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UNION SPACES AND GENERALIZED CLOSED SETS

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ABSTRACT. Generalized closed sets in union spaces are introduced and many of Levine's results for generalized closed sets in topological spaces are strengthened. Also, continuous functions between union spaces are introduced, thereby unifying many important classes of functions between topological spaces. Some preservation theorems of Baker for generalized closed sets by functions between topological spaces and of Thivagar, Israel, and Ravi for bitopological generalized closed sets are unified and strengthened. Finally, a decomposition of continuity for functions between union spaces is found.

1. Introduction. A topology on a set X is a collection τ of subsets of X satisfying the following three conditions: a) $\varnothing$, $X \in \tau$, b) $\mathcal{S} \subseteq \tau \Rightarrow \cup\mathcal{S} \in \tau$ and c) $\mathcal{F} \subseteq \tau$ finite $\Rightarrow \cap\mathcal{F} \in \tau$. Maki [**13**] introduced a minimal structure for a set X as a family m_X of subsets of X satisfying condition a). See also [**14**]. Noiri and Popa have applied minimal structures widely to unify results for various generalized continuity forms [**17, 18, 20, 22, 23**]. Minimal structures τ which also satisfy b) have been found especially useful. In this paper we continue the unification theme by adopting only b) for our "union" structures. If $\mathcal{C}$ is a union structure, the case $\mathcal{S} = \varnothing$ admits $\varnothing = \cup\mathcal{S} \in \mathcal{C}$. Thus, a union structure $\mathcal{C}$ fails to be a minimal structure if $X \notin \mathcal{C}$. For example, if $X = \{a, b\}$ then $\mathcal{C} = \{\varnothing, \{a\}\}$ is a union structure but not a minimal structure.

Definition 1. A subcollection $\mathcal{C} \subseteq \mathcal{P}(X)$ is a union structure for X if $\mathcal{C}$ is closed under arbitrary union. In this case, we say that $(X, \mathcal{C})$ is a union space. We denote by $\mathcal{C}^c$ (read $\mathcal{C}$ complements) the collection of complements in X of members of $\mathcal{C}$. Members of $\mathcal{C}$ are called $\mathcal{C}$-open subsets of X, and members of $\mathcal{C}^c$ are called $\mathcal{C}$-closed subsets of X. If $\mathcal{C}$

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is a cover for X in the sense that, for each $x \in X$ there exists a $U \in \mathcal{C}$ with $x \in U$, then $(X, \mathcal{C})$ is called a covered union space.

Union structures properly generalize topologies and minimal structures with property b) for the latter structures are precisely the covered union space structures. For, if $\mathcal{C}$ is a covering union structure on X, then $\cup\mathcal{C} = X \in \mathcal{C}$ so that $\mathcal{C}$ is a minimal structure for X. Union structures also generalize the families of $\tau_{1,2}$-open subsets of bitopological spaces (X, τ_1, τ_2) introduced by Ravi and Thivagar in [**24**] and refined by Ravi, Thivagar, and Ekici in [**26**]. A subset A of a bitopological space (X, τ_1, τ_2) is $\tau_{1,2}$-open if $A = U \cup V$ for some $U \in \tau_1$ and $V \in \tau_2$. The collection of all $\tau_{1,2}$-open subsets of (X, τ_1, τ_2) is denoted $(1,2)^*$-$\mathcal{O}(X)$. It is clear that $(1,2)^*$-$\mathcal{O}(X)$ is a union structure on X since the topologies τ_1 and τ_2 are each union structures on X. The following example shows that union structures strictly generalize families of $\tau_{1,2}$-open subsets of bitopological spaces.

Example 2. Let $X = \{a, b, c\}$ and $\mathcal{C} = \{\varnothing, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Clearly, $\mathcal{C}$ is a union structure on X. We show that topologies τ_1 and τ_2 do not exist on X for which $\mathcal{C} = (1,2)^*$-$\mathcal{O}(X)$. Indeed, if $\mathcal{C} = (1,2)^*$-$\mathcal{O}(X)$ for some topologies τ_1 and τ_2 on X, then $\tau_1 \cup \tau_2 \subseteq (1,2)^*$-$\mathcal{O}(X) = \mathcal{C}$ implies that neither (X, τ_1) nor (X, τ_2) contains an isolated point. It follows that $\{a, b\} \in \tau_1$ or $\{a, b\} \in \tau_2$. Without loss of generality, suppose that $\{a, b\} \in \tau_1$. Then $\{a, c\} \in \tau_2$. For, otherwise, $\{a, c\} \in \tau_1$ implies that $\{a\} = \{a, b\} \cap \{a, c\} \in \tau_1$ admitting an isolated point in (X, τ_1). Finally, either $\{b, c\} \in \tau_1$ or $\{b, c\} \in \tau_2$ but, in the first case, b is isolated in (X, τ_1) and, in the second case, c is an isolated point in (X, τ_2).

In any topological space (X, τ), the families $\mathcal{PO}(X, \tau)$, $\mathcal{SO}(X, \tau)$, τ^α, $b\mathcal{O}(X, \tau)$, $\beta\mathcal{O}(X, \tau)$, τ_s, τ_θ and $b\theta\mathcal{O}(X, \tau)$ of preopen sets, semi-open sets, α-open sets [**15**], b-open sets [**2**], β-open sets, δ-open sets (unions of regular open sets), θ-open sets and b-θ-open sets [**21**], respectively, are much studied examples of union structures. The respective interior and closure operators for most of these families have played a significant role in these studies. For example, the preinterior of a subset A of X, $\mathrm{pint}\,(A)$, is the union of all preopen subsets of A, and the preclosure of A, $\mathrm{pcl}\,(A)$ is the intersection of all preclosed supersets of A where a preclosed set is one whose complement is preopen. A unified approach to interior and closure given next is fundamentally important to the study of union structures.

Definition 3. If $(X, \mathcal{C})$ is a union space and $A \subseteq X$, $\mathcal{C}\text{-int}\,(A) = \cup\{U \in \mathcal{C} \mid U \subseteq A\}$ and $\mathcal{C}\text{-cl}\,(A) = \cap\{F \in \mathcal{C}^c \mid A \subseteq F\}$.

Special cases of $\mathcal{C}$-int and $\mathcal{C}$-cl include pint and *pcl*, sint and *scl*, α-int and α-cl, bint and *bcl*, β int and β cl, δ-int and δ- cl, θ-int and θ-cl and $b\theta$-int and $b\theta$-cl when $\mathcal{C}$ is the family $\mathcal{PO}(X,\tau)$, $\mathcal{SO}(X,\tau)$, τ^α, $b\mathcal{O}(X,\tau)$, $\beta\mathcal{O}(X,\tau)$, τ_s, τ_θ or $b\theta\mathcal{O}(X,\tau)$, respectively for a topological space (X,τ). In particular, for any subset A of a space (X,τ) it is known that $pint(A) = A \cap \text{int}\,(\text{cl}\,(A))$, $\text{sint}\,(A) = A \cap \text{cl}\,(\text{int}\,(A))$, α-$\text{int}\,(A) = A \cap \text{int}\,(\text{cl}\,(\text{int}\,(A)))$ $b\text{int}\,(A) = \text{sint}\,(A) \cup \text{pint}\,(A)$, $\beta\text{int}\,(A) = A \cap \text{cl}\,(\text{int}\,(\text{cl}\,(A)))$, $\delta\text{-int}\,(A) = \cup\{U \subseteq A \mid U = \text{int}\,(\text{cl}\,(U))\}$, θ-$\text{int}\,(A) = \cup\{U \subseteq A \mid \text{ for all } x \in U, \text{ there exists a } V \in \tau \text{ with } x \in V \text{ and } \text{cl}\,(V) \subseteq U\}$ and $b\theta\text{-int}\,(A) = \cup\{U \subseteq A \mid \text{ for all } x \in U, \text{ there exists a } V \in b\mathcal{O}(X,\tau) \text{ with } x \in V \text{ and } b\text{cl}\,(V) \subseteq U\}$. Here, $b\text{cl}\,(V) = X - b\text{int}\,(X - V)$.

In a recent paper, Thivagar, Israel and Ravi [**28**] extended the results of Baker [**3**] from a topological setting to a bitopological setting. In the third section of this paper, we will simultaneously extend the results of both papers by lifting these results to the setting of union spaces. But first, in the second section we lift the generalized closed sets introduced by Levine [**10**] to the setting of union spaces. We now close the introduction with a list of properties of $\mathcal{C}$-int and $\mathcal{C}$-cl as operators on a union space $(X, \mathcal{C})$. Let $A, B \subseteq X$.

