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DIFFERENTIAL FRÉCHET *-ALGEBRAS AND CHARACTERIZATION OF SMOOTH FUNCTIONS ON R

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ABSTRACT. The purpose of this article is to search for smooth $*$-algebras which have properties similar to the properties of the algebra of smooth complex valued functions defined on the real line.

1. Introduction. One of the most important ideas of mathematics is the duality between commutative algebra and geometry. For example, the famous Gelfand-Naimark theorem states that the category of locally compact Hausdorff spaces is equivalent to the dual of the category of commutative C^*-algebras.

There is a trend in noncommutative geometry to search for a smooth dense $*$-subalgebra of C^*-algebras or Fréchet $*$-algebras whose properties are close to the properties of the algebra of smooth functions on certain domains. In [**6, 9**] the concept of smooth $*$-algebras is defined using derivations. On the other hand, Blackadar and Cuntz [**4**] have developed an abstract theory of differential structure in a C^*-algebra based on the notion of differential seminorms with values in the convolution algebra $\ell^1(N)$.

In this article, we introduce the concept of differential F^*-algebras of rank 1 (and of rank 2), which is generated by a single self-adjoint element and provided with some additional conditions. Then, it is proven that these algebras characterize the algebra of all smooth complex valued functions defined on closed and bounded intervals (and on the real line).

The paper is organized as follows. In Section 2, we consider differential F^*-algebras of rank 1 provided with additional conditions and es-

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tablish its isomorphism to $C^\infty([a,b])$. In Section 3, we express Fréchet $*$-algebras as double inverse limits of Banach $*$-algebras. Section 4 contains some technical lemmas needed in the next section. Finally, in Section 5 we define differential F^*-algebras of rank 2 provided with additional conditions and establish the characterization of $C^\infty(\mathbf{R})$.

Throughout this paper all algebras are assumed to be *unital* and Hausdorff.

2. Differential F^*-algebras of rank 1. A Fréchet $*$-algebra A is a complete metrizable topological algebra with continuous involution whose topology is determined by a countable family of separating submultiplicative $*$-seminorms $\{\rho_n\}$. Let $N_n = \rho_n^{-1}(0)$; then N_n is a $*$-ideal of A and the completion of A/N_n is a Banach $*$-algebra denoted by A_n. If ρ_n is a C^*-seminorm on A, then A_n is a C^*-algebra.

Let $(A, \{\rho_n : n \in \mathbf{Z}^+\})$ be a Fréchet $*$-algebra. Then the inverse limit and the inverse system is formed as follows:

For $n < m$, then $\rho_n \leq \rho_m$. Thus, we have a continuous onto $*$-homomorphism: $\phi_{nm} : A/N_m \to A/N_n$, which extends to the completion. Then $\{A_n, \phi_{nm}\}$ is an inverse system of Banach $*$-algebra which defines an inverse limit $\lim_{\leftarrow n} A_n$, which is a subalgebra of the product $\prod A_n$, and it is isomorphic to A, i.e., $A \simeq \lim_{\leftarrow n} A_n$ (cf. [**3, 8, 10**]).

From this it follows easily that, if A is a Fréchet $*$-algebra, then A has an idempotent a if and only if each A_n has an idempotent a_n, where $a = (a_n)$. Moreover, if A is generated by a single self adjoint element x, then each A_n is generated by x_n (cf. [**8, 10**]).

A linear map $D : A \to A$ is called a derivation, if it satisfies $D(ab) = (Da)b + a(Db)$ for all $a, b \in A$. A closable $*$-derivation D is a linear $*$-derivation mapping, from a dense subalgebra B of A into A such that if $\{a_n\}_n$ is a sequence in B such that $a_n \to 0$ and $D(a_n) \to b$, then $b = 0$.

Brooks in [**5**] introduced F^*-algebra; it is a Fréchet $*$-algebra $(A, \{\rho_n\})$, such that $\{\rho_n\}$ are ascending C^*-seminorms.

Definition 2.1. A Fréchet $*$-algebra $(A_\infty, \{\rho_n : n \in \mathbf{Z}^+\})$ is called a *differential F^*-algebra of rank* 1, if ρ_0 is a C^*-seminorm on A_∞ and, for any $n \geq 1$,

$$\rho_n(a) = \sum_{k=0}^{n} \frac{1}{k!}\, \rho_0(\delta^k(a)), \quad a \in A_\infty,$$

where $\delta : A_\infty \to A_\infty$ is a closed $*$-derivation, such that $\delta(N_0) \subset N_0$.

