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ON CERTAIN q-PHILLIPS OPERATORS

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ABSTRACT. In the present paper we propose certain q-Phillips operators. We also establish the approximation properties of these operators and estimate convergence results. Furthermore, we study the Voronovskaja-type asymptotic formula for the q-Phillips operators.

1. Introduction. R.S. Phillips [**26**] defined the well-known positive linear operators

$$P_n(f;x) = n\sum_{k=1}^{\infty} e(-nx)\,\frac{n^k x^k}{k!}\int_0^{\infty} e(-nt)\,\frac{n^{k-1}t^{k-1}}{(k-1)!}f(t)\,dt + e(-nx)\,f(0),$$

where $x \in [0,\infty)$. Some approximation properties of these operators were studied by Gupta and Srivastava [**12**] and by May [**21**]. The Bézier variant of these Phillips operators were proposed and studied by Gupta [**10**], where the rate of convergence for the Bézier variant of Phillips operators for bounded variation functions was discussed. Recently, intensive research has been conducted on operators based on q-integers, see [**1–3, 14, 17, 20, 22–31**. The q-Bernstein polynomials $B_{n,q}(f;x)$, $n = 1,2,\ldots$, $0 < q < \infty$, were introduced by G.M. Phillips in [**25**]. While for $q = 1$ these polynomials coincide with the classical ones, for $q \neq 1$ we obtain new polynomials possessing interesting properties, see [**14, 22, 23**]. In [**27**], Trif introduced the q-Meyer-König and Zeller operators for each positive integer n, and $f \in C[0,1]$. Like the classical operators, the q-Bernstein operators and the q-Meyer-König and Zeller operators share some good properties such as the shape-preserving properties and monotonicity for convex function. Very recently, Gupta [**9**] introduced and studied approximation properties of q-Durrmeyer operators. Gupta and Wang [**13**]

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introduced the q-Durrmeyer type operators and studied estimation of the rate of convergence for continuous functions in terms of modulus of continuity. In [**7, 10**] those authors studied some direct local and global approximation theorems for q-Durrmeyer operators $M_{n,q}$ for $0 < q < 1$. Some other analogues of the Bernstein-Durrmeyer operators related to the q-Bernstein basis functions $p_{n,k}(q;x)$ have been studied by Derriennic [**4**]. Very recently, Mahmudov in [**19**] introduced the following q-Szász-Mirakjan operator

$$\mathcal{S}_{n,q}(f;x) = \frac{1}{\prod_{j=0}^{\infty}(1+(1-q)q^j[n]x)} \sum_{k=0}^{\infty} f\left(\frac{[k]}{q^{k-2}[n]}\right) q^{[k(k-1)]/2} \frac{[n]^k x^k}{[k]!},$$

where $x \in [0,\infty)$, $0 < q < 1$, $f \in C[0,\infty)$ and investigated their approximation properties.

In this paper we introduce the following so called q-Phillips operators.

Definition 1. For $f \in R^{[0,\infty)}$,we define the following q-parametric Phillips operators

(1.1)
$$\mathcal{P}_{n,q}(f;x) = [n]\sum_{k=1}^{\infty} q^{k-1} S_{n,k}(q;qx) \times \int_0^{\infty/(1-q)} S_{n,k-1}(q;t) f(t)\, d_q t + e_q(-[n]qx) f(0),$$

where $x \in [0,\infty)$ and $S_{n,k}(q;x) = e_q(-[n]x)q^{[k(k-1)]/2}([n]^k x^k)/[k]!$.

These operators generalize the sequence of classical Phillips operators. Very recently Gupta [**8**] proposed another sequence of q-Phillips operators based on q-Szasz basis functions considered in [**3**] as

$$P_n^q(f(t);x) = \frac{[n]_q}{b_n}\sum_{k=1}^{\infty} s_{n,k}^q(x) \times \int_0^{qb_n/(1-q^n)} q^{1/2} s_{n,k-1}^q(t) f\left(q^{-1}t\right) d_q t + E_q\left(-[n]_q \frac{x}{b_n}\right) f(0),$$

where

$$s_{n,k}^{q}\left(x\right)=\frac{\left(\left[n\right]_{q}x\right)^{k}}{q^{(k+1)/2}\left[k\right]_{q}!\left(b_{n}\right)^{k}}E_{q}\left(-\left[n\right]_{q}\frac{x}{b_{n}}\right).$$

In this paper we study the approximation properties of the q-Phillips operators defined by (1.1), establish some local approximation results for continuous functions in terms of modulus of continuity and obtain inequalities for the weighted approximation error of q-Phillips operators. Furthermore, we study Voronovskaja-type asymptotic formula for the q-Phillips operators.