1. $\mathcal{C}\text{-int}\,(A) \subseteq A \subseteq \mathcal{C}\text{-cl}\,(A)$.
2. $\mathcal{C}\text{-int}\,(A) \in \mathcal{C}$ and $\mathcal{C}\text{-cl}\,(A) \in \mathcal{C}^c$.
3. $\mathcal{C}\text{-int}\,(A) = A \Leftrightarrow A \in \mathcal{C}$ and $\mathcal{C}\text{-cl}\,(A) = A \Leftrightarrow A \in \mathcal{C}^c$.
4. $\mathcal{C}\text{-cl}\,(A) = \{x \in X \mid x \in U \in \mathcal{C} \Rightarrow U \cap A \neq \varnothing\}$.
5. $\mathcal{C}\text{-int}\,(X - A) = X - \mathcal{C}\text{-cl}\,(A)$ and $\mathcal{C}\text{-cl}\,(X - A) = X - \mathcal{C}\text{-int}\,(A)$.
6. $A \subseteq B \Rightarrow \mathcal{C}\text{-int}\,(A) \subseteq \mathcal{C}\text{-int}\,(B)$ and $\mathcal{C}\text{-cl}\,(A) \subseteq \mathcal{C}\text{-cl}\,(B)$.
7. $\mathcal{C}\text{-int}\,(A \cap B) \subseteq \mathcal{C}\text{-int}\,(A) \cap \mathcal{C}\text{-int}\,(B)$ and $\mathcal{C}\text{-cl}\,(A) \cup \mathcal{C}\text{-cl}\,(B) \subseteq \mathcal{C}$-cl.$(A \cup B)$.

The proofs are straightforward, and the reader can find examples showing that the set inclusions in the seventh property are generally not equalities.

2. C-g-closed sets. The following generalizes g-closed sets of Levine [**10**] and the $(1,2)^*$-g-closed sets of Ravi, Thivagar, and Ekici [**27**].

Definition 4. A subset A of a union space $(X,\mathcal{C})$ is $\mathcal{C}$-g-closed if $\mathcal{C}\text{-cl}\,(A) \subseteq U$ whenever $U \in \mathcal{C}$ and $A \subseteq U$. A set $A \subseteq X$ is $\mathcal{C}$-g-open if $X - A$ is $\mathcal{C}$-g-closed.

Clearly every $\mathcal{C}$-closed subset of X is $\mathcal{C}$-g-closed, and hence every $\mathcal{C}$-open set is $\mathcal{C}$-g-open. Example 5.5 of [**10**] shows that there are topological spaces in which non-closed g-closed sets exist. Since every topology is a union structure, $\mathcal{C}$-g-closedness is strictly weaker than $\mathcal{C}$-closedness in union spaces. The following generalizes results of Levine [**10**] as well as results of Ravi, Thivagar and Jinjinli [**25**].

Theorem 5. *The following are equivalent for a subset A of a union space $(X,\mathcal{C})$.*

1. *A is $\mathcal{C}$-g-closed.*
2. *$\mathcal{C}\text{-cl}\,(A) - A$ contains no nonempty $\mathcal{C}$-closed set.*
3. *$\mathcal{C}\text{-cl}\,(A) - A$ is $\mathcal{C}$-g-open.*

Proof. We show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$. Suppose that A is $\mathcal{C}$-g-closed and $F \subseteq \mathcal{C}\text{-cl}\,(A) - A$ for some $\mathcal{C}$-closed set F. Then, $U = X - F$ is $\mathcal{C}$-open with $A \subseteq U$. Since A is $\mathcal{C}$-g-closed, $F \subseteq \mathcal{C}\text{-cl}\,(A) \subseteq U = X - F$ implies that $F = \varnothing$ so that $1 \Rightarrow 2$. Now suppose that $A \subseteq X$ and $\mathcal{C}\text{-cl}\,(A) - A$ contains no nonempty $\mathcal{C}$-closed set. If $F \in \mathcal{C}^c$ and $F \subseteq \mathcal{C}\text{-cl}\,(A) - A$, then $F = \varnothing$ implies that $F \subseteq \mathcal{C}\text{-int}\,(\mathcal{C}\text{-cl}\,(A) - A)$ so that $\mathcal{C}\text{-cl}\,(A) - A$ must be $\mathcal{C}$-g-open and so $2 \Rightarrow 3$. Now suppose that $A \subseteq X$ for which $\mathcal{C}\text{-cl}\,(A) - A$ is $\mathcal{C}$-g-open, and let $U \in \mathcal{C}$ with $A \subseteq U$. Now, $\mathcal{C}\text{-cl}\,(A) \cap (X - U) = F$ is $\mathcal{C}$-closed and $F \subseteq \mathcal{C}\text{-cl}\,(A) - A$. Since $\mathcal{C}\text{-cl}\,(A) - A$ is $\mathcal{C}$-g-open, $F \subseteq \mathcal{C}\text{-int}\,(\mathcal{C}\text{-cl}\,(A) - A) = \varnothing$. But, $F = \varnothing$ implies that $\mathcal{C}\text{-cl}\,(A) \subseteq U$. It follows that A is $\mathcal{C}$-g-closed, and therefore $3 \Rightarrow 1$. $\square$

Classically, a subset A of a topological space is locally closed if it can be expressed as the intersection of an open set with a closed set. We now introduce a union space version of local closedness.

Definition 6. A subset A of a union space $(X, \mathcal{C})$ is $\mathcal{C}$-locally closed if $A = U \cap F$ for some $U \in \mathcal{C}$ and some $F \in \mathcal{C}^c$.

Proposition 7. *A subset A of a union space $(X, \mathcal{C})$ is $\mathcal{C}$-locally closed if and only if $\mathcal{C}$-$\mathrm{cl}\,(A) - A$ is $\mathcal{C}$-closed.*

Proof. (Sufficiency). If $\mathcal{C}$-$\mathrm{cl}\,(A) - A$ is $\mathcal{C}$-closed, then $U = X - (\mathcal{C}$-$\mathrm{cl}\,(A) - A) \in \mathcal{C}$ and $A \subseteq U$. Also, $A \subseteq \mathcal{C}$-$\mathrm{cl}\,(A) \in \mathcal{C}^c$. Since $A \subseteq U \cap \mathcal{C}$-$\mathrm{cl}\,(A) \subseteq A$, we have $A = U \cap \mathcal{C}$-$\mathrm{cl}\,(A)$ is $\mathcal{C}$-locally closed.

(Necessity). Suppose that $A = U \cap F$ for some $U \in \mathcal{C}$ and some $F \in \mathcal{C}^c$. Then $A \subseteq F$ implies $\mathcal{C}$-$\mathrm{cl}\,(A) \subseteq F$ so that $A = U \cap \mathcal{C}$-$\mathrm{cl}\,(A)$. Thus, $\mathcal{C}$-$\mathrm{cl}\,(A) - A = (\mathcal{C}$-$\mathrm{cl}\,(A) - A) \cap (X - U) \in \mathcal{C}^c$. $\square$

Corollary 8. *A subset A of a union space $(X, \mathcal{C})$ is $\mathcal{C}$-closed if and only if it is both $\mathcal{C}$-locally closed and $\mathcal{C}$-g-closed.*

Proof. The necessity is clear. For the sufficiency, let $A \subseteq X$ be both $\mathcal{C}$-locally closed and $\mathcal{C}$-g-closed. By the proposition, above $\mathcal{C}$-$\mathrm{cl}\,(A) - A$ is a closed subset of itself so that by 2 of the theorem above $\mathcal{C}$-$\mathrm{cl}\,(A) - A = \varnothing$. Thus, $A = \mathcal{C}$-$\mathrm{cl}\,(A) \in \mathcal{C}^c$. $\square$

It follows that $\mathcal{C}$-local closedness and $\mathcal{C}$-g-closedness are independent since each is strictly weaker than $\mathcal{C}$-closedness.

