, $\overline{M}$ is idempotent.*

(2) *If $p = 4n+3$, then $M^2 \not\ni x^2 + p$ and $M \supset M^2 + P$, that is, $\overline{M}$ is not idempotent.*

Proof. (1) It follows that $M^2 = \pi^2 D + xR$, because $D = \pi D + \sigma(\pi)D$. Suppose that $x^2 + p \in M^2$. Then $p \in \pi^2 D$ and so $\sigma(\pi)D = \pi D$ follows, a contradiction. Hence $M^2 \not\ni x^2 + p$. Since $\pi D = M \cap D \supseteq (M^2+P) \cap D \supseteq M^2 \cap D = \pi^2 D$, we have either $(M^2+P)\cap D = \pi D$ or $(M^2+P)\cap D = \pi^2 D$. If $(M^2+P)\cap D = \pi^2 D$, then $M^2 + P \ni \pi^2 + x^2 - (x^2+p) = \pi^2 - p$, which implies $p \in \pi^2 D$, a contradiction as above. So $(M^2+P)\cap D = \pi D$, and thus $M^2 + P \supseteq \pi D + xR = M$. Hence, $M = M^2 + P$ follows.

(2) It is easy to see that $M^2 \not\ni x^2 + p$ since $M^2 = p^2D + pxR + x^2R$. Suppose that $M = M^2 + P$. Then $x \in M^2 + P$ and write $x = p^2 d + pxf(x) + x^2 g(x) + (x^2+p)h(x)$, where $d \in D$, $f(x) = \sum f_i x^i$, $g(x) = \sum g_i x^i$ and $h(x) = \sum h_i x^i$, where $f_i, g_i, h_i \in D$. Then $1 = p\sigma(f_0) + ph_1$, a contradiction. Hence, $M \supset M^2 + P$.

Next, we will study a maximal ideal M with $M \not\ni x$.

Lemma 3.3. *Let M be a maximal ideal of R with $M \ni x^2 + p$ and $M \not\ni x$. Then*:

(1) *There is a prime number q ($\neq p$) and a monic polynomial $f(x) \in M$ with $M = f(x)R + qR$.*

(2) *If $\deg f(x) \geq 2$, then $M = P + qR$, $M^2 \not\ni x^2 + p$ and $\overline{M}$ is not idempotent.*

(3) *If $\deg f(x) = 1$, then $q = 2$ and either $M = (x + 1)R + 2R$ or $M = (x + i)R + 2R$.*

Proof. (1) Since $M \cap D$ is a non-zero σ-prime ideal, there is a prime number q with $M \cap D = qD$. Set $\widetilde{R} = R/qD[x; \sigma] = \widetilde{D}[x; \widetilde{\sigma}]$, where $\widetilde{D} = D/qD = (\mathbf{Z}/q\mathbf{Z}) \oplus (\mathbf{Z}/q\mathbf{Z})i$, a semi-simple Artinian ring. Since $\widetilde{M} = M/qD[x; \sigma] \not\ni \widetilde{x}$, it follows from [**3**, Lemma 2.6] that $\widetilde{M} = \widetilde{f(x)}\widetilde{R}$ for some monic polynomial $\widetilde{f(x)}$, where $f(x) \in M$. So $M = f(x)R+qR$, and we may suppose that $f(x)$ is monic. It is clear that $q \neq p$, because $x \notin M$ and $x^2 + p \in M$.

(2) If $\deg f(x) \geq 2$, then $\widetilde{x}^2 + \widetilde{p} = \widetilde{f(x)}\widetilde{d}$ for some $d \in D$, and so $\widetilde{d} = \widetilde{1}$. Hence, $\widetilde{M} = (\widetilde{x}^2 + \widetilde{p})\widetilde{R}$, and thus $M = (x^2+p)R+qR = P+qR$. Suppose that $x^2 + p \in M^2$. Then $\widetilde{M} = \widetilde{M^2}$, a contradiction, because $\widetilde{M}$ is principal. Hence, $x^2 + p \notin M^2$. Since $M^2 + P = q^2R + P$, it follows that $\overline{M} = \overline{q}\overline{R} \supset \overline{M}^2 = \overline{q}^2\overline{R}$ and so $\overline{M}$ is not idempotent.