2. Moments.

Throughout the paper we employ the standard notations of q-calculus. q-integer and q-factorial are defined by

$$[n]:=\begin{cases}(1-q^{n})/(1-q) & \text{if } q\in R^{+}\backslash\{1\},\\ n & \text{if } q=1\end{cases}\qquad \text{for } n\in N \text{ and } [0]=0,$$
$$[n]!:=[1]\,[2]\ldots[n]\quad \text{for } n\in N \text{ and } [0]!=1.$$

For integers $0\leq k\leq n$, the q-binomial is defined by

$$\begin{bmatrix} n\\ k\end{bmatrix}:=\frac{[n]!}{[k]!\,[n-k]!}.$$

There are two q-analogues of the exponential function e^{z}, see [**15**],

$$e_{q}\left(z\right)=\sum_{k=0}^{\infty}\frac{z^{k}}{[k]!}=\frac{1}{\left(1-\left(1-q\right)z\right)_{q}^{\infty}},\quad |z|<\frac{1}{1-q},\ |q|<1,$$

and

$$E_{q}\left(z\right)=\prod_{j=0}^{\infty}\left(1+\left(1-q\right)q^{j}z\right)=\sum_{k=0}^{\infty}q^{k(k-1)/2}\frac{z^{k}}{[k]!} \tag{2.1}$$

$$=\left(1+\left(1-q\right)z\right)_{q}^{\infty},\quad |q|<1,$$

where $(1-x)_{q}^{\infty}=\prod_{j=0}^{\infty}(1-q^{j}x)$.

We set

(2.2)
$$\begin{aligned} S_{n,k}(q;x) &= \frac{1}{E_q([n]x)} q^{k(k-1)}/2 \frac{[n]^k x^k}{[k]!} \\ &= e_q(-[n]x) q^{k(k-1)/2} \frac{[n]^k x^k}{[k]!}, \qquad n = 1, 2, \ldots . \end{aligned}$$

It is clear that $S_{n,k}(q;x) \geq 0$ for all $q \in (0,1)$ and $x \in [0,\infty)$ and, moreover,

$$\sum_{k=0}^{\infty} s_{n,k}(q;x) = e_q(-[n]x) \sum_{k=0}^{\infty} q^{(k(k-1))/2} \frac{([n]x)^k}{[k]!} = 1.$$

The q-Jackson integrals and the q-improper integrals are defined as

$$\begin{aligned} \int_0^a f(t)\, d_q t &:= a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n, \qquad a > 0, \\ \int_0^{\infty/A} f(t)\, d_q t &:= (1-q) \sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, \quad A > 0. \end{aligned}$$

The two q-Gamma functions are defined as

$$\begin{aligned} \Gamma_q(x) &= \int_0^{1/(1-q)} t^{x-1} E_q(-qt)\, d_q t, \\ \gamma_q^A(x) &= \int_0^{\infty} A(1-q) t^{x-1} e_q(-t)\, d_q t. \end{aligned}$$

For every A, $x > 0$, one has

$$\Gamma_q(x) = K(A;x) \gamma_q^A(x),$$

where $K(A;x) = 1/(1+A)A^x(1+(1/A))_q^x(1+A)_q^{1-x}$. In particular, for any positive integer n,

$$K(A;n) = q^{n(n-1)/2} \quad \text{and} \quad \Gamma_q(n) = q^{n(n-1)/2} \gamma_q^A(n),$$

see [**5**].