Proposition 9. *If A is a $\mathcal{C}$-g-closed subset of a union space $(X, \mathcal{C})$ and $A \subseteq B \subseteq \mathcal{C}$-$\mathrm{cl}\,(A)$, then B is $\mathcal{C}$-g-closed.*

Proof. Since A is $\mathcal{C}$-g-closed, $\mathcal{C}$-$cl(A) - A$ contains no nonempty $\mathcal{C}$-closed set. Since $\mathcal{C}$-$\mathrm{cl}\,(B) - B \subseteq \mathcal{C}$-$\mathrm{cl}\,(A) - A$ we have that B must be $\mathcal{C}$-g-closed since $\mathcal{C}$-$\mathrm{cl}\,(B) - B$ contains no nonempty $\mathcal{C}$-closed set. $\square$

It is interesting to note that the family $\mathcal{GO}(X,\tau)$ of g-open subsets of a topological space (X,τ) has property **C**, i.e., is closed under finite intersection, but strongly fails to have property **B** in the sense that even a finite union of two g-open sets may fail to be g-open [**10**]. Since g-openness is a special case of $\mathcal{C}$-g-openness in a union space, evidently the collection $\mathcal{GO}(X,\mathcal{C})$ of $\mathcal{C}$-g-open subsets of the union space $(X,\mathcal{C})$ is not closed generally under finite union. We now show that, unlike the topological case, $\mathcal{GO}(X,\mathcal{C})$ also fails to be closed generally under finite intersection. Equivalently, a finite union of $\mathcal{C}$-g-closed sets may fail to be $\mathcal{C}$-g-closed.

Example 10. Let $X = \{a,b,c\}$, and let $\mathcal{C} = \{\varnothing,\{a,b\},\{a,c\},\{b,c\}, X\}$. Then $\mathcal{C}$ is a union structure on X with $\mathcal{C}^c = \{\varnothing,\{a\},\{b\},\{c\},X\}$. Then $\{a\}$ and $\{c\}$ are $\mathcal{C}$-g-closed being $\mathcal{C}$-closed. Yet, $\{a\}\cup\{c\} = \{a,c\}$ is not $\mathcal{C}$-g-closed since $\{a,c\} \subseteq \{a,c\} \in \mathcal{C}$ and $\mathcal{C}\text{-cl}\,(\{a,c\}) = X \nsubseteq \{a,c\}$.

One partially positive result is that the union of a $\mathcal{C}$-open set with a $\mathcal{C}$-g-open set is a $\mathcal{C}$-g-open set. Equivalently, we have the following.

Proposition 11. *If A is a C-g-closed subset of a union space $(X,\mathcal{C})$ and $F \in \mathcal{C}^c$, then $A\cap F$ is $\mathcal{C}$-g-closed.*

Proof. Suppose that U is $\mathcal{C}$-open and $A\cap F \subseteq U$. Then $A \subseteq U\cup(X-F)$ and $U\cup(X-F)$ is $\mathcal{C}$-open. Therefore, $\mathcal{C}\text{-cl}\,(A) \subseteq U\cup(X-F)$ so that $A\cap F \subseteq \mathcal{C}\text{-cl}\,(A)\cap F \subseteq U$. Since $\mathcal{C}\text{-cl}\,(A)\cap F$ is $\mathcal{C}$-closed, $\mathcal{C}$-cl $(A\cap F) \subseteq \mathcal{C}\text{-cl}\,(A)\cap F \subseteq U$. Evidently, $A\cap F$ is $\mathcal{C}$-g-closed. ◻

Proposition 12. *Each subset A of a union space $(X,\mathcal{C})$ is $\mathcal{C}$-g-closed if and only if $\mathcal{C} = \mathcal{C}^c$.*

Proof. (Sufficiency). Suppose that $\mathcal{C} = \mathcal{C}^c$, and let A be any subset of X. Since $\mathcal{C}\text{-cl}\,(A) - A$ contains no nonempty $\mathcal{C}$-open subset it also contains no nonempty $\mathcal{C}$-closed set. Thus, A must be $\mathcal{C}$-g-closed.

(Necessity). Assume that every subset of X is $\mathcal{C}$-g-closed, and let U be any $\mathcal{C}$-open subset of X. Then $U \subseteq U$ and U is $\mathcal{C}$-g-closed implies that

$\mathcal{C}$-cl$(U) \subseteq U$ so that $U = \mathcal{C}$-cl$(U) \in \mathcal{C}^c$. Thus, $\mathcal{C} \subseteq \mathcal{C}^c$. Now, if $F \in \mathcal{C}^c$, then $X - F \in \mathcal{C}$ and since $\mathcal{C} \subseteq \mathcal{C}^c$, we have that $F = X - (X - F) \in \mathcal{C}$ so that $\mathcal{C} = \mathcal{C}^c$. □

There are natural extensions of the classical separation axioms for topological spaces to union spaces. We mention a few even though we are primarily concerned with low separation due to our focus on $\mathcal{C}$-g-closed sets.

Definition 13. A union space $(X, \mathcal{C})$ has $\mathcal{C}$-T_0 separation if, for any two distinct elements x and y of X, a $\mathcal{C}$-open set U exists such that $|U \cap \{x, y\}| = 1$.

Definition 14. A union space $(X, \mathcal{C})$ has $\mathcal{C}$-$T_{1/2}$ separation if each $\mathcal{C}$-g-closed subset of X is $\mathcal{C}$-closed.

Definition 15. A union space $(X, \mathcal{C})$ has $\mathcal{C}$-T_1 separation if $\{x\}$ is $\mathcal{C}$-closed for each $x \in X$.

Definition 16. A union space $(X, \mathcal{C})$ has $\mathcal{C}$-T_2 separation (or is a $\mathcal{C}$-Hausdorff space) if for the pair $\{x, y\}$ of distinct elements of X, disjoint $\mathcal{C}$-open sets U_x and U_y exist with $x \in U_x$ and $y \in U_y$.

Definition 17. A union space $(X, \mathcal{C})$ is $\mathcal{C}$-regular if, for each $x \in X$ and each $\mathcal{C}$-closed set $F \subseteq X - \{x\}$, disjoint $\mathcal{C}$-open sets U and V exist such that $x \in U$ and $F \subseteq V$. The union space has $\mathcal{C}$-T_3 separation if it is both $\mathcal{C}$-T_1 and $\mathcal{C}$-regular.

Definition 18. A union space $(X, \mathcal{C})$ is $\mathcal{C}$-normal if, for each pair of disjoint $\mathcal{C}$-closed subsets E and F, disjoint $\mathcal{C}$-open sets U and V exist such that $E \subseteq U$ and $F \subseteq V$. The union space has $\mathcal{C}$-T_4 separation if it is both $\mathcal{C}$-T_1 and $\mathcal{C}$-normal.

Clearly, for any union space $(X, \mathcal{C})$, $\mathcal{C}$-$T_4 \Rightarrow \mathcal{C}$-$T_3 \Rightarrow \mathcal{C}$-$T_2 \Rightarrow \mathcal{C}$-$T_1 \Rightarrow \mathcal{C}$-$T_0$. Further, examples from the case when $\mathcal{C}$ is a topology warrant that none of these implications is reversible. We now show, analogous to the topological case, that $\mathcal{C}$-$T_{1/2}$ is intermediate to $\mathcal{C}$-T_1 and $\mathcal{C}$-T_0.

Theorem 19. *For any union space $(X, \mathcal{C})$, $\mathcal{C}$-$T_1 \Rightarrow \mathcal{C}$-$T_{1/2} \Rightarrow \mathcal{C}$-$T_0$.*

Proof. Assume that $(X, \mathcal{C})$ is $\mathcal{C}$-T_1, and let A be a $\mathcal{C}$-g-open subset of X. We must show that A is $\mathcal{C}$-open. If $A = \varnothing$, then $A \in \mathcal{C}$. If $A \neq \varnothing$ and $x \in A$, then since $\{x\}$ is $\mathcal{C}$-closed, $x \in \mathcal{C}\text{-int}\,(A)$. Thus, $A \subseteq \mathcal{C}\text{-int}\,(A)$, and we have $A = \mathcal{C}\text{-int}\,(A) \in \mathcal{C}$. This implies that $(X, \mathcal{C})$ is $\mathcal{C}$-$T_{1/2}$. Now suppose that $(X, \mathcal{C})$ is not $\mathcal{C}$-T_0. A pair $\{x, y\}$ of distinct points x and y in X exists such that, for all $U \in \mathcal{C}$, $U \cap \{x, y\} \neq \varnothing \Rightarrow \{x, y\} \subseteq U$. It follows that both $x \in \mathcal{C}\text{-cl}\,(\{y\})$ and $y \in \mathcal{C}\text{-cl}\,(\{x\})$. In fact, $\mathcal{C}\text{-cl}\,(\{x\}) = \mathcal{C}\text{-cl}\,(\{y\})$ since each is contained in the other. Let $A = \mathcal{C}\text{-cl}\,(\{x\}) - \{x\}$. Then $y \in A$ since $y \neq x$. We show that $(X, \mathcal{C})$ is not $\mathcal{C}$-$T_{1/2}$ by showing that A is $\mathcal{C}$-g-closed but not $\mathcal{C}$-closed. For all $U \in \mathcal{C}$, $x \in U$ implies $y \in U$ so that $U \cap A \neq \varnothing$. This implies that A is not $\mathcal{C}$-closed since $x \in \mathcal{C}\text{-cl}\,(A) - A$. Now suppose that U is $\mathcal{C}$-open and $A \subseteq U$. Then, $y \in A$ implies that $y \in U$ which in turn implies that $x \in U$. Therefore, $\mathcal{C}\text{-cl}\,(\{x\}) \subseteq U$. But, $A \subseteq \mathcal{C}\text{-cl}\,(\{x\}) \subseteq U$ implies that $\mathcal{C}\text{-}cl(A) \subseteq \mathcal{C}\text{-cl}\,(\{x\}) \subseteq U$ so that A is $\mathcal{C}$-g-closed. $\square$

Examples showing that the implications of this theorem are irreversible were given by Levine [**10**] in the case that $\mathcal{C}$ is a topology. Levine also gave an example showing that $T_{1/2}$ separation in topological spaces behaves non-standardly. In particular, he showed that $T_{1/2}$ separation is not finitely productive in the sense that a product of two $T_{1/2}$ spaces will generally fail to be $T_{1/2}$. Of course, this prevents $\mathcal{C}$-$T_{1/2}$ from being finitely productive if union product spaces are defined in a way that generalizes topological product spaces. Such a definition is now given.

