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EXISTENCE OF POSITIVE PERIODIC SOLUTIONS IN NEUTRAL NONLINEAR EQUATIONS WITH FUNCTIONAL DELAY

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ABSTRACT. We use Krasnoselskii's fixed point theorem to show that the nonlinear neutral differential equation with functional delay

$$x'(t) = -a(t)x(t) + c(t)x'(t - g(t)) + q(t, x(t - g(t)))$$

which arises in the study of the blood cell, has a positive periodic solution. We apply our results to models in biomathematics.

1. Introduction. Motivated by the papers [**16, 18, 22**], and the references therein, we consider the nonlinear neutral differential equation with functional delay

$$x'(t) = -a(t)x(t) + c(t)x'(t - g(t)) + q(t, x(t - g(t))) \tag{1.1}$$

which arises in a food-limited population models (see [**3–6, 7, 9–11**]), [**17**] and blood cell models, (see [**1, 21**]). For system (1.1), there may be a stable equilibrium point of the population. In the case the equilibrium point becomes unstable, a nontrivial periodic solution may exist. Then oscillation of solutions occurs. The existence of such a stable periodic solution is of quite fundamental importance biologically since it concerns the long time survival of species. The study of such phenomena has become an essential part of qualitative theory of differential equations. For historical background, basic theory of periodicity, and discussions of applications of (1.1) to a variety of dynamical models, we refer the interested reader to [**13–15, 17, 18, 21, 23, 26**].

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One of the most used models, a prototype of (1.1), is the system of Volterra integrodifferential equations (see [**26**])

$$\dot{N}(t) = -\gamma(t)N(t) + \alpha(t)\int_0^{\infty} B(s)e^{-\beta(t)N(t-s)}ds$$

where $N(t)$ is the number of red blood cell at time t and $\alpha, \beta, \gamma \in C(\mathbf{R}, r)$ are T-periodic, $B \in L^1(\mathbf{R}^+)$ and is piecewise continuous. This is a generalized model of the red cell system introduced by Wazewska-Czyzewska and Lasota [**25**]

$$\dot{n}(t) = -\gamma n(t) + \alpha e^{-\beta n(t-r)}$$

where α, β, γ, r are constants with $r > 0$. In [**21**] the authors established criteria for the existence of positive periodic solutions for the periodic neutral logistic equation, with distributed delays,

$$\begin{aligned} x'(t) = x(t)\Bigg[a(t) - \sum_{i=1}^{n} a_i(t)\int_{-T_i}^{0} x(t+\theta)\,\mathrm{d}\mu_i(\theta) \\ - \sum_{j=1}^{m} b_j(t)\int_{-\widehat{T}_j}^{0} x'(t+\theta)\,\mathrm{d}\nu_j(\theta)\Bigg], \end{aligned} \tag{1.2}$$

where the coefficients a, a_i, b_j are continuous and periodic functions, with the same period. The values $T_i, \widehat{T}_j$ are positive, and the functions μ_i, ν_j are nondecreasing with $\int_{-T_i}^{0} \mathrm{d}\mu_i = 1$ and $\int_{-\widehat{T}_j}^{0} \mathrm{d}\nu_j = 1$. Equation (1.2) is of Logistic form, and hence the methods that were used to obtain the existence of positive periodic solutions will not work for our model (1.1). For example, in the above equation, the authors used the transformation $x(t) = e^{N(t)}$ and put (1.2) in the form

$$\begin{aligned} N'(t) = \Bigg[a(t) - \sum_{i=1}^{n} a_i(t)\int_{-T_i}^{0} e^{N(t+\theta)}\,\mathrm{d}\mu_i(\theta) \\ - \sum_{j=1}^{m} b_j(t)\int_{-\widehat{T}_j}^{0} N'(t+\theta)e^{N(t+\theta)}\,\mathrm{d}\nu_j(\theta)\Bigg]. \end{aligned} \tag{1.3}$$

Then, they used the argument of a priori bound to show the existence of a periodic solution of (1.3), which in returns implies the existence of a positive periodic solution of (1.2). The same is true for the paper of [**20**]. Equation (1.1) represents a generalization of the hematopoiesis and blood cell production models, (see [**12, 24, 27**]).