Example 2.2. Let $A = C([a,b])$ be the algebra of all continuous complex valued functions on the interval $[a,b]$. It is a commutative unital C^*-algebra with the sup norm $|| \, ||_0$. The collection $P([a,b])$ of all polynomials in t is dense in $C([a,b])$ by the Weierstrass theorem. The n times continuously differentiable functions $C^n([a,b])$ are commutative unital Banach $*$-algebras when provided with the n-norm

$$||f||_n = \sum_{k=0}^{n} \frac{1}{k!} \left\| \frac{d^k f}{dt^k} \right\|_0, \quad f \in C^n([a,b]), \text{ with convention } \frac{d^0 f}{dt^0} = f.$$

$C^\infty([a,b])$, which is an inverse limit of $C^n([a,b])$, is in fact a differential F^*-algebra of rank 1.

Lemma 2.3. *Let $(A_\infty, \{\rho_n : n \in \mathbf{Z}^+\})$ be a differential F^*-algebra of rank 1. Then the Banach $*$-algebra A_n forms a nested sequence*

$$\cdots A_n \subset A_{n-1} \subset \cdots A_2 \subset A_1 \subset A_0.$$

Proof. Consider the continuous homomorphism $\phi : A_n \to A_{n-1}$, and let $a \in \ker(\phi) \subset A_n$. Then there is a sequence $\{a_j\}$ in A_∞ fundamental in ρ_n (it defines a) such that $\lim_j \rho_{n-1}(a_j) = 0$. Set $b_j = \delta^{n-1}(a_j)$. As $\rho_n(a_j) = \rho_{n-1}(a_j) + (1/n!)\, \rho_0(\delta(b_j))$, the sequence $\{\delta(b_j)\}$ is fundamental in ρ_0. Since $\lim_j \rho_0(b_j) = \lim_j \rho_0(\delta^{n-1}(a_j)) \leq (n-1)! \lim_j \rho_{n-1}(a_j) = 0$, and since δ is closable in ρ_0, we have $\delta(b_j) \to 0$ in ρ_0, so that $\lim_j \rho_n(a_j) = 0$. Therefore, $a = 0$. $\square$

Lemma 2.4. *Let $(A_\infty, \{\rho_n : n \in \mathbf{Z}^+\})$ be a differential F^*-algebra of rank 1. Then $N_0 = \{0\}$, ρ_0 is a C^*-norm on A_∞ and the Banach $*$-algebra A_0 is a C^*-algebra.*

Proof. As $\delta(N_0) \subset N_0$, $\rho_n(a) = 0$ for all $a \in N_0$, and all n. As the family of seminorms $\{\rho_n\}$ is separating, hence $N_0 = \{0\}$. $\square$

The author in [**2**] introduced the concept of a *special* C^**-algebra*; it is a C^*-algebra A with no nontrivial idempotent, and it is generated by a single self-adjoint element x. The next theorem characterizes those Fréchet $*$-algebras that are isomorphic to $C^\infty([a,b])$.

Theorem 2.5. *Let* $(A_\infty, \{\rho_n : n \in Z^+\})$ *be a differential* F^**-algebra of rank* 1 *generated by a single self adjoint element* x *such that* $\delta(x) = 1$. *Suppose that* A_∞ *has no nontrivial idempotents. Then* A_∞ *is isomorphic to* $C^\infty([a,b])$, *where* $\mathrm{sp}_{A_\infty}(x) \simeq [a,b]$.

Proof. The remark before Definition 2.1 and Lemma 2.4 implies that A_0 is a special C^*-algebra. By [**2**], A_0 is isometric $*$-isomorphic to $C([a,b])$, where $[a,b]$ is homeomorphic to the spectrum of x. The Gelfand mapping gives the isometric $*$-isomorphism $\Gamma : (C([a,b]), \|\cdot\|_0) \to (A_0, \rho_0)$, defined by $\Gamma(f) = f(x)$ and $\|f\|_0 = \rho_0(\Gamma(f)) = \rho_0(f(x))$.