In this section, we will calculate $\mathcal{P}_{n,q}(t^i;x)$ for $i=0,1,2$. By the definition of the q-Gamma function, γ_q^1, we have

$$\begin{aligned}
\int_0^{\infty/(1-q)} & t^s S_{n,k}(q;t)\, d_q t \\
&= \int_0^{\infty/(1-q)} t^s e_q(-[n]\,t) q^{k(k-1)/2} \frac{[n]^k\, t^k}{[k]!} d_q t \\
&= \frac{1}{[n]^{s+1}} \frac{1}{[k]!} q^{k(k-1)/2} \\
&\quad \times \int_0^{\infty/(1-q)} ([n]\,t)^{k+s}\, e_q(-[n]\,t)\,[n]\, d_q t \\
&= \frac{1}{[n]^{s+1}} \frac{1}{[k]!} q^{k(k-1)/2} \int_0^{\infty/(1-q)} (u)^{k+s}\, e_q(-u) d_q u \\
&= \frac{1}{[n]^{s+1}} \frac{1}{[k]!} q^{k(k-1)/2} \gamma_q^1(k+s+1) \\
&= \frac{1}{[n]^{s+1}} \frac{1}{[k]!} q^{k(k-1)/2} \frac{[k+s]!}{q^{(k+s+1)(k+s)/2}} \\
&= \frac{1}{[n]^{s+1}} \frac{[k+s]!}{[k]!} \frac{1}{q^{(2k+s)(s+1)/2}}.
\end{aligned}$$

Lemma 1. *We have*

$$\begin{aligned}
\mathcal{P}_{n,q}(1;x) = 1, \qquad \mathcal{P}_{n,q}(t;x) = x, \\
\mathcal{P}_{n,q}(t^2;x) = \frac{1}{q^2} x^2 + \frac{(1+q)}{q^2\,[n]} x, \\
\mathcal{P}_{n,q}((t-x)^2;x) = \left(\frac{1}{q^2} - 1\right) x^2 + \frac{(1+q)}{q^2\,[n]} x.
\end{aligned}$$

Proof. For $f(t)=1$,

$$\begin{aligned}
\mathcal{P}_{n,q}(1;x) = [n] \sum_{k=1}^{\infty} q^{k-1} S_{n,k}(q;qx) \int_0^{\infty/(1-q)} S_{n,k-1}(q;t)\, d_q t \\
+ e_q\left(-[n]\,qx\right)
\end{aligned}$$

$$= [n]\sum_{k=1}^{\infty} q^{k-1} S_{n,k}(q;qx)\frac{1}{[n]}\frac{1}{q^{k-1}} + e_q\left(-[n]\,qx\right)$$
$$= \sum_{k=1}^{\infty} S_{n,k}(q;qx) + e_q\left(-[n]\,qx\right) = \sum_{k=0}^{\infty} S_{n,k}(q;qx) = 1.$$

For $f(t) = t$,

$$\begin{aligned}
\mathcal{P}_{n,q}(t;x) &= [n]\sum_{k=1}^{\infty} q^{k-1} S_{n,k}(q;qx)\int_0^{\infty/(1-q)} tS_{n,k-1}(q;t)\,d_qt \\
&= [n]\sum_{k=1}^{\infty} q^{k} S_{n,k}(q;qx)\frac{[k]}{[n]^2}\frac{1}{q^{2k-1}} \\
&= \sum_{k=0}^{\infty} S_{n,k}(q;qx)\frac{[k]}{[n]}\frac{1}{q^{k}} \\
&= \frac{1}{q^2}\sum_{k=0}^{\infty} S_{n,k}(q;qx)\frac{[k]}{[n]}\frac{1}{q^{k-2}} = \frac{1}{q^2}q^2x = x.
\end{aligned}$$

For $f(t) = t^2$,

$$\begin{aligned}
\mathcal{P}_{n,q}(t^2;x) &= [n]\sum_{k=1}^{\infty} q^{k-1} S_{n,k}(q;qx)\int_0^{\infty/(1-q)} t^2S_{n,k-1}(q;t)\,d_qt \\
&= \sum_{k=1}^{\infty} S_{n,k}(q;qx)\frac{[k+1]\,[k]}{[n]^2}\frac{1}{q^{2k+1}} \\
&= \sum_{k=0}^{\infty} S_{n,k}(q;qx)\frac{[k+1]\,[k]}{[n]^2}\frac{1}{q^{2k+1}} \\
&= \sum_{k=0}^{\infty} S_{n,k}(q;qx)\frac{\left([k]+q^k\right)[k]}{[n]^2}\frac{1}{q^{2k+1}} \\
&= \sum_{k=0}^{\infty} S_{n,k}(q;qx)\frac{[k]^2}{[n]^2}\frac{1}{q^{2k+1}} \\
&\quad + \sum_{k=0}^{\infty} S_{n,k}(q;qx)\frac{q^k\,[k]}{[n]^2}\frac{1}{q^{2k+1}} \\
&= \frac{1}{q^5}\sum_{k=0}^{\infty} S_{n,k}(q;qx)\frac{[k]^2}{[n]^2}\frac{1}{q^{2k-4}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{[n]\,q^3}\sum_{k=0}^{\infty}S_{n,k}(q;qx)\frac{[k]}{[n]}\frac{1}{q^{k-2}}\\
&=\frac{1}{q^5}\left(q^3x^2+\frac{q^3}{[n]}x\right)+\frac{1}{[n]\,q}x\\
&=\frac{1}{q^2}x^2+\frac{1}{q^2\,[n]}x+\frac{1}{[n]\,q}x\\
&=\frac{1}{q^2}x^2+\frac{(1+q)}{q^2\,[n]}x. \qquad \square
\end{aligned}
$$