2. Preliminaries. The Krasnoselskii fixed point theorem has been extensively used in differential and functional differential equations, and by Burton in proving the existence of periodic solutions. Also, Burton was the first to use the theorem to obtain stability results regarding solutions of integral equations and functional differential equations. For a collection of different types of results, we refer the reader to [**2**] and the references therein. The author is unaware of any results regarding the use of Krasnoselskii to prove the existence of a positive periodic solution.

Theorem 2.1 (Krasnoselskii). *Let* $\mathbf{M}$ *be a closed convex nonempty subset of a Banach space* $(\mathbf{B}, \|\cdot\|)$. *Suppose that* A *and* B *map* $\mathbf{M}$ *into* $\mathbf{B}$ *such that*

(i)] A *is compact and continuous,*

(ii) B *is a contraction mapping.*

(iii) $x, y \in \mathbf{M}$ *implies* $Ax + By \in \mathbf{M}$.

Then a $z \in \mathbf{M}$ *exists with* $z = Az + Bz$.

For $T > 0$, define $P_T = \{\phi \in C(\mathbf{R}, \mathbf{R}), \phi(t+T) = \phi(t)\}$, where $C(\mathbf{R}, \mathbf{R})$ is the space of all real valued continuous functions. Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x\| = \max_{t \in [0,T]} |x(t)| = \max_{t \in \mathbf{R}} |x(t)|.$$

In this paper we assume that

$$a(t+T) = a(t), \quad c(t+T) = c(t), \quad g(t+T) = g(t), \quad g(t) \geq g^* > 0 \tag{2.1}$$

with $c(t)$ continuously differentiable, $g(t)$ twice continuously differentiable and g^* constant. In [**5**], $a(t)$ is assumed to be positive, while here we only ask that

$$\int_0^T a(s)\, ds > 0. \tag{2.2}$$

It is interesting to note that equation (1.1) becomes of advanced type when $g(t) < 0$. Since we are searching for periodic solutions, it is natural to ask that $q(t,x)$ be continuous in both arguments and periodic in t. Also, we assume that, for all t, $0 \le t \le T$,

$$g'(t) \neq 1. \tag{2.3}$$

Lemma 2.2. *Suppose that* (2.1)–(2.3) *hold. If* $x(t) \in P_T$, *then* $x(t)$ *is a solution of equation* (1.1) *if and only if*

$$\begin{aligned} x(t) &= \frac{c(t)}{1-g'(t)}x(t-g(t)) \\ &\quad + \int_t^{t+T} \Big[-r(u)x(u-g(u)) + q(u, x(u-g(u)))\Big] \frac{e^{\int_t^u a(s)ds}}{e^{\int_0^T a(s)ds} - 1}\, du \end{aligned} \tag{2.4}$$

where

$$r(t) = \frac{(c'(t) + c(t)a(t))(1-g'(t)) + g''(t)c(t)}{(1-g'(t))^2}. \tag{2.5}$$

Proof. Let $x(t) \in P_T$ be a solution of (1.1). Multiply both sides of (1.1) with $e^{\int_0^t a(s)ds}$ and then integrate from t to $t+T$ to obtain:

$$\begin{aligned} \int_t^{t+T} \Big[x(u)e^{\int_0^u a(s)ds}\Big]' du = \int_t^{t+T} \Big[&c(u)x'(u-g(u)) \\ &+ q(u, x(u), x(u-g(u)))\Big] e^{\int_0^u a(s)ds} du. \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned} &x(t+T)e^{\int_0^{t+T} a(s)ds} - x(t)e^{\int_0^t a(s)ds} \\ &\qquad = \int_t^{t+T} \Big[c(u)x'(u-g(u)) + q(u, x(u-g(u)))\Big] e^{\int_0^u a(s)ds} du. \end{aligned}$$