If $p(x) = \sum_{j=0}^m \alpha_j x^j \in \mathcal{P}(x)$ is a polynomial in x, then $\delta(p(x)) = \sum_{j=0}^m j\alpha_j x^{j-1} = p'(x)$. Let $p(t)$ be a polynomial in $C([a,b])$. Then one can easily show that $\Gamma(d^m p)/(dt^m) = \delta^m(p(x))$ true for any m. Therefore:

$$\left\|\frac{d^m p}{dt^m}\right\|_0 = \rho_0\left(\Gamma\left(\frac{d^m p}{dt^m}\right)\right) = \rho_0(\delta^m(p(x)). \tag{1}$$

For each n, Let $\Gamma_n : C^n([a,b]) \to A_n$ be the restriction map of Γ to the subalgebra $C^n([a,b])$, which is provided with the n-norm, as in Example 2.2. Clearly, Γ_n is continuous and, if p is any polynomial in $C^n([a,b])$, then from (1) we get:

$$\|p\|_n = \sum_{k=0}^n \frac{1}{k!}\left\|\frac{d^k p}{dt^k}\right\|_0 = \sum_{k=0}^n \frac{1}{k!}\rho_0(\delta^k(p(x))) = \rho_n(p(x)) = \rho_n(\Gamma_n(p)).$$

The collection $P([a,b])$ is dense in $C^n([a,b])$, and as A_∞ is generated by x, thus x_n is dense in A_n. Consequently, Γ_n is an isometric $*$-isomorphism, and $A_\infty \simeq \lim_{\leftarrow n} A_n \simeq \lim_{\leftarrow n} C^n([a,b]) \simeq C^\infty([a,b])$. $\square$

3. The case of double inverse limits. In this section we deal with Fréchet $*$-algebras where the seminorms are indexed by double indices, that is why we call them of rank 2, such Fréchet $*$-algebras are isomorphic to a double inverse limit of Banach $*$-algebras.

Consider the Fréchet $*$-algebra $(A, \{\rho_{n,i} : (n,i) \in \mathbf{Z}^+ \times N\})$, where the separating family of submultiplicative $*$-seminorms are doubly indexed $\{\rho_{n,i}\}$. Denote by A_i^n the Banach $*$-algebra which is the completion of $A/N_{i,n}$. Then (cf. [**3, 10**])

$$A \simeq \lim_{\leftarrow (n,i)} A_i^n \simeq \lim_{\leftarrow n}(\lim_{\leftarrow i} A_i^n) \simeq \lim_{\leftarrow i}(\lim_{\leftarrow n} A_i^n). \tag{2}$$

Let $F^n \simeq \lim_{\leftarrow i} A_i^n$; then F^n is a Fréchet $*$-algebra and any $a^n \in F^n$ is of the form $a^n = (g_i^n(a))_i$, where $g_i^n : A \to A/N_{i,n}$ is a canonical continuous $*$-homomorphism. The metric d on F^n is of the form (cf. [**7**])

$$d(a^n, b^n) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\rho_{n,i}(a-b)}{1+\rho_{n,i}(a-b)}.$$

Lemma 3.1. *Let $(A, \{\rho_{n,i} : (n,i) \in \mathbf{Z}^+ \times N\})$ be a Fréchet $*$-algebra. If A is generated by a single self adjoint element x, then each A_i^n is generated by a single self adjoint element $g_i^n(x)$. Moreover, for each n, F^n is generated by $x^n = (g_i^n(x))$.*

Proof. One can easily show that $g_i^n(x)$ is dense in A_i^n. To show x^n is dense in F^n, let $a^n \in F^n$, given $\varepsilon > 0$. Some k exists such that $1/2^{k-1} < \varepsilon$. Also a polynomial $p(x)$ exists such that $\rho_{n,k}(a - p(x)) < 1/2^{k+1}$. Thus, $\rho_{n,j}(a - p(x)) < 1/2^{k+1}$ for $j = 0, \dots, k$. Hence,

$$\sum_{m=0}^{k} \frac{1}{2^m} \frac{\rho_{n,m}(a-p(x))}{1+\rho_{n,m}(a-p(x))} < \frac{1}{2^k},$$

and

$$\sum_{m=k+1}^{\infty} \frac{1}{2^m} \frac{\rho_{n,m}(a-p(x))}{1+\rho_{n,m}(a-p(x))} \le \frac{1}{2^k},$$

which gives $d(x^n, a^n) < \varepsilon$. $\square$

4. Differential F^*-algebras of rank 2. This section deals with some technical lemmas which are needed in the next section, where the concept of characterization is dealt with.

Definition 4.1. A Fréchet $*$-algebra $(A_\infty, \{\rho_{n,i} : (n,i) \in \mathbf{Z}^+ \times N\})$ is called a *differential F^*-algebra of rank* 2, if the following holds:

1. For $n = 0$, the collection, $\{\rho_{0,i} : i \in \mathbf{N}\}$ is a C^*-seminorm, and $\rho_{n,i}$'s are an increasing sequence of $*$-seminorms.