Lemma 2. *For all* $0<q<1$ *the following identity holds*:

$$
\mathcal{P}_{n,q}(t^m;x)=\frac{1}{[n]^m\,q^{(m^2-m)/2}}\sum_{s=1}^{m}C_s(m)\,[n]^s\sum_{k=0}^{\infty}\frac{[k]^s}{[n]^s}\frac{1}{q^{km}}S_{n,k}(q;qx).
$$

Proof. We have

$$
\begin{aligned}
\mathcal{P}_{n,q}(t^m;x)&=[n]\sum_{k=1}^{\infty}q^{k-1}S_{n,k}(q;qx)\int_0^{\infty/(1-q)}t^mS_{n,k-1}(q;t)\,d_qt\\
&=[n]\sum_{k=1}^{\infty}q^{k-1}S_{n,k}(q;qx)\frac{1}{[n]^{m+1}}\frac{1}{[k-1]!}\\
&\qquad\times q^{[(k-1)(k-2)]/2}\frac{[k-1+m]!}{q^{(k+m)(k-1+m)/2}}\\
&=\sum_{k=1}^{\infty}\frac{[k-1+m]\cdots[k]}{[n]^m}\frac{1}{q^{(m^2+2mk-m)/2}}S_{n,k}(q;qx)\\
&=\sum_{k=0}^{\infty}\frac{[k-1+m]\cdots[k]}{[n]^m\,q^{(m^2+2mk-m)/2}}S_{n,k}(q;qx).
\end{aligned}
$$

Using $[k+s]=[s]+q^s[k]$, we obtain

$$
[k]\,[k+1]\cdots[k+m-1]=\prod_{s=0}^{m-1}\left([s]+q^s\,[k]\right)=\sum_{s=1}^{m}C_s\,(m)\,[k]^s
$$

where $C_s(m) > 0$, $s = 1, 2, \dots, m$ are the constants independent of k. Hence,

$$\begin{aligned}
\mathcal{P}_{n,q}(t^m; x) &= \frac{1}{[n]^m q^{(m^2-m)/2}} \sum_{k=0}^{\infty} \frac{1}{q^{km}} \sum_{s=1}^{m} C_s(m) [k]^s S_{n,k}(q; qx) \\
&= \frac{1}{[n]^m q^{(m^2-m)/2}} \sum_{k=0}^{\infty} \sum_{s=1}^{m} C_s(m) [k]^s \frac{1}{q^{km}} S_{n,k}(q; qx) \\
&= \frac{1}{[n]^m q^{(m^2-m)/2}} \sum_{s=1}^{m} C_s(m) [n]^s \\
&\qquad \sum_{k=0}^{\infty} \left(\frac{[k]}{[n]}\right)^s \frac{1}{q^{km}} S_{n,k}(q; qx). \qquad \square
\end{aligned}$$

3. Approximation properties. Let $C_B[0, \infty)$ be the space of all real-valued continuous bounded functions f on $[0, \infty)$, endowed with the norm $\|f\| = \sup_{x \in [0,\infty)} |f(x)|$. The Peetre's K-functional is defined by

$$K_2(f; \delta) = \inf_{g \in C^2[0,\infty)} \{\|f - g\| + \delta \|g''\|\},$$

where $C_B^2[0, \infty) := \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$. By [**1**, Theorem 2.4], there exists an absolute constant $M > 0$ such that

$$K_2(f, \delta) \leq M \omega_2(f; \sqrt{\delta}), \tag{3.1}$$

where $\delta > 0$ and the second order modulus of smoothness is defined as

$$\omega_2(f; \sqrt{\delta}) = \sup_{0 < h \leq \delta} \sup_{x \in [0,\infty)} |f(x + 2h) - 2f(x + h) + f(x)|,$$

where $f \in C_B[0, \infty)$ and $\delta > 0$. Also, we let

$$\omega(f; \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0,\infty)} |f(x + h) - f(x)|.$$