By dividing both sides of the above equation by $e^{\int_0^{t+T} a(s)ds}$ and the fact that $x(t+T) = x(t)$, we obtain

$$\begin{aligned} x(t) = \Big(1 - e^{-\int_t^{t+T} a(s)ds}\Big)^{-1} \int_t^{t+T} \Big[&c(u))x'(u-g(u)) \\ &+ q(u, x(u), x(u-g(u)))\Big] e^{-\int_u^{t+T} a(s)ds} du. \end{aligned} \tag{2.6}$$

Rewrite

$$\begin{aligned}\int_t^{t+T} & c(u)x'(u-g(u))e^{-\int_u^{t+T} a(s)ds}du \\ &= \int_t^{t+T} \frac{c(u)x'(u-g(u))(1-g'(u))}{(1-g'(u))}\, e^{-\int_u^{t+T} a(s)ds}du.\end{aligned}$$

By performing an integration by parts on the above integral with

$$U = \frac{c(u)}{1-g'(u)}e^{-\int_u^{t+T} a(s)ds}$$

and

$$dV = x'(u-g(u))(1-g'(u))\, du,$$

we obtain

$$\begin{aligned}\int_t^{t+T} & c(u)x'(u-g(u))e^{-\int_u^{t+T} a(s)ds}du \\ &= \frac{c(t)}{1-g'(t)}x(t-g(t))\Big(1-e^{-\int_t^{t+T} a(s)ds}\Big) \\ &\quad - \int_t^{t+T} r(u)e^{-\int_u^{t+T} a(s)ds}x(u-g(u))\, du\end{aligned}$$

where $r(u)$ is given by (2.5). Finally, due to the integration over one period and the periodicity of all functions, we have that

$$\begin{aligned}\frac{e^{-\int_u^{t+T} a(s)ds}}{1-e^{-\int_t^{t+T} a(s)ds}} &= \frac{e^{-\int_u^{t+T} a(s)ds}}{e^{-\int_t^{t+T} a(s)ds}(e^{\int_t^{t+T} a(s)ds}-1)} \\ &= \frac{e^{\int_t^u a(s)ds}}{e^{\int_t^{t+T} a(s)ds}-1} = \frac{e^{\int_t^u a(s)ds}}{e^{\int_0^T a(s)ds}-1}.\end{aligned}$$

This completes the proof. □

To simplify notation, we let

$$M = \frac{e^{\int_0^{2T} |a(s)|ds}}{e^{\int_0^T a(s)ds}-1}, \tag{2.7}$$

and

$$m = \frac{e^{-\int_0^{2T} |a(s)|ds}}{e^{\int_0^T a(s)ds} - 1}. \tag{2.8}$$

Let

$$G(t,u) = \frac{e^{\int_u^t a(s)ds}}{e^{\int_0^T a(s)ds} - 1}. \tag{2.9}$$

It is easy to see that, for all $(t,u) \in [0,2T] \times [0,2T]$,

$$m \leq G(t,u) \leq M,$$

and, for all $t, u \in \mathbf{R}$, we have

$$G(t+T, u+T) = G(t,u).$$

3. Main results. In this section we obtain the existence of a positive periodic solution by considering the two cases:

$$0 \leq \frac{c(t)}{1-g'(t)} < 1, \qquad -1 \leq \frac{c(t)}{1-g'(t)} \leq 0.$$

For some non-negative constant L and a positive constant K, we define the set

$$\mathbf{M} = \{\phi \in P_T : L \leq ||\phi|| \leq K\},$$

which is a closed convex and bounded subset of the Banach space P_T. In addition, we assume that there are constants α, β with $0 \leq \beta \leq \alpha < 1$ such that