(3) Suppose that $\deg f(x) = 1$. Then $\widetilde{f(x)} = \widetilde{x} + \widetilde{\alpha}$ for some nonzero $\widetilde{\alpha} \in \widetilde{D}$. Since $\widetilde{M} = (\widetilde{x}+\widetilde{\alpha})\widetilde{R}$ is an ideal, we have $\widetilde{i}(\widetilde{x}+\widetilde{\alpha}) = (\widetilde{x}+\widetilde{\alpha})\widetilde{\beta}$ for some $\beta = a + bi \in D$ with $\widetilde{\beta} \neq \widetilde{0}$, and so $\widetilde{i} = \widetilde{\sigma}(\widetilde{\beta})$ and $\widetilde{i}\widetilde{\alpha} = \widetilde{\alpha}\widetilde{\beta}$. Thus, $\widetilde{a} = \widetilde{0}$ and $2\widetilde{b} = \widetilde{0}$. Hence $q = 2$ follows. Then note that $\widetilde{D}[x; \widetilde{\sigma}] = \widetilde{D}[x]$, the polynomial ring over $\widetilde{D}$.

Since $\widetilde{D} = \{\widetilde{0}, \widetilde{1}, \widetilde{i}, \widetilde{i+1}\}$, $f(x)$ is one of $\{\underbrace{x + 1}, x + \underbrace{i, x} + i + 1\}$. Let $M = (x + i + 1)R + 2R$. Then $\widetilde{M} \ni (x + i + 1)(x - i - 1) = \widetilde{x}^2$, and so $M \ni x$. Hence, we do not need to consider the maximal ideal $(x + i + 1)R + 2R$. If $M = (x + 1)R + 2R$, then it is easy to see that $M \not\ni x$, because $\widetilde{M} = (x + \widetilde{1})\widetilde{R}$. Let $p = 2l + 1$ (note $p \neq 2$). Then $M \ni (x + 1)^2 + 2(l - x) = x^2 + p$. Similarly, we can prove that $(x + i)R + 2R \not\ni x$ and $(x + i)R + 2R \ni x^2 + p$.

From the proof of Lemma 3.3, we have:

Remark. $M = (x+1)R+2R$ and $N = (x+i)R+2R$ are both maximal ideals of R containing $x^2 + p$.

Lemma 3.4. *If $p = 4n + 3$, then $\overline{R}$ is not a hereditary prime ring.*

Proof. Let $M = (x + 1)R + 2R$ be a maximal ideal of R. Then $M^2 \ni (x + 1)^2 - 2(x + 1) + 4(n + 1) = x^2 + p$. Hence, $\overline{R}$ is not a hereditary prime ring by Theorem 2.10. $\square$

Lemma 3.5. *If $p = 4n + 1$, then $\overline{R}$ is a hereditary prime ring, but not a Dedekind prime ring.*

Proof. Let $M = (x+1)R+2R$ and $N = (x+i)R+2R$ be the maximal ideals of R. By Lemmas 3.2, 3.3 and Theorem 2.10, it suffices to prove that $M^2 \not\ni x^2 + p$ and $N^2 \not\ni x^2 + p$.

First we will prove that $M^2 \not\ni x^2 + p$. Suppose, on the contrary, that $M^2 \ni x^2+p$. Then, since $M^2 = (x+1)^2R+2(x+1)R+4R$, considering $R/4R$, and using the same notation in R, we may suppose that

$$x^2 + 1 = (x^2 + 2x + 1)f(x) + 2(x + 1)g(x)$$

for some $f(x) = f_nx^n +\cdots+ f_1x+ f_0$ and $g(x) = g_{n+1}x^{n+1} +\cdots+ g_1x+ g_0$, where $f_i, g_j \in D$. Comparing the coefficients of x^j $(0 \le j \le n + 2)$, we have

$$\begin{aligned}
&1 = f_0 + 2g_0,\\
&0 = 2\sigma(f_0) + f_1 + 2\sigma(g_0) + 2g_1,\\
&1 = f_0 + 2\sigma(f_1) + f_2 + 2\sigma(g_1) + 2g_2,\\
&0 = f_{j-2} + 2\sigma(f_{j-1}) + f_j + 2\sigma(g_{j-1}) + 2g_j \ (2 \le j \le n),\\
&0 = f_{n-1} + 2\sigma(f_n) + 2\sigma(g_n) + 2g_{n+1},\\
&0 = f_n + 2\sigma(g_{n+1}).
\end{aligned}$$