Definition 20. Let $\{(X_\gamma, \mathcal{C}_\gamma) \mid \gamma \in \Gamma\}$ be a nonempty family of nonempty union spaces. The union product space $\prod_{\gamma\in\Gamma}(X_\gamma, \mathcal{C}_\gamma) = (X, \mathcal{C})$ with $X = \prod_{\gamma\in\Gamma} X_\gamma$ and $\mathcal{C} = \prod_{\gamma\in\Gamma} \mathcal{C}_\gamma$ is the family of all unions of collections of sets of the form $\prod_{\gamma\in\Gamma} U_\gamma$ where for some finite subset $F \subseteq \Gamma$, $U_\gamma = X_\gamma$ if $\gamma \notin F$ and $U_\gamma \in \mathcal{C}_\gamma$ for $\gamma \in F$.

Note that the product union structure as defined here is a minimal structure since both $\varnothing \in \mathcal{C}$ and $X \in \mathcal{C}$. It is easy to see that, for each

$\gamma \in \Gamma$ the projection function $p_\gamma : X \to X_\gamma$ is $\mathcal{C}\mathcal{C}_\gamma$-open in the sense that $p_\gamma(U) \in \mathcal{C}_\gamma$ for each $U \in \mathcal{C}$ if $X_\gamma \in \mathcal{C}_\gamma$. In any case, we have the following lemma and theorem.

Lemma 21. *If $(X, \mathcal{C}) = \prod_{\gamma\in\Gamma}(X_\gamma, \mathcal{C}_\gamma)$ is a union product space, and for each $\gamma \in \Gamma$, $A_\gamma \subseteq X_\gamma$ is $\mathcal{C}_\gamma$-g-closed, then $A = \prod_{\gamma\in\Gamma} A_\gamma$ is $\mathcal{C}$-g-closed.*

Proof. Suppose A is not $\mathcal{C}$-g-closed. Then $\mathcal{C}$-cl$\,(A) - A$ contains a nonempty $\mathcal{C}$-closed subset F. For $x = (x_\gamma) \in F$, $\mathcal{C}$-cl$\,(\{x\}) = \prod_{\gamma\in\Gamma} \mathcal{C}_\gamma$-cl$\,(\{x_\gamma\}) \subseteq F \subseteq \mathcal{C}$-cl$\,(A) - A$. Now, for each $\gamma \in \Gamma$, $\mathcal{C}_\gamma$-cl$\,(\{x_\gamma\}) \subseteq \mathcal{C}_\gamma$-cl$\,(A_\gamma)$ and $\mathcal{C}_\gamma$-cl$\,(A_\gamma) - A_\gamma$ contains no nonempty $\mathcal{C}_\gamma$-closed set. Thus, $\mathcal{C}_\gamma$-cl$\,(\{x_\gamma\}) \cap A_\gamma \neq \varnothing$ for each $\gamma \in \Gamma$. By the axiom of choice, choose $y_\gamma \in \mathcal{C}_\gamma$-cl$\,(\{x_\gamma\}) \cap A_\gamma$ for each γ and note that, for $y = (y_\gamma) \in X$, $y \in \mathcal{C}$-cl$\,(\{y\}) = \prod_{\gamma\in\Gamma} \mathcal{C}_\gamma$-cl$\,(\{y_\gamma\}) \subseteq \prod_{\gamma\in\Gamma} \mathcal{C}_\gamma$-cl$\,(\{x_\gamma\}) = \mathcal{C}$-cl$\,(\{x\}) \subseteq \mathcal{C}$-cl$\,(A) - A$ implies that $y \in A = \prod_{\gamma\in\Gamma} A_\gamma$. This contradicts the fact that each $y_\gamma \in A_\gamma$. $\square$

Theorem 22. *If $\{(X_\gamma, \mathcal{C}_\gamma) \mid \gamma \in \Gamma\}$ is a nonempty family of nonempty union spaces, then each factor union space $(X\gamma, \mathcal{C}_\gamma)$ is $\mathcal{C}_\gamma$-$T_{1/2}$ if the union product space $\prod_{\gamma\in\Gamma}(X_\gamma, \mathcal{C}_\gamma)$ is $\mathcal{C}$-$T_{1/2}$.*

Proof. Suppose that $(X, \mathcal{C}) = \prod_{\gamma\in\Gamma}(X_\gamma, \mathcal{C}_\gamma)$ is $\mathcal{C}$-$T_{1/2}$, and let A_{γ_0} be a $\mathcal{C}_{\gamma_0}$-g-closed subset of X_{γ_0} for some $\gamma_0 \in \Gamma$. Since each X_{γ_0} is $\mathcal{C}_{\gamma_0}$-closed, we have $A = \prod_{\gamma\in\Gamma} A_\gamma$ is $\mathcal{C}$-g-closed by the lemma if $A_\gamma = X_\gamma$ for each $\gamma \in \Gamma - \{\gamma_0\}$. Then A is C-closed since $(X, \mathcal{C})$ is $\mathcal{C}$-$T_{1/2}$ and $X - A$ is $\mathcal{C}$-open. But, $X - A = \prod_{\gamma\in\Gamma} B_\gamma$ where $B_\gamma = X_\gamma$ for $\gamma \in \Gamma - \{\gamma_0\}$ and $B_{\gamma_0} = X_{\gamma_0} - A_{\gamma_0}$. Since $X - A$ is a union of $\mathcal{C}$-open sets of the form $\prod_{\gamma\in\Gamma} U_\gamma$ where for some finite subset $F \subseteq \Gamma$, $U_\gamma = X_\gamma$ if $\gamma \notin F$ and $U_\gamma \in \mathcal{C}_\gamma$ for $\gamma \in F$, and since the image of such a set under the projection p_γ for $\gamma \in F$ is a $\mathcal{C}_\gamma$-open set U_γ, we have that $p_{\gamma_0}(X - A) = X_{\gamma_0} - A_{\gamma_0} \in \mathcal{C}_{\gamma_0}$. For, $A_{\gamma_0} \neq \varnothing$ implies that $\gamma_0 \in F$ for all such sets $\prod_{\gamma\in\Gamma} U_\gamma$ whose union forms $X - A$. On the other hand, if $A_{\gamma_0} = \varnothing$, then X_{γ_0} is a union of $\mathcal{C}_{\gamma_0}$-open sets in which case $X_{\gamma_0} \in \mathcal{C}_{\gamma_0}$ so that $A_{\gamma_0} = \varnothing$ is $\mathcal{C}_{\gamma_0}$-closed. In either case, we have that $A_{\gamma_0} \in \mathcal{C}^c_{\gamma_0}$ so that $(X_{\gamma_0}, \mathcal{C}_{\gamma_0})$ is a $\mathcal{C}_{\gamma_0}$-$T_{1/2}$ union space. $\square$

Levine [**10**] showed that generally an infinite product of g-open subsets is not a g-open subset of the product space. However, a finite product of g-open subsets, with one set chosen from each of the finitely many factor spaces, is a g-open subset of the finite product space.