$$0 \leq \beta \leq \frac{c(t)}{1-g'(t)} \leq \alpha < 1, \tag{3.1}$$

and for all $u \in \mathbf{R}, \rho \in \mathbf{M}$

$$\frac{(1-\beta)L}{mT} \leq q(s,\rho) - r(s)\rho \leq \frac{(1-\alpha)K}{MT}, \tag{3.2}$$

where M and m are defined by (2.7) and (2.8), respectively. To apply Theorem 2.1, we will need to construct two mappings; one is contraction and the other is compact. Thus, we set the map $\mathbf{A} : \mathbf{M} \to P_T$

$$(3.3)\quad (\mathbf{A}\varphi)(t) = \int_t^{t+T} G(t,s)[q(s,\varphi(s-g(s))) - r(s)\varphi(s-g(s))]\,ds, \quad t \in \mathbf{R}.$$

In a similar way, we set the map $\mathbf{B} : \mathbf{M} \to P_T$

$$(3.4)\qquad (\mathbf{B}\varphi)(t) = \frac{c(t)}{1-g'(t)}\varphi(t-g(t)), t \in \mathbf{R}.$$

It is clear from condition (3.1) that $\mathbf{B}$ defines a contraction mapping under the supremum norm.

Lemma 3.1. *If* (2.1)–(2.3), (3.1) *and* (3.2) *hold, then the operator* $\mathbf{A}$ *is completely continuous on* $\mathbf{M}$.

Proof. For $t \in [0,T]$ which implies that $u \in [t, t+T] \subseteq [0, 2T]$, and for $\varphi \in \mathbf{M}$, we have by (3.3) that

$$\begin{aligned}|(\mathbf{A}\varphi)(t)| &\leq \| \int_t^{t+T} G(t,s)[q(s,\varphi(s-g(s))) - r(s)\varphi(s-g(s))]\,ds\| \\ &\leq TM\frac{(1-\alpha)K}{MT}.\end{aligned}$$

From the estimation of $|\mathbf{A}\varphi(t)|$, it follows that

$$||\mathbf{A}\varphi(t)|| \leq \left(1 - \frac{c(t)}{1-g'(t)}\right)K \leq Q_1,$$

for some positive constant Q_1. This shows that $\mathbf{A}(\mathbf{M})$ is uniformly bounded. It is left to show that $\mathbf{A}(\mathbf{M})$ is equicontinuous. Let $\varphi \in \mathbf{M}$. Then a differentiation of (3.3) with respect to t yields

$$(\mathbf{A}\varphi)'(t) = G(t,t+T)[q(t,\varphi(t-g(t))) - r(t)\varphi(t-g(t))] + a(t)(\mathbf{A}\varphi)(t).$$

Hence, by taking the supremum norm in the above expression, we have

$$||(\mathbf{A}\varphi)'|| \leq \frac{Q_1}{T} + ||a(t)||Q_1.$$

Thus, the estimation on $|(\mathbf{A}\varphi)'(t)|$ implies that $\mathbf{A}(\mathbf{M})$ is equicontinuous. Then, using the Ascoli-Arzela theorem, we obtain that A is a compact map. Due to the continuity of all terms in (3.3) for $t \in [0, T]$, we have that $\mathbf{A}$ is continuous. This completes the proof of Lemma 3.1. ◻

Theorem 3.2. *If* (2.1)–(2.3), (3.1) *and* (3.2) *hold, then equation* (1.1) *has a positive periodic solution z satisfying $L \le z \le K$.*