Here, if $\deg f(x) = 0$, then $f_1 = f_2 = g_2 = 0$, and if $\deg f(x) = 1$, then $f_2 = 0$. Adding the coefficients of x^{2j} and x^{2j+1}, respectively, we have the following equations:

Case 1. n is an even number, say, $n = 2l$.

$$2 = 2\Big(\sum_{j=0}^{l} f_{2j} + \sum_{j=1}^{l} \sigma(f_{2j-1})\Big) + 2\Big(\sum_{j=0}^{l} g_{2j} + \sum_{j=1}^{l+1} \sigma(g_{2j-1})\Big) \tag{1}$$

and

$$0 = 2\Big(\sum_{j=0}^{l} \sigma(f_{2j}) + \sum_{j=1}^{l} f_{2j-1}\Big) + 2\Big(\sum_{j=0}^{l} \sigma(g_{2j}) + \sum_{j=1}^{l+1} g_{2j-1}\Big). \tag{2}$$

Set $\alpha = \sum_{j=0}^{l} f_{2j}$, $\beta = \sum_{j=1}^{l} f_{2j-1}$, $\gamma = \sum_{j=0}^{l} g_{2j}$ and $\delta = \sum_{j=1}^{l+1} g_{2j-1}$. Then, adding (1) to (2), we have $2 = 2(\alpha+\sigma(\alpha)+\beta+\sigma(\beta)+\gamma+\sigma(\gamma)+\delta+\sigma(\delta)) = 4c$ for some $c \in \mathbf{Z}$, a contradiction. Hence, $M^2 \not\ni x^2 + p$.

Case 2. $n = 2l + 1$.

$$2 = 2\Big(\sum_{j=0}^{l} f_{2j} + \sum_{j=1}^{l+1} \sigma(f_{2j-1})\Big) + 2\Big(\sum_{j=0}^{l+1} g_{2j} + \sum_{j=1}^{l+1} \sigma(g_{2j-1})\Big) \tag{3}$$

and

$$0 = 2\Big(\sum_{j=0}^{l} \sigma(f_{2j}) + \sum_{j=1}^{l+1} f_{2j-1}\Big) + 2\Big(\sum_{j=0}^{l+1} \sigma(g_{2j}) + \sum_{j=1}^{l+1} g_{2j-1}\Big). \tag{4}$$

Adding (3) to (4), we have $2 = 4d$ for some $d \in \mathbf{Z}$, a contradiction. Hence, $M^2 \not\ni x^2 + p$.

Next, suppose that $N^2 \ni x^2+p$. Since $N^2 = (x^2-1)R+2(x+i)R+4R$, as before, we may suppose that

$$x^2+1 = (x^2-1)h(x) + 2(x+i)k(x)$$

for some $h(x) = h_n x^n + \cdots + h_1 x + h_0$ and $k(x) = k_{n+1}x^{n+1} + \cdots + k_1 x + k_0$, where $h_i, k_j \in D$. Comparing the coefficients of x^j $(0 \le j \le n+2)$, we have

$$\begin{aligned}
1 &= -h_0 + 2k_0 i,\\
0 &= -h_1 + 2\sigma(k_0) + 2k_1 i,\\
1 &= (h_0 - h_2) + 2\sigma(k_1) + 2k_2 i,\\
0 &= h_{j-2} - h_j + 2\sigma(k_{j-1}) + 2k_j i \quad (3 \le j \le n),\\
0 &= h_{n-1} + 2\sigma(k_n) + 2k_{n+1} i,\\
0 &= h_n + 2\sigma(k_{n+1}).
\end{aligned}$$

Here, if $n = 0$, then $h_1 = h_2 = k_2 = 0$ and, if $n = 1$, then $h_2 = h_3 = k_3 = 0$. Adding the coefficients of x^{2j} and x^{2j+1}, respectively, we have the following equations:

Case 1. $n = 2l$.

$$2 = 2i\bigg(\sum_{j=0}^{l} k_{2j}\bigg) + 2\bigg(\sum_{j=0}^{l} \sigma(k_{2j+1})\bigg) \tag{5}$$

$$0 = 2\bigg(\sum_{j=0}^{l} \sigma(k_{2j})\bigg) + 2i\bigg(\sum_{j=0}^{l} k_{2j+1}\bigg). \tag{6}$$

Operating σ to (6) and multiplying it by i,

$$0 = 2i\bigg(\sum_{j=0}^{l} k_{2j}\bigg) + 2\bigg(\sum_{j=0}^{l} \sigma(k_{2j+1})\bigg) \tag{7}$$

Adding (5) to (7), we have $2 = 4i(\sum_{j=0}^{l} k_{2j}) + 4\sigma(\sum_{j=0}^{l} k_{2j+1})$, a contradiction.