Proposition 23. *If $(X,\mathcal{C})$ and $(Y,\mathcal{D})$ are union spaces, A is a $\mathcal{C}$-g-open subset of X and B is a $\mathcal{D}$-g-open subset of Y, then $A\times B$ is an $\mathcal{E}$-g-open subset of the union product space $(Z,\mathcal{E})$ where $Z=X\times Y$ and $\mathcal{E}$ is the collection of all unions of families of subsets of Z of the form $U\times V$ with $U\in\mathcal{C}$ and $V\in\mathcal{D}$.*

Proof. Let F be an $\mathcal{E}$-closed subset of $A\times B$. We must show that $F\subseteq \mathcal{E}\text{-int}\,(A\times B)$. But, it is easily seen that $\mathcal{E}\text{-int}\,(A\times B)=\mathcal{E}\text{-int}\,(A)\times\mathcal{E}\text{-int}\,(B)$. Now, for any $z=(x,y)\in F$, $\mathcal{E}\text{-cl}\,(\{z\})=\mathcal{C}\text{-cl}\,(\{x\})\times\mathcal{D}\text{-cl}\,(\{y\})\subseteq F\subseteq A\times B$ implies that $\mathcal{C}\text{-cl}\,(\{x\})\subseteq A$ and $\mathcal{D}\text{-cl}\,(\{y\})\subseteq B$. Therefore, $\mathcal{C}\text{-cl}\,(\{x\})\subseteq\mathcal{C}\text{-int}\,(A)$ and $\mathcal{D}\text{-cl}\,(\{y\})\subseteq\mathcal{D}\text{-int}\,(B)$, and hence $z=(x,y)\in\mathcal{E}\text{-int}\,(A\times B)$. Evidently, $F\subseteq\mathcal{E}\text{-int}\,(A\times B)$ and $A\times B$ is $\mathcal{E}$-g-open. $\square$

The proof of the following flows naturally along classical lines and is therefore omitted.

Theorem 24. *A union space $(X,\mathcal{C})$ is $\mathcal{C}$-T_2 if and only if the diagonal $\Delta(X^2)=\{(x,x)\mid x\in X\}$ is $\mathcal{C}\prod\mathcal{C}$-closed in the union product space $(X,\mathcal{C})\times(X,\mathcal{C})$.*

What about low separation in subspaces of union spaces?

Definition 25. If $(X,\mathcal{C})$ is a union space and $A\subseteq X$, then the union subspace A is the union space $(A,\mathcal{C}_A)$ where $\mathcal{C}_A$ is the union structure defined by $\mathcal{C}_A=\{U\cap A\mid U\in\mathcal{C}\}$.

The following is easily proven.

Lemma 26. *If $(X,\mathcal{C})$ is a union space with $B\subseteq A\subseteq X$, then $\mathcal{C}_A\text{-cl}\,(B)=A\cap\mathcal{C}\text{-cl}\,(B)$.*

It is clear that a union subspace A of a C-T_1 union space $(X,\mathcal{C})$ is $\mathcal{C}_A$-T_1 and A is C_A-T_0 if $(X,\mathcal{C})$ is $\mathcal{C}$-T_0. But what about union subspaces of $\mathcal{C}$-$T_{1/2}$ union spaces? A partial answer will be given following our next result which simultaneously generalizes a result of Levine [**10**] and a result of Ravi, Thivagar and Jinjinli [**27**].

Theorem 27. *If $(X,\mathcal{C})$ is a union space, A is a $\mathcal{C}$-g-closed subset of X, and B is a $\mathcal{C}_A$-g-closed subset of A, then B is a $\mathcal{C}$-g-closed subset of X.*

Proof. Suppose that $B \subseteq U$ for some $U \in \mathcal{C}$. Then $B \subseteq U \cap A \in \mathcal{C}_A$ implies that $\mathcal{C}_A\text{-cl}\,(B) = A \cap \mathcal{C}\text{-cl}\,(B) \subseteq U \cap A$. Therefore, $A \subseteq U \cup (X - \mathcal{C}\text{-cl}\,(B)) \in \mathcal{C}$ implies that $\mathcal{C}\text{-cl}\,(A) \subseteq U \cup (X - \mathcal{C}\text{-cl}\,(B))$. But, $B \subseteq A$ implies that $\mathcal{C}\text{-cl}\,(B) \subseteq \mathcal{C}\text{-cl}\,(A)$ which implies that $\mathcal{C}\text{-cl}\,(B) \subseteq U$ since $\mathcal{C}\text{-cl}\,(B) \cap (X - \mathcal{C}\text{-cl}\,(B)) = \varnothing$. Evidently, B is $\mathcal{C}$-g-closed. $\square$

Corollary 28. *If $(X,\mathcal{C})$ is a $\mathcal{C}$-$T_{1/2}$ union space and A is a $\mathcal{C}$-closed subset of X, then the union subspace $(A,\mathcal{C}_A)$ has $\mathcal{C}_A$-$T_{1/2}$ separation.*

Proof. Suppose that B is a $\mathcal{C}_A$-g-closed subset of $(A,\mathcal{C}_A)$. Then, since A is $\mathcal{C}$-closed or equivalently $\mathcal{C}$-g-closed since $(X,\mathcal{C})$ is $\mathcal{C}$-$T_{1/2}$, we have that B is $\mathcal{C}$-g-closed, and hence $\mathcal{C}$-closed in X. Therefore, $B = B \cap A$ is $\mathcal{C}_A$-closed in A. Evidently, $(A,\mathcal{C}_A)$ is a $\mathcal{C}_A$-$T_{1/2}$ union subspace of $(X,\mathcal{C})$. $\square$

The corollary states that $T_{1/2}$ for union spaces is weakly hereditary in the sense that it is inherited by closed union subspaces. To prove that in topological spaces $T_{1/2}$ is not fully hereditary, an example of a $T_{1/2}$ but not T_1 space (X,τ) is needed having a subspace (A,τ_A) which is T_0 but not $T_{1/2}$.

3. Functions. Continuity, central to topological theory, requires only that preimages of open sets be open. Many related classes of functions in topological theory have been investigated each requiring that preimages of one type of set belong to another class of sets. These classes of sets that are so connected by the function are often union structures. We generalize all classes of functions of this type.

Definition 29. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is $\mathcal{CD}$-continuous if $f^{-1}(V) \in \mathcal{C}$ for each $V \in \mathcal{D}$.

Equivalently, f is $\mathcal{CD}$-continuous if preimages of $\mathcal{D}$-closed sets are $\mathcal{C}$-closed. Many studied classes of $\mathcal{CD}$-continuous functions, such as the classes of pre-, semi-, α- and β-continuous functions arise by letting (X, τ) and $(Y, \mathcal{D})$ be topological spaces and letting $\mathcal{C}$ be a special family of subsets of X such as $\mathcal{PO}(X, \tau)$, $\mathcal{SO}(X, \tau)$, τ^α or $\beta\mathcal{O}(X, \tau)$. Other classes of functions that have been studied allow $\mathcal{D}$ to be a special family of subsets of a space (Y, σ) but require $\mathcal{C}$ to be a topology. These include the classes of strongly α-irresolute functions [**11**] and strongly semi-continuous functions [**1**]. Still other classes of interest allow both $\mathcal{C}$ and $\mathcal{D}$ to represent special families of subsets of spaces (X, τ) and (Y, σ), respectively, such as the class of irresolute functions [**8**], the class of α-irresolute functions [**12**], the class of almost irresolute functions [**5**], the class of almost α-irresolute functions [**5**] and the class of semi α-irresolute functions [**4**]. All of these classes as well as their generalizations by functions between spaces having minimal structures satisfying property **B** [**17, 19, 20, 22, 23**] are unified and subsumed by the class of $\mathcal{CD}$-continuous functions between union spaces $(X, \mathcal{C})$ and $(Y, \mathcal{D})$.

Definition 30. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is $\mathcal{CD}$-closed if $f(F)$ is $\mathcal{D}$-closed whenever F is $\mathcal{C}$-closed.

Definition 31. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is $\mathcal{CD}$-open if $f(U) \in \mathcal{D}$ whenever $U \in \mathcal{C}$.

Definition 32. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is contra $\mathcal{CD}$-continuous if $f^{-1}(F) \in \mathcal{C}$ whenever F is $\mathcal{D}$-closed.

The following generalizes both a-continuity of Baker [**3**] and a generalized version of Baker's a-continuity for functions between bitopological spaces due to Thivagar, Israel and Ravi [**28**].

Definition 33. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is $\mathcal{CD}$-a-continuous ($\mathcal{CD}$-approximately continuous) if $\mathcal{C}\text{-cl}\,(A) \subseteq f^{-1}(V)$ whenever $V \in \mathcal{D}$ and A is a $\mathcal{C}$-g-closed set with $A \subseteq f^{-1}(V)$.

Equivalently, f is $\mathcal{C}\mathcal{D}$-a-continuous if $f^{-1}(F) \subseteq \mathcal{C}\text{-int}\,(A)$ whenever $F \in \mathcal{D}^c$ and A is a $\mathcal{C}$-g-open set with $f^{-1}(F) \subseteq A$.