Lemma 3. *Let $f \in C_B[0, \infty)$. Then, for all $f \in C_B^2[0, \infty)$, we have*

$$|\mathcal{P}_{n,q}(f; x) - f(x)| \leq \left\{ \left(\frac{1 - q^2}{q^2}\right) x^2 + \frac{(1 + q)}{q^2 [n]} x \right\} \|f''\|. \tag{3.2}$$

Proof. Let $x \in [0,\infty)$ and $f \in C_B^2[0,\infty)$. Using Taylor's formula,

$$f(t)-f(x)=(t-x)f'(x)+\int_x^t (t-u)f''(u)\,du,$$

we can write

$$\begin{aligned}
\mathcal{P}_{n,q}(f;x)-f(x) &= \mathcal{P}_{n,q}((t-x)f'(x);x)\\
&\quad+\mathcal{P}_{n,q}\left(\int_x^t (t-u)f''(u)\,du;x\right)\\
&= f'(x)\mathcal{P}_{n,q}((t-x);x)\\
&\quad+\mathcal{P}_{n,q}\left(\int_x^t (t-u)f''(u)\,du;x\right)\\
&\quad-\int_x^x (x-u)f''(u)\,du\\
&= \mathcal{P}_{n,q}\left(\int_x^t (t-u)f''(u)\,du;x\right).
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
\left|\int_x^t (t-u)f''(u)\,du\right| &\le \int_x^t |t-u|\,|f''(u)|\;du\\
&\le \|f''\|\int_x^t |t-u|\;du\\
&\le (t-x)^2\,\|f''\|\,,
\end{aligned}$$

we conclude that

$$\begin{aligned}
|\mathcal{P}_{n,q}(f;x)-f(x)| &= \left|\mathcal{P}_{n,q}\left(\int_x^t (t-u)f''(u)\,du;x\right)\right|\\
&\le \mathcal{P}_{n,q}((t-x)^2\,\|f''\|\,;x)\\
&= \left\{\left(\frac{1-q^2}{q^2}\right)x^2+\frac{(1+q)}{q^2\,[n]}x\right\}\|f''\|\,. \qquad \square
\end{aligned}$$

Lemma 4. *For $f \in C[0,\infty)$, we have*

$$\|\mathcal{P}_{n,q}f\| \le \|f\|\,.$$

Theorem 1. *Let $f \in C_B[0,\infty)$. Then, for every $x \in [0,\infty)$, there exists a constant $M > 0$ such that*

$$|\mathcal{P}_{n,q}(f;x) - f(x)| \leq M\omega_2(f; \sqrt{\delta_n(x)}),$$

where

$$\delta_n(x) = \left(\frac{1-q^2}{q^2}\right)x^2 + \frac{(1+q)}{q^2\,[n]}x.$$

Proof. Now, taking into account the boundedness of $\mathcal{P}_{n,q}$, we get

$$\begin{aligned}|\mathcal{P}_{n,q}(f;x) - f(x)| &= |\mathcal{P}_{n,q}(f;x) - \mathcal{P}_{n,q}(g,x) - f(x) \\ &\qquad\qquad\qquad + g(x) + \mathcal{P}_{n,q}(g,x) - g(x)| \\ &\leq |\mathcal{P}_{n,q}(f-g;x) - (f-g)(x)| \\ &\quad + |\mathcal{P}_{n,q}(g;x) - g(x)| \\ &\leq |\mathcal{P}_{n,q}(f-g;x) + (f-g)(x)| \\ &\quad + |\mathcal{P}_{n,q}(g;x) - g(x)| \\ &\leq 2\,\|f-g\| + \left\{\left(\frac{1-q^2}{q^2}\right)x^2 + \frac{(1+q)}{q^2\,[n]}x\right\}\|g''\| \\ &\leq 2\left(\|f-g\| + \delta_n(x)\,\|g''\|\right).\end{aligned}$$