Proof. Let $\varphi, \psi \in \mathbf{M}$. Then, by (3.3) and (3.4), we have that

$$\begin{aligned}(\mathbf{B}\varphi)(t) &+ (\mathbf{A}\psi)(t)\\ &= \frac{c(t)}{1-g'(t)}\varphi(t-g(t))\\ &\quad + \int_t^{t+T} G(t,s)[q(s,\varphi(s-g(s)) - r(s)\varphi(s-g(s))]\,ds\\ &\le \alpha K + MT\frac{(1-\alpha)K}{MT} = K.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathbf{B}\varphi)(t) &+ (\mathbf{A}\psi)(t)\\ &= \frac{c(t)}{1-g'(t)}\varphi(t-g(t))\\ &\quad + \int_t^{t+T} G(t,s)[q(s,\varphi(s-g(s)) - r(s)\varphi(s-g(s))]\,ds\\ &\ge \beta L + m\int_t^{t+T} [q(s,\varphi(s-g(s)) - r(s)\varphi(s-g(s))]\,ds\\ &\ge \beta L + mT\frac{(1-\beta)L}{mT} = L.\end{aligned}$$

This shows that $\mathbf{B}\varphi + \mathbf{A}\psi \in \mathbf{M}$. All the hypotheses of Theorem 2.1 are satisfied, and therefore equation (1.1) has a periodic solution, say z, residing in $\mathbf{M}$. This completes the proof. ◻

For the next theorem, we assume that there are negative constants α, β with $-1 < \beta \le \alpha \le 0$, such that

$$-1 < \beta \le \frac{c(t)}{1-g'(t)} \le \alpha \le 0, \tag{3.5}$$

and, for all $u \in \mathbf{R}$, $\rho \in \mathbf{M}$,

$$\frac{(L-\beta K)}{mT} \leq q(s,\rho) - r(s)\rho \leq \frac{(K-\alpha L)}{MT}, \tag{3.6}$$

where M and m are defined by (2.7) and (2.8), respectively.

Theorem 3.3. *If* (2.1)–(2.3), (3.5) *and* (3.6) *hold, then equation* (1.1) *has a positive periodic solution z satisfying $L \leq z \leq K$.*

The proof follows along the lines of Theorem 3.2, and hence we omit it here.

4. Example. The neutral differential equation

$$x'(t) = -\frac{1}{2}\sin^2(t)x(t) + \frac{1}{50}x'(t-\pi) + \frac{cos^2(t)}{x^2(t-\pi)+100} + \frac{1}{25} \tag{4.1}$$

has a positive π-periodic solution x satisfying

$$\frac{1}{10} \leq x \leq 2.$$

To see this, we let

$$q(s,\rho) = \frac{\cos^2(s)}{\rho^2+100} + \frac{1}{25}, \qquad r(s) = \frac{1}{2}\sin^2(s) \quad \text{and} \quad T = g(t) = \pi.$$

Then,

$$\frac{c(t)}{1-g'(t)} = \frac{1}{50} < 1,$$

and

$$r(t) = \frac{1}{100}\sin^2(t).$$

A simple calculation yields

$$4.030 < M < 4.032 \quad \text{and} \quad 1.74 < m < 1.75.$$

Let $K = 2$, and $L = 1/10$, and define the set $\mathbf{M} = \{1/10 \le \upsilon \le 2\}$. Then, for $\rho \in [1/10, 2]$, we have

$$\begin{aligned} q(s,\rho) - r(s)\rho &= \frac{\cos^2(s)}{\sigma^2 + 100} + \frac{1}{100}\sin^2(s)\rho + \frac{1}{25} \\ &\le \frac{1}{100} + \frac{1}{50} + \frac{1}{25} \\ &= 0.07 < \frac{(1 - [c(t)/1 - g'(t)])K}{MT}. \end{aligned}$$

On the other hand,

$$\begin{aligned} q(s,\rho) - r(s)\rho &= \frac{\cos^2(u)}{\sigma^2 + 100} + \frac{1}{100}\sin^2(u)\rho + \frac{1}{25} \\ &> \frac{1}{25} > \frac{(1 - [c(t)/1 - g'(t)])L}{mT}. \end{aligned}$$

We see that all the conditions of Theorem 3.2 are satisfied, and hence equation (4.1) has a positive π-periodic solution x satisfying $1/10 \le x \le 2$.

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