Definition 34. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is $\mathcal{C}\mathcal{D}$-a-closed if $f(F) \subseteq \mathcal{D}\text{-int}\,(B)$ whenever $F \in \mathcal{C}^c$ and B is a $\mathcal{D}$-g-open set with $f(F) \subseteq B$.

Definition 35. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is contra $\mathcal{C}\mathcal{D}$-a-continuous if $\mathcal{C}\text{-cl}\,(A) \subseteq f^{-1}(F)$ whenever $F \in \mathcal{D}^c$ and A is a $\mathcal{C}$-g-closed set with $A \subseteq f^{-1}(F)$.

Definition 36. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is $\mathcal{C}\mathcal{D}$-g-continuous if $f^{-1}(F)$ is $\mathcal{C}$-g-closed for each $F \in \mathcal{D}^c$.

Definition 37. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is contra $\mathcal{C}\mathcal{D}$-g-continuous if $f^{-1}(V)$ is $\mathcal{C}$-g-closed for each $V \in \mathcal{D}$.

Definition 38. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is $\mathcal{C}\mathcal{D}$-g-closed if $f(F)$ is $\mathcal{D}$-g-closed for each $F \in \mathcal{C}^c$.

Definition 39. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is $\mathcal{C}\mathcal{D}$-g-irresolute if $f^{-1}(B)$ is $\mathcal{C}$-g-closed whenever B is $\mathcal{D}$-g-closed.

Easily, every $\mathcal{C}\mathcal{D}$-continuous function is both $\mathcal{C}\mathcal{D}$-a-continuous and $\mathcal{C}\mathcal{D}$-g-continuous. Likewise, every $\mathcal{C}\mathcal{D}$-closed function is both $\mathcal{C}\mathcal{D}$-a-closed and $\mathcal{C}\mathcal{D}$-g-closed. The two examples below demonstrate the independence of $\mathcal{C}\mathcal{D}$-a-continuity and $\mathcal{C}\mathcal{D}$-g-continuity. But first we give the following.

Proposition 40. *Let $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ be a function between union spaces. Then f is contra $\mathcal{C}\mathcal{D}$-continuous implies that f is $\mathcal{C}\mathcal{D}$-a-continuous, and f is $\mathcal{C}\mathcal{D}$-continuous implies that f is contra $\mathcal{C}\mathcal{D}$-a-continuous.*

Proof. If f is contra $\mathcal{C}\mathcal{D}$-continuous and $V \in \mathcal{D}$, then $f^{-1}(V)$ is $\mathcal{C}$-closed so that, for each $\mathcal{C}$-g-closed set A with $A \subseteq f^{-1}(V)$, we must have $\mathcal{C}$-cl$\,(A) \subseteq \mathcal{C}$-cl$\,(f^{-1}(V)) = f^{-1}(V)$. On the other hand, if f is $\mathcal{CD}$-continuous and $F \in \mathcal{D}^c$, then $f^{-1}(F) \in \mathcal{C}^c$ so that, again, $\mathcal{C}$-cl$\,(A) \subseteq f^{-1}(F)$ whenever A is $\mathcal{C}$-g-closed and $A \subseteq f^{-1}(F)$. $\square$

Since contra $\mathcal{CD}$-continuity and $\mathcal{CD}$-continuity are independent and each implies $\mathcal{CD}$-a-continuity, the class of $\mathcal{CD}$-a-continuous functions properly contains each of the classes of $\mathcal{CD}$-continuous functions and contra $\mathcal{CD}$-continuous functions. Similarly, contra $\mathcal{CD}$-a-continuous functions exist which are not $\mathcal{CD}$-continuous.

Example 41. Let $X = \{a, b\}$ be the two-point Sierpinski space with topology $\mathcal{C} = \{\varnothing, \{a\}, X\}$, and let $f : (X, \mathcal{C}) \to (X, \mathcal{C})$ be the non-identity bijection defined by $f(x) \neq x$. Then f is contra $\mathcal{CC}$-continuous, and hence $\mathcal{CC}$-a-continuous. But f is not $\mathcal{CC}$-g-continuous since $f^{-1}(\{a\}) = \{b\}$ is not $\mathcal{C}$-g-open. Indeed, $\{b\} \subseteq \{b\}$ and $\{b\} \in \mathcal{C}^c$, yet $\{b\} \nsubseteq \mathcal{C}$-int$\,(\{b\}) = \varnothing$.

Example 42. Let $X = \{a, b\}$, $\mathcal{C} = \{\varnothing, X\}$, $\mathcal{D} = \{\varnothing, \{a\}, X\}$, and let $f : (X, \mathcal{C}) \to (X, \mathcal{D})$ be the identity bijection. Since $\mathcal{C} = \mathcal{C}^c$, f is $\mathcal{CD}$-g-continuous. However, f is not $\mathcal{CD}$-a-continuous since $\{a\}$ is $\mathcal{C}$-g-closed and $\mathcal{D}$-open with $\{a\} \subseteq f^{-1}(\{a\})$, and yet $X = \mathcal{C}$-cl$\,(\{a\}) \nsubseteq f^{-1}(\{a\})$.

Because the union spaces in these two examples are topological spaces, the stronger result that a-continuity and g-continuity are independent has been obtained. Further, since the functions of these examples are bijections, we have that a-closedness and g-closedness of functions are independent. The following characterization of $T_{1/2}$ separation in union spaces extends and adds to results in each of [**3, 28**].

Theorem 43. *The following are equivalent for a union space* $(X, \mathcal{C})$.

1. $(X, \mathcal{C})$ *is* $\mathcal{C}$-$T_{1/2}$.

2. *For every union space* $(Y, \mathcal{D})$ *each function* $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ *is* $\mathcal{CD}$-*a-continuous.*

3. *For every union space* $(Y, \mathcal{D})$ *each function* $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ *is contra* $\mathcal{CD}$-*a-continuous.*

Proof. We show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$. If $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ and $(X, \mathcal{C})$ is $\mathcal{C}$-$T_{1/2}$, then, for any $V \in \mathcal{D}$ and any $\mathcal{C}$-g-closed set A with $A \subseteq f^{-1}(V)$, then A is $\mathcal{C}$-closed implies that $\mathcal{C}\text{-cl}\,(A) = A \subseteq f^{-1}(V)$ so that f is $\mathcal{CD}$-a-continuous. Now, if $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is a function and $F \in \mathcal{D}^c$, define $f_0 : (X, \mathcal{C}) \to (Y, \{\varnothing, F\})$ by $f_0(x) = f(x)$ for all $x \in X$. Since $(Y, \{\varnothing, F\})$ is a union space, by hypothesis, f_0 is $\mathcal{C}\{\varnothing, F\}$-$a$-continuous. Thus, for each C-g-closed subset A of X, $A \subseteq f_0^{-1}(F) = f^{-1}(F) \Rightarrow \mathcal{C}\text{-cl}\,(A) \subseteq f^{-1}(F)$. Evidently, f is contra $\mathcal{CD}$-a-continuous. Finally, let A be a C-g-closed subset of X, let $Y = X$, $\mathcal{D} = \{\varnothing, X - A\}$ and $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ the identity function. Since f is contra $\mathcal{CD}$-a-continuous and $A \in \mathcal{D}^c$ with $A \subseteq f^{-1}(A)$, we have that $\mathcal{C}\text{-cl}\,(A) \subseteq f^{-1}(A) = A$ so that $A = \mathcal{C}\text{-cl}\,(A) \in \mathcal{C}^c$. Evidently, $(X, \mathcal{C})$ is $\mathcal{C}$-$T_{1/2}$. □

The following extends results of [**3, 28**]. The proof is similar to that of the foregoing theorem and will be omitted.

Theorem 44. *A union space $(Y, \mathcal{D})$ is $\mathcal{D}$-$T_{1/2}$ if and only if, for every union space $(X, \mathcal{C})$, every function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is $\mathcal{CD}$-a-closed.*

The following further extends results of [**3, 28**].