Now, taking the infimum on the right-hand side over all $g \in C_B^2[0,\infty)$, and using (3.1), we get the following result

$$|\mathcal{P}_{n,q}(f;x) - f(x)| \leq 2K_2(f;\delta_n(x)) \leq M\omega_2(f;\sqrt{\delta_n(x)}). \qquad \square$$

Theorem 2. *Let $0 < \alpha \leq 1$ and E be any bounded subset of the interval $[0,\infty)$. Then, if $f \in C_B[0,\infty)$ is locally* $\mathrm{Lip}\,(\alpha)$, *i.e., the condition*

$$|f(y) - f(x)| \leq L\,|y-x|^{\alpha},\ y \in E \quad \text{and } x \in [0,\infty), \tag{3.3}$$

holds, then, for each $x \in [0,\infty)$, we have

$$|\mathcal{P}_{n,q}(f;x) - f(x)| \leq L\left\{\delta_n^{\alpha/2}(x) + 2\,(d\,(x,E))^{\alpha}\right\},$$

where L is a constant depending on α and f; and $d(x,E)$ is the distance between x and E defined as

$$d\left(x,E\right)=\inf\left\{\left|t-x\right|:t\in E\right\}.$$

Proof. Let $\overline{E}$ denote the closure of E in $[0,\infty)$. Then, there exists a point $x_0\in\overline{E}$ such that $|x-x_0|=d(x,E)$. Using the triangle inequality

$$\left|f\left(t\right)-f\left(x\right)\right|\leq\left|f\left(t\right)-f\left(x_0\right)\right|+\left|f\left(x\right)-f\left(x_0\right)\right|,$$

we get, by (3.3),

$$\begin{aligned}\left|\mathcal{P}_{n,q}(f;x)-f(x)\right|&\leq\mathcal{P}_{n,q}(\left|f\left(t\right)-f\left(x_0\right)\right|;x)\\&\quad+\mathcal{P}_{n,q}(\left|f\left(x\right)-f\left(x_0\right)\right|;x)\\&\leq L\left\{\mathcal{P}_{n,q}(\left|t-x_0\right|^{\alpha};x)+\left|x-x_0\right|^{\alpha}\right\}\\&\leq L\left\{\mathcal{P}_{n,q}(\left|t-x\right|^{\alpha}+\left|x-x_0\right|^{\alpha};x)+\left|x-x_0\right|^{\alpha}\right\}\\&=L\left\{\mathcal{P}_{n,q}(\left|t-x\right|^{\alpha};x)+2\left|x-x_0\right|^{\alpha}\right\}.\end{aligned}$$

Using the Hölder inequality with $p=2/\alpha$, $q=2/(2-\alpha)$, we find that

$$\begin{aligned}\left|\mathcal{P}_{n,q}(f;x)-f(x)\right|&\leq L\Bigg\{\left[\mathcal{P}_{n,q}(\left|t-x\right|^{\alpha p};x)\right]^{1/p}\left[\mathcal{P}_{n,q}(1^q;x)\right]^{1/q}\\&\qquad\qquad+2\left(d\left(x,E\right)\right)^{\alpha}\Bigg\}\\&=L\left\{\left[\mathcal{P}_{n,q}(\left|t-x\right|^{2};x)\right]^{\alpha/2}+2\left(d\left(x,E\right)\right)^{\alpha}\right\}\\&\leq L\left\{\left[\left(\frac{1-q^2}{q^2}\right)x^2+\frac{(1+q)}{q^2\left[n\right]}x\right]^{\alpha/2}+2\left(d\left(x,E\right)\right)^{\alpha}\right\}\\&=L\left\{\delta_n^{\alpha/2}(x)+2\left(d\left(x,E\right)\right)^{\alpha}\right\}.\qquad\square\end{aligned}$$

We consider the following classes of functions:

$$C_m\left[0,\infty\right):=\left\{f\in C\left[0,\infty\right):\exists M_f>0\ \left|f\left(x\right)\right|<M_f\left(1+x^m\right)\right.$$
$$\left.\text{and }\ \left\|f\right\|_m:=\sup_{x\in[0,\infty)}\frac{\left|f\left(x\right)\right|}{1+x^m}\right\},$$

$$C_m^* [0,\infty) := \left\{ f \in C_m [0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^m} < \infty \right\}, \quad m \in \mathbf{N}.$$

Next, we obtain a direct approximation theorem in $C_1^*[0,\infty)$ and an estimation in terms of the weighted modulus of continuity. It is known that, if f is not uniformly continuous on the interval $[0,\infty)$, then the usual first modulus of continuity $\omega(f,\delta)$ does not tend to zero, as $\delta \to 0$. For every $f \in C_m^*[0,\infty)$, the weighted modulus of continuity is defined as follows

$$\Omega_m(f,\delta) = \sup_{\substack{x\geq 0 \\ 0<h\leq\delta}} \frac{|f(x+h) - f(x)|}{1+(x+h)^m},$$

see [**16**].