2. For $n \geq 1$, the $*$-seminorms are defined by

$$\rho_{n,i}(a) = \sum_{k=0}^{n} \frac{1}{k!} \rho_{0,i}(\delta^k(a)), \quad a \in A_\infty, \text{ with convention } \delta^0(a) = a,$$

where $\delta : A_\infty \to A_\infty$ is a $*$-derivation. Moreover, for each i, the map $\delta : A_\infty / N_{0,i} \to A_i^0$, is a closable $*$-derivation.

Lemma 4.2. *Let $(A_\infty, \{\rho_{n,i} : (n,i) \in \mathbf{Z}^+ \times N\})$ be a differential F^*-algebra of rank* 2*. Then, for each i, the following holds*:

1. *We have a nested sequence of Banach $*$-algebras*

$$\cdots A_i^n \subset A_i^{n-1} \subset \cdots A_i^2 \subset A_i^1 \subset A_i^0.$$

2. *Each A_i^n is spectrally invariant in A_i^0.*

Proof. 1. Similar to the proof of Lemma 2.3.

2. The Banach $*$-algebra A_i^n is dense in the C^*-algebra A_i^0. Thus, by [**8**, Lemma 3], A_i^n is a Q-algebra (i.e. the group of all invertible elements is open in A_i^n) with respect to the $\rho_{0,i}$ norm. As A_i^n is $\rho_{0,i}$ dense in A_i^0, the result follows from [**9**]. $\square$

Lemma 4.3. *Let A be a differential F^*-algebra of rank* 2*, and let $a \in A_i^0$. Then, for $n \geq 1$,*

$$a \in A_i^n \iff \delta a, \delta^2 a, \ldots, \delta^n a \in A_i^0.$$

Proof. Let $a \in A_i^n$. Then $\delta a \in A_i^{n-1}$. Lemma 4.2 implies that $\delta a \in A_i^0$. Similarly, for any k with $1 < k \leq n$, then $\delta^k(a) \in A_i^0$. Conversely, let $\delta^k a \in A_i^0$ for $0 \leq k \leq n$. Then, for each k, a sequence $\{\delta^k(a_j)\}$ exists in A_∞ fundamental in $\rho_{0,i}$ (it defines $\delta^k(a)$). From Definition 4.1, the sequence $\{a_j\}$ in A_∞ is fundamental in $\rho_{n,i}$. As

$$\lim_j \rho_{n,i}(a_j) = \lim_j \sum_{k=0}^{n} \frac{1}{k!}\,\rho_{0,i}(\delta^k(a_j)) = \rho_{n,i}(a).$$

Therefore, $a \in A_i^n$. $\square$

5. Characterization of smooth functions on the real line. This section contains the main result, in which we establish the characterization of smooth complex valued functions defined on the real line.

Lemma 5.1. *Let A_∞ be a differential F^*-algebra of rank 2, notations as in Section 3. Then, for each $a \in A_\infty$,*

$$\mathrm{sp}_{A\infty}(a) = \mathrm{sp}_{F^0}(a^0), \text{ where } a^0 = (g_i^0(a))_i.$$

Proof. For any $a \in A_\infty$, the spectrum of a is $\mathrm{sp}_{A_\infty}(a) = \cup_i \cup_n \mathrm{sp}_{A_i^n}(g_i^n(a))$ (cf. [**9, 10**]). By Lemma 4.2, A_i^n is spectrally invariant in A_i^0; moreover, $F^0 \simeq \lim_{\leftarrow i} A_i^0$ is a Fréchet $*$-algebra. Thus, $\mathrm{sp}_{A\infty}(a) = \cup_i \mathrm{sp}_{A_i^0}(g_i^0(a)) = \mathrm{sp}_{F^0}(a^0)$. $\square$

The author introduced the concept of special F^*-algebras in [**2**]; it is an F^*-algebra A with no nontrivial idempotent, and it is generated by a single self adjoint element x such that $\mathrm{sp}_A(x)$ contains no boundary points. For example, $C(\mathbf{R})$ is a special F^*-algebra.

Lemma 5.2. *Let A_∞ be a differential F^*-algebra of rank 2 generated by a single self adjoint element x such that $\delta(x) = 1$ and $\mathrm{sp}(x)$ contains no boundary points. Suppose that A_∞ has no nontrivial idempotent. Then F^0 is a special F^*-algebra.*

Proof. Lemma 3.1 implies that F^0 is generated by a single self adjoint element $x^0 = (g_i^0(x))_i$, and it has no nontrivial idempotent. Moreover, $\mathrm{sp}_{F^0}(x^0)$ contains no boundary points by Lemma 5.1. $\square$