Theorem 45. *If $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ are union spaces and $\mathcal{C} = \mathcal{C}^c$, then the following are equivalent for any function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$.*

1. *f is $\mathcal{CD}$-a-continuous.*
2. *f is contra $\mathcal{CD}$-continuous.*
3. *f is $\mathcal{CD}$-continuous.*
4. *f is contra $\mathcal{CD}$-a-continuous.*

Proof. We show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$. Let $V \in \mathcal{D}$ and $A = f^{-1}(V)$. Since $\mathcal{C} = \mathcal{C}^c$, A is $\mathcal{C}$-g-closed so that $\mathcal{C}\text{-cl}\,(A) \subseteq f^{-1}(V) = A$, and this implies that $A = f^{-1}(V) \in \mathcal{C}^c$. Evidently, f is contra $\mathcal{CD}$-continuous. But, f is contra $\mathcal{CD}$-continuous and $\mathcal{C} = \mathcal{C}^c$ implies that $f^{-1}(V) \in \mathcal{C}^c = \mathcal{C}$ for each $V \in \mathcal{D}$ so that f is $\mathcal{CD}$-continuous. By Proposition 40 every $\mathcal{CD}$-continuous function is contra $\mathcal{CD}$-a-continuous. And, in conclusion, if f is contra $\mathcal{CD}$-a-

continuous and $F \in \mathcal{D}^c$, then by Proposition 12 $f^{-1}(F)$ is $\mathcal{C}$-g-closed and $f^{-1}(F) \subseteq f^{-1}(F)$ implies $\mathcal{C}\text{-cl}\,(f^{-1}(F)) \subseteq f^{-1}(F)$, and hence $f^{-1}(F) \in \mathcal{C}^c$. It follows that f is $\mathcal{CD}$-continuous and hence $\mathcal{CD}$-a-continuous. □

The following extends Theorem 14 of [**3**].

Theorem 46. *If $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is $\mathcal{CD}$-a-continuous and A is a $\mathcal{C}$-g-closed subset of X, then $f\,|_A : (A, \mathcal{C}_A) \to (Y, \mathcal{D})$ is $\mathcal{C}_A\mathcal{D}$-a-continuous.*

Proof. Let B be a $\mathcal{C}_A$-g-closed subset of $(f\,|_A)^{-1}(V)$ for $V \in \mathcal{D}$. Since A is $\mathcal{C}$-g-closed, we have that B is also $\mathcal{C}$-g-closed and $B \subseteq A \cap f^{-1}(V) \subseteq f^{-1}(V)$. So, $\mathcal{C}\text{-cl}\,(B) \subseteq f^{-1}(V)$ implies that $\mathcal{C}_A\text{-cl}\,(B) = A \cap \mathcal{C}\text{-cl}\,(B) \subseteq A \cap f^{-1}(V) = (f\,|_A)^{-1}(V)$. □

Theorem 47. *If $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is $\mathcal{CD}$-g-continuous and A is a $\mathcal{C}$-closed subset of X, then $f\,|_A : (A, \mathcal{C}_A) \to (Y, \mathcal{D})$ is $\mathcal{C}_A\mathcal{D}$-g-continuous.*

Proof. Let F be a $\mathcal{D}$-closed subset of Y. Then $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is $\mathcal{CD}$-g-continuous implies that $f^{-1}(F)$ is $\mathcal{C}$-g-closed so that, if also A is $\mathcal{C}$-closed, we have that $A \cap f^{-1}(F) = (f\,|_A)^{-1}(F)$ is $\mathcal{C}$-g-closed. Now, if $W \in \mathcal{C}_A$ and $(f\,|_A)^{-1}(F) \subseteq W$, then since $W = U \cap A$ for some $U \in \mathcal{C}$, it follows that $\mathcal{C}\text{-cl}\,((f\,|_A)^{-1}(F)) \subseteq U$ so that $\mathcal{C}_A\text{-cl}\,((f\,|_A)^{-1}(F)) = A \cap \mathcal{C}\text{-cl}\,((f\,|_A)^{-1}(F)) \subseteq U \cap A = W$. □

The following results on composite functions extend results found in [**3, 28**].

Proposition 48. *If $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ between union spaces is $\mathcal{CD}$-closed and $g : (Y, \mathcal{D}) \to (Z, \mathcal{E})$ between union spaces is $\mathcal{DE}$-a-closed, then $g \circ f$ is $\mathcal{CE}$-a-closed.*

Proof. Let $F \in \mathcal{C}^c$, and suppose that $(g \circ f)(F) = g(f(F)) \subseteq C$ for some $\mathcal{E}$-g-open subset C of Z. Since f is $\mathcal{CD}$-closed $f(F) \in \mathcal{D}^c$ and then, since g is $\mathcal{DE}$-a-closed, $g(f(F)) = (g \circ f)(F) \subseteq \mathcal{E}\text{-int}\,(C)$. □

Theorem 49. *If $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ between union spaces is $\mathcal{CD}$-a-closed and $g : (Y, \mathcal{D}) \to (Z, \mathcal{E})$ between union spaces is $\mathcal{DE}$-open and $\mathcal{DE}$-g-irresolute, then $g \circ f$ is $\mathcal{CE}$-a-closed.*

Proof. Let $F \in \mathcal{C}^c$, and suppose that $(g \circ f)(F) = g(f(F)) \subseteq C$ for some $\mathcal{E}$-g-open subset C of Z. Since g is $\mathcal{DE}$-g-irresolute, $g^{-1}(C)$ is $\mathcal{D}$-g-open and $f(F) \subseteq g^{-1}(C)$. Now, since f is $\mathcal{CD}$-a-closed, $f(F) \subseteq \mathcal{D}$-int$(g^{-1}(C))$ and we have that $(g \circ f)(F) \subseteq g(\mathcal{D}\text{-int}(g^{-1}(C)))$. Since g is also $\mathcal{DE}$-open, $g(\mathcal{D}\text{-int}(g^{-1}(C))) \in \mathcal{E}$ and $g(\mathcal{D}\text{-int}(g^{-1}(C))) \subseteq C$ implies that $g(\mathcal{D}\text{-int}(g^{-1}(C))) \subseteq \mathcal{E}\text{-int}(C)$, and it follows therefore that $(g \circ f)(F) \subseteq \mathcal{E}\text{-int}(C)$. □

Proposition 50. *If $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ between union spaces is $\mathcal{CD}$-a-continuous and $g : (Y, \mathcal{D}) \to (Z, \mathcal{E})$ between union spaces is $\mathcal{DE}$-continuous, then $g \circ f$ is $\mathcal{CE}$-a-continuous.*

Proof. Let $W \in \mathcal{E}$, and let $A \subseteq (g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ for some $\mathcal{C}$-g-closed subset A of X. Since g is $\mathcal{DE}$-continuous $g^{-1}(W) \in \mathcal{D}$ and since also f is $\mathcal{CD}$-a-continuous $\mathcal{C}\text{-cl}(A) \subseteq f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$. □

The following is an easy extension of the topological version, and the proof is omitted.

Proposition 51. *If $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ between union spaces is $\mathcal{CD}$-g-continuous and $g : (Y, \mathcal{D}) \to (Z, \mathcal{E})$ between union spaces is $\mathcal{DE}$-continuous, then $g \circ f$ is $\mathcal{CE}$ -g-continuous.*

Corollary 52. *If $f_\gamma : (X, \mathcal{C}) \to (Y_\gamma, \mathcal{D}_\gamma)$ is a function between union spaces for each $\gamma \in \Gamma$ and $f : (X, \mathcal{C}) \to \prod_{\gamma \in \Gamma}(Y_\gamma, \mathcal{D}_\gamma)$ is the function defined by $f(x) = (f_\gamma(x))$, then each f_γ is $\mathcal{CD}_\gamma$-a-continuous if f is $\mathcal{C}\prod_{\gamma \in \Gamma}\mathcal{D}_\gamma$-a-continuous.*

Proof. For each $\delta \in \Gamma$, the projection function $\pi_\delta : \prod_{\gamma \in \Gamma}(Y_\gamma, \mathcal{D}_\gamma) \to (Y_\delta, \mathcal{D}_\delta)$ is $\prod_{\gamma \in \Gamma}\mathcal{D}_\gamma\mathcal{D}_\delta$-continuous and $f_\delta = \pi_\delta \circ f$. □

The proof of the next result is similar and is therefore omitted.

Corollary 53. *If $f_\gamma : (X, \mathcal{C}) \to (Y_\gamma, \mathcal{D}_\gamma)$ is a function between union spaces for each $\gamma \in \Gamma$ and $f : (X, \mathcal{C}) \to \prod_{\gamma\in\Gamma}(Y_\gamma, \mathcal{D}_\gamma)$ is the function defined by $f(x) = (f_\gamma(x))$, then each f_γ is $\mathcal{C}\mathcal{D}_\gamma$-g-continuous if f is $\mathcal{C}\prod_{\gamma\in\Gamma}\mathcal{D}_\gamma$-g-continuous.*

The following inverse and forward preservation results extend theorems found in [**3, 28**].