Lemma 5. *Let* $f \in C_m^*[0,\infty)$, $m \in \mathbf{N}$. *Then*

(1) $\Omega_m(f,\delta)$ *is a monotone increasing function of* δ,

(2) $\lim_{\delta\to 0^+} \Omega_m(f,\delta) = 0$,

(3) *for any* $\alpha \in [0,\infty)$, $\Omega_m(f,\alpha\delta) \leq (1+\alpha)\Omega_m(f,\delta)$.

In the next theorem we give an expression of the approximation error with the operators $\mathcal{P}_{n,q}$ by means of Ω_1.

Theorem 3. *If* $f \in C_1^*[0,\infty)$, *then the inequality*

$$\|\mathcal{P}_{n,q}(f) - f\|_2 \leq k(q)\,\Omega_1\left(f; \frac{1}{\sqrt{[n]}}\right),$$

where k *is a constant independent of* f *and* n.

Proof. From the definition of $\Omega_1(f,\delta)$ and Lemma 5, we may write

$$\begin{aligned} |f(t) - f(x)| &\leq (1+x+|t-x|)\left(\frac{|t-x|}{\delta}+1\right)\Omega_1(f,\delta) \\ &\leq (1+2x+t)\left(\frac{|t-x|}{\delta}+1\right)\Omega_1(f,\delta). \end{aligned}$$

Then

$$\begin{aligned}\left|\mathcal{P}_{n,q}(f;x)-f(x)\right| &\leq \mathcal{P}_{n,q}\left(\left|f(t)-f(x)\right|;x\right)\\ &\leq \Omega_1(f,\delta)\left(\mathcal{P}_{n,q}\left((1+2x+t);x\right)\right.\\ &\quad \left.+\,\mathcal{P}_{n,q}\bigg((1+2x+t)\frac{|t-x|}{\delta};x\bigg)\right).\end{aligned}$$

Applying the Cauchy-Schwartz inequality to the second term, we get

$$\begin{aligned}&\mathcal{P}_{n,q}\bigg((1+2x+t)\frac{|t-x|}{\delta};x\bigg)\\ &\qquad\leq\left(\mathcal{P}_{n,q}\left((1+2x+t)^2;x\right)\right)^{1/2}\left(\mathcal{P}_{n,q}\bigg(\frac{|t-x|^2}{\delta^2};x\bigg)\right)^{1/2}.\end{aligned}$$

Consequently,

$$\begin{aligned}\left|\mathcal{P}_{n,q}(f;x)-f(x)\right| &\leq \Omega_1(f,\delta)\left(\mathcal{P}_{n,q}\left((1+2x+t);x\right)\right.\\ &\quad +\left(\mathcal{P}_{n,q}\left((1+2x+t)^2;x\right)\right)^{1/2}\\ &\quad \left.\times\left(\mathcal{P}_{n,q}\bigg(\frac{|t-x|^2}{\delta^2};x\bigg)\right)^{1/2}\right).\end{aligned}\tag{3.4}$$

On the other hand, there is a positive constant $K(q)$ such that

$$\mathcal{P}_{n,q}\left((1+2x+t);x\right)=1+3x\leq 3(1+x),$$

$$\begin{aligned}\left(\mathcal{P}_{n,q}\left((1+2x+t)^2;x\right)\right)^{1/2} &= \bigg((1+2x)^2+2(1+2x)x\\ &\quad +\frac{1}{q^2}x^2+\frac{(1+q)}{q^2[n]}x\bigg)^{1/2}\\ &\leq K(q)(1+x),\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}\left(\mathcal{P}_{n,q}\bigg(\frac{|t-x|^2}{\delta^2};x\bigg)\right)^{1/2} &= \frac{1}{\delta q}\sqrt{(1-q^2)x^2+\frac{(1+q)}{[n]}x}\\ &= \frac{1}{\delta q}\sqrt{\frac{(1+q)(1-q^n)}{[n]}x^2+\frac{(1+q)x}{[n]}}\\ &\leq \frac{2}{\delta q\sqrt{[n]}}\sqrt{x^2+x}\leq\frac{2}{\delta q\sqrt{[n]}}(1+x).\end{aligned}\tag{3.6}$$