For A_∞ as in Lemma 5.2, then F^0 is a special F^*-algebra. Consequently, by [**2**], the character of F^0, $\mathcal{M}(F^0) = \cup_i \mathcal{M}(A_i^0)$ is homeomorphic to $\mathbf{R}$, where each $\mathcal{M}(A_i^0)$ is homeomorphic to a closed and bounded interval. Denote by $C(\mathcal{M}(A_i^0))$ the algebra of all continuous complex valued functions defined on $\mathcal{M}(A_i^0)$. Then we have the following isomorphisms (cf. [**12**])

$$C(\mathcal{M}(F^0)) \simeq C(\lim_{\to i} \mathcal{M}(A_i^0)) \simeq \lim_{\leftarrow i} C(\mathcal{M}(A_i^0)). \tag{3}$$

For each i, the C^n-elements of $C(\mathcal{M}(A_i^0))$ are defined by:
(4)

$$C^n(\mathcal{M}(A_i^0)) = \Big\{ f \in C(\mathcal{M}(A_i^0)) \ni \frac{d^k f}{dt^k} \in C(\mathcal{M}(A_i^0)),\ k = 1, \dots, n \Big\}.$$

It is a Banach $*$-algebra when provided with the $*$-norm as in Example 2.2. Clearly, these Banach $*$-algebras are nested, i.e.,

$$\cdots C^n(\mathcal{M}(A_i^0)) \subset \cdots \subset C^1(\mathcal{M}(A_i^0)) \subset C(\mathcal{M}(A_i^0)).$$

The next proposition establishes the isomorphism between $C^n(\mathcal{M}(A_i^0))$ and A_i^n.

Proposition 5.3. *Let A_∞ be as in Lemma* 5.2. *Then $A_i^n \simeq C^n(\mathcal{M}(A_i^0))$.*

Proof. Let $\Gamma : C(\mathcal{M}(A_i^0)) \to A_i^0$ be the Gelfand isomorphism, as in Theorem 2.5. Let $B_i^n = \{a \in A_i^0 \ni \delta^k(a) \in A_i^0,\ k = 1, 2, \dots, n\}$. Then B_i^n is isomorphic to A_i^n as Banach $*$-algebras, when B_i^n is provided with the $*$-norm $\gamma_n(a) = \sum_{k=0}^n 1/k!\, \rho_{0,i}(\delta^k(a))$.

For each n, let $\Gamma_n : C^n(\mathcal{M}(A_i^0)) \to B_i^n$ be the restriction map of Γ. The differentiation operator d/dt in $C^n(\mathcal{M}(A_i^0))$ will correspond to the derivation operator δ in B_i^n. The isomorphism of Γ_n is established by following step by step the proof of Theorem 2.5. $\square$

The next two theorems are the main theorems of this article, as they establish the characterization of $C^\infty(\mathbf{R})$.

Theorem 5.4. *Let A_∞ be a differential F^*-algebra of rank* 2 *generated by a single self adjoint element x such that $\delta(x) = 1$ and* $\mathrm{sp}(x)$ *contains no boundary points. Suppose that A_∞ has no nontrivial idempotent. Then $A_\infty \simeq C^\infty(\mathbf{R})$.*

Proof. Lemma 5.2 and [**2**] imply that $\mathcal{M}(F^0)$ is homeomorphic to $\mathbf{R}$. Write A_∞ as a double inverse limit of Banach $*$-algebras:

$$\begin{aligned} A_\infty &\simeq \lim_{\leftarrow n}\lim_{\leftarrow i} A_i^n \\ &\simeq \lim_{\leftarrow n}\lim_{\leftarrow i} C^n(\mathcal{M}(A_i^0)),, \text{ by Proposition 5.3} \\ &\simeq \lim_{\leftarrow n} C^n(\lim_{\rightarrow i} \mathcal{M}(A_i^0)), \text{ by (3)} \\ &\simeq \lim_{\leftarrow n} C^n(\mathcal{M}(F^0)), \qquad \text{by (3)} \\ &\simeq \lim_{\leftarrow n} C^n(\mathbf{R}) \simeq C^\infty(\mathbf{R}). \qquad \square \end{aligned}$$

Theorem 5.5. $A_\infty \simeq C^\infty(\mathbf{R})$ *if and only if A_∞ is as in Theorem* 5.4.

Proof. $C^\infty(\mathbf{R})$ is a differential F^*-algebra of rank 2. Connectedness of $\mathbf{R}$ implies that $C^\infty(\mathbf{R})$ has no nontrivial idempotent; moreover, a theorem of Nachbin [**11**] implies that the coordinate function $p(t) = t$, for $t \in \mathbf{R}$, generates $C^\infty(\mathbf{R})$. The converse follows from Theorem 5.4. $\square$

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