Theorem 54. *If $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is both $\mathcal{CD}$-g-continuous and $\mathcal{CD}$-a-closed, then f is $\mathcal{CD}$-g-irresolute.*

Proof. Let B be $\mathcal{D}$-g-closed with $f^{-1}(B) \subseteq U$ and $U \in \mathcal{C}$. Then, $X - U \in \mathcal{C}^c$ and $X - U \subseteq X - f^{-1}(B) = f^{-1}(Y - B) \Rightarrow f(X - U) \subseteq \mathcal{D}$-int$\,(Y - B) = Y - \mathcal{D}$-cl$\,(B)$ since f is $\mathcal{CD}$-a-closed. It follows that $X - U \subseteq X - f^{-1}(\mathcal{D}\text{-cl}\,(B))$ so that $f^{-1}(B) \subseteq f^{-1}(\mathcal{D}\text{-}cl(B)) \subseteq U$. Since f is also $\mathcal{CD}$-g-continuous, $f^{-1}(\mathcal{D}\text{-cl}\,(B))$ is $\mathcal{C}$-g-closed and, thus, $\mathcal{C}$-cl$\,(f^{-1}(B)) \subseteq \mathcal{C}$-cl$\,(f^{-1}(\mathcal{D}\text{-cl}\,(B))) \subseteq U$. Therefore, $f^{-1}(B)$ is $\mathcal{C}$-g-closed. $\square$

Theorem 55. *If $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is both $\mathcal{CD}$-a-continuous and $\mathcal{CD}$-closed, then $f(A)$ is $\mathcal{D}$-g-closed whenever A is $\mathcal{C}$-g-closed.*

Proof. Let A be a $\mathcal{C}$-g-closed subset of X, and let $V \in \mathcal{D}$ with $f(A) \subseteq V$. Then, $A \subseteq f^{-1}(V) \Rightarrow \mathcal{C}$-cl$\,(A) \subseteq f^{-1}(V)$ since f is $\mathcal{CD}$-a-continuous. Since f is also $\mathcal{CD}$-closed, $f(\mathcal{C}\text{-cl}\,(A)) \in \mathcal{D}^c$ so that $f(A) \subseteq f(\mathcal{C}\text{-cl}\,(A)) \Rightarrow \mathcal{D}$-cl$\,(f(A)) \subseteq \mathcal{D}$-cl$\,(f(\mathcal{C}\text{-cl}\,(A))) = f(\mathcal{C}$-cl$\,(A)) \subseteq f(f^{-1}(V)) \subseteq V$. Evidently, $f(A)$ is $\mathcal{D}$-g-closed. $\square$

We conclude this paper with two decomposition results, one for $\mathcal{CD}$-continuity and one for contra $\mathcal{CD}$-continuity. In [**6**], a consequence of Theorem 30 (or Theorem 34) is that every function between topological spaces which is both g-continuous and $\mathcal{G}^*$-continuous is continuous, where a function is $\mathcal{G}^*$-continuous if each preimage of an open set is contained in the interior of each of its g-open supersets. In other words, the $\mathcal{G}^*$-continuous functions are precisely the contra a-continuous functions. A union space version follows.

Theorem 56. *A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ between union spaces is $\mathcal{CD}$-continuous if and only if f is both $\mathcal{CD}$-g-continuous and contra $\mathcal{CD}$-a-continuous.*

Proof. The necessity is clear by Proposition 40. For the sufficiency, let $F \in \mathcal{D}^c$ and note that $f^{-1}(F)$ is $\mathcal{C}$-g-closed since f is $\mathcal{CD}$-g-continuous. But, f is contra $\mathcal{CD}$-a-continuous and $f^{-1}(F) \subseteq f^{-1}(F) \Rightarrow \mathcal{C}\text{-cl}\,(f^{-1}(F)) \subseteq f^{-1}(F) \Rightarrow f^{-1}(F) \in \mathcal{C}^c$. Thus, f is $\mathcal{CD}$-continuous. ◻

The following generalizes $\mathcal{LC}$-continuity introduced in [**9**] for functions between topological spaces.

Definition 57. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is $\mathcal{CD}$-$\mathcal{LC}$-continuous if $f^{-1}(V)$ is C-locally closed for each $V \in \mathcal{D}$.

Definition 58. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is contra $\mathcal{CD}$-$\mathcal{LC}$-continuous if $f^{-1}(F)$ is C-locally closed for each $F \in \mathcal{D}^c$.

Lemma 59. *Every (contra) $\mathcal{CD}$-$\mathcal{LC}$-continuous function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ between union spaces is (contra) $\mathcal{CD}$-a-continuous.*

Proof. It suffices to show that, for each $\mathcal{C}$-g-closed subset A of X, and for each $\mathcal{C}$-locally closed subset $U \cap F$ where $U \in \mathcal{C}$ and $F \in \mathcal{C}^c$, we have that $A \subseteq U \cap F \Rightarrow \mathcal{C}\text{-cl}\,(A) \subseteq U \cap F$. But, $A \subseteq U \Rightarrow \mathcal{C}\text{-cl}\,(A) \subseteq U$ since A is $\mathcal{C}$-g-closed, and $A \subseteq F \Rightarrow \mathcal{C}\text{-cl}\,(A) \subseteq \mathcal{C}\text{-cl}\,(F) = F$ since $F \in \mathcal{C}^c$. It follows that $\mathcal{C}\text{-cl}\,(A) \subseteq U \cap F$ whenever $A \subseteq U \cap F$. ◻

Corollary 60. *If $(X, \mathcal{C})$ is a covered union space and $(Y, \mathcal{D})$ is any union space, then a function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is $\mathcal{CD}$-continuous if and only if f is both $\mathcal{CD}$-g-continuous and contra $\mathcal{CD}$-$\mathcal{LC}$-continuous.*

Proof. For the necessity, we note that every $\mathcal{CD}$-continuous function from a covered union space $(X, \mathcal{C})$ into a union space $(Y, \mathcal{D})$ is contra $\mathcal{CD}$-$\mathcal{LC}$-continuous since every element of $\mathcal{C}^c$ is $\mathcal{C}$-locally closed. In particular, if $F \in \mathcal{C}^c$, then since $X \in \mathcal{C}$, we have that $F = X \cap F$ is $\mathcal{C}$-locally closed. The sufficiency follows from Theorem 56 and Lemma 59. $\square$

The following example shows that the covering condition in the hypothesis of Corollary 60 cannot be dropped.

Example 61. Let $f : (X, \mathcal{C}) \to (X, \mathcal{C})$ be the identity function with $X = \{a, b\}$ and $\mathcal{C} = \{\varnothing, \{a\}\}$. Clearly, f is $\mathcal{CC}$-continuous. However, f is not contra $\mathcal{CC}$-$\mathcal{LC}$-continuous since $X \in \mathcal{C}^c$, yet $X = f^{-1}(X)$ is not $\mathcal{C}$-locally closed.

This example also shows that the converse to Lemma 59 does not hold since the function of the example is contra $\mathcal{CC}$-a-continuous by Proposition 40. The following stronger example shows that contra a-continuity does not imply contra $\mathcal{LC}$-continuity for functions between topological spaces.

Example 62. Let $X = \{1, 2, \ldots\}$ have the cofinite topology τ, let $Y = \{0, 1\}$ have the Sierpinski topology $\sigma = \{\varnothing, \{0\}, Y\}$, and let $f : X \to Y$ be the characteristic function of the subset $E = \{2, 4, \ldots\}$ of even positive integers, i.e., $f(x) = 1$ if and only if $x \in E$. Easily the family of locally closed subsets of (X, τ) is $\tau \cup \tau^c$ and f is not contra $\mathcal{LC}$-continuous since $f^{-1}(\{1\}) = E$ is not locally closed. However, since (X, τ) is a T_1 space, and hence a $T_{1/2}$ space, f is contra a-continuous by Theorem 43.

Theorem 63. *A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ is contra $\mathcal{CD}$-continuous if and only if f is both contra $\mathcal{CD}$-g-continuous and $\mathcal{CD}$-a-continuous.*

Proof. The necessity follows from Proposition 40 and the fact that $\mathcal{C}$-closed sets are $\mathcal{C}$-g-closed. For the sufficiency, let the function f be both contra $\mathcal{CD}$-g-continuous and $\mathcal{CD}$-a-continuous, and let $V \in \mathcal{D}$. Then, $f^{-1}(V)$ is $\mathcal{C}$-g-closed since f is contra $\mathcal{CD}$-g-continuous and, since also f is $\mathcal{CD}$-a-continuous and $f^{-1}(V) \subseteq f^{-1}(V)$, we have that $\mathcal{C}\text{-cl}\,(f^{-1}(V)) \subseteq f^{-1}(V)$ so that $f^{-1}(V) \in \mathcal{C}^c$. Evidently, f is contra $\mathcal{CD}$-continuous. $\square$

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