Case 2. $n = 2l+1$.

$$2 = 2i\bigg(\sum_{j=0}^{l+1} k_{2j}\bigg) + 2\bigg(\sum_{j=0}^{l} \sigma(k_{2j+1})\bigg) \tag{8}$$

$$0 = 2\bigg(\sum_{j=0}^{l+1} \sigma(k_{2j})\bigg) + 2i\bigg(\sum_{j=0}^{l} k_{2j+1}\bigg) \tag{9}$$

Thus, by the same method as in the case $n = 2l$, $2 = 4i(\sum_{j=0}^{l+1} k_{2j}) + 4\sigma(\sum_{j=0}^{l} k_{2j+1})$, a contradiction. Hence $N^2 \not\ni x^2 + p$, which completes the proof. □

Lemma 3.6. *Let $S = \{2^i \mid i = 0, 1, 2, \dots\}$ be the central multiplicative set in R, and let M be a maximal ideal of R with $M \cap S = \varnothing$ and $M \supset P$. Then*:

(1) *$M^2 \supseteq P$ if and only if $M_S^2 \supseteq P_S$.*

(2) *$M^2 + P = M$ if and only if $(M^2 + P)_S = M_S$.*

Proof. (1) If $M^2 \supseteq P$, then it is clear that $(M^2)_S \supseteq P_S$. Conversely, suppose $M_S^2 \supseteq P_S$. Then there is an $s \in S$ with $sP \subseteq M^2$. Since $sR + M = R$, we have $P = (sR + M)P \subseteq M^2$.

(2) This is proved in the same way as in (1).

Summarizing Lemmas 3.1–3.6, we have:

Proposition 3.7. *Let p be a prime number and $P = (x^2 + p)R$. Then*:

(1) *If $p = 2$, then $\overline{R}$ is not a hereditary prime ring.*

(2) *If $p = 4n + 3$, then $\overline{R}$ is not a hereditary prime ring and $\overline{R}_S = R_S/P_S$ is a Dedekind prime ring, where $S = \{2^i \mid i = 0, 1, 2, \dots\}$.*

(3) *If $p = 4n+1$, then $\overline{R}$ is a hereditary prime ring but not a Dedekind prime ring.*

Proof. (1) This follows from Lemma 3.1.

(2) By Lemma 3.4, $\overline{R}$ is not a hereditary prime ring. Let M be a maximal ideal of R with $M \supset P$ and $M \cap S = \varnothing$. Then, by Lemmas 3.2, 3.3 and 3.6, $(M^2)_S \not\supseteq P_S$ and $\overline{M}_S \supset \overline{M^2}_S$. Hence, $\overline{R}_S$ is a Dedekind prime ring by [**15**, (5.6.3)].

(3) $\overline{R}$ is a hereditary prime ring but not Dedekind by Lemma 3.5.

We will end the paper with two remarks.

(1) Let $P = \mathfrak{p}[x; \sigma]$ be a minimal prime ideal of R, where $\mathfrak{p}$ is a non-zero σ-prime ideal of D. Then there is a prime number p with

$\mathfrak{p} = pD$. If $p = 4n + 1$, then $\overline{R} = R/P$ is a hereditary prime ring but not Dedekind. If $p = 4n + 3$, then $\overline{R} = R/P$ is a Dedekind prime ring.

(2) Let $P' = (x^2 + 1/2)K[x; \sigma] \in \mathrm{Spec}_0(K[x; \sigma])$, where $K = \mathbf{Q} \oplus \mathbf{Q}i$ and $\mathbf{Q}$ is the field of rational numbers. Then $P = P' \cap R = (2x^2 + 1)R \in \mathrm{Spec}_0(R)$ and $2x^2 + 1$ is not a monic polynomial (as was mentioned in the introduction, Hillman only considered monic polynomials).

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