Now from (3.4), (3.5) and (3.6) we have

$$\begin{aligned}|\mathcal{P}_{n,q}(f;x)-f(x)| &\leq \Omega_1(f,\delta)\left(3(1+x)+K(q)\frac{2(1+x)^2}{q\delta\sqrt{[n]}}\right)\\ &\leq (1+x^2)\,\Omega_1(f,\delta)\left(3K_1+K(q)\frac{4}{q\delta\sqrt{[n]}}\right),\end{aligned}$$

where

$$K_1=\sup_{x\geq 0}\frac{1+x}{1+x^2}.$$

If we take $\delta=[n]^{-1/2}$, then from the above inequality we obtain the desired result. □

4. Voronovskaja-type theorem. In this section, we proceed to state and prove a Voronovskaja-type theorem for the q-Phillips operators. We first prove the following lemma:

Lemma 6. *Let* $0<q<1$. *We have*

$$\begin{aligned}\mathcal{P}_{n,q}(t^3;x)&=\frac{1}{q^6}x^3+\frac{[2]\,[3]}{[n]\,q^6}x^2+\frac{[2]\,[3]}{[n]^2\,q^5}x\\ \mathcal{P}_{n,q}(t^4;x)&=\frac{1}{q^{12}}x^4+\frac{[2]\,[3]\,(1+q^2)}{[n]\,q^{12}}x^3\\ &\quad+\frac{[2]\,[3]^2\,(1+q^2)}{[n]^2\,q^{11}}x^2+\frac{[2]^2\,[3]\,(1+q^2)}{[n]^3\,q^9}x.\end{aligned}$$

Proof. Simple calculations show that

$$\begin{aligned}\mathcal{P}_{n,q}(t^3;x)&=\frac{1}{[n]^3\,q^3}\sum_{k=0}^{\infty}\frac{[k+2]\,[k+1]\,[k]}{q^{3k}}S_{n,k}(q;qx)\\ &=\frac{1}{[n]^3\,q^3}\sum_{k=0}^{\infty}\frac{[k]^3+q^k(2+q)\,[k]^2+q^{2k}(1+q)\,[k]}{q^{3k}}\\ &\qquad\times S_{n,k}(q;qx)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{[n]^3 q^3}\Bigg\{ \sum_{k=0}^{\infty} \frac{[k]^3}{q^{3k}} S_{n,k}(q;qx) \\
&\qquad\qquad\qquad + \sum_{k=0}^{\infty} \frac{(2+q)\,[k]^2}{q^{2k}} S_{n,k}(q;qx) \\
&\qquad\qquad\qquad\qquad + \sum_{k=0}^{\infty} \frac{(1+q)\,[k]}{q^{k}} S_{n,k}(q;qx) \Bigg\} \\
&= \frac{1}{q^3} \sum_{k=0}^{\infty} \frac{[k]^3}{[n]^3 q^{3k}} S_{n,k}(q;qx) \\
&\quad + \frac{(2+q)}{[n]\,q^3} \sum_{k=0}^{\infty} \frac{[k]^2}{[n]^2 q^{2k}} S_{n,k}(q;qx) \\
&\quad + \frac{(1+q)}{[n]^2 q^3} \sum_{k=0}^{\infty} \frac{[k]}{[n]\,q^{k}} S_{n,k}(q;qx) \\
&= \frac{1}{q^9} \sum_{k=0}^{\infty} \frac{[k]^3}{[n]^3 q^{3k-6}} S_{n,k}(q;qx) \\
&\quad + \frac{(2+q)}{[n]\,q^7} \sum_{k=0}^{\infty} \frac{[k]^2}{[n]^2 q^{2k-4}} S_{n,k}(q;qx) \\
&\quad + \frac{(1+q)}{[n]^2 q^5} \sum_{k=0}^{\infty} \frac{[k]}{[n]\,q^{k-2}} S_{n,k}(q;qx) \\
&= \frac{1}{q^9} \sum_{k=0}^{\infty} \left(\frac{[k]}{[n]\,q^{k-2}}\right)^3 S_{n,k}(q;qx) \\
&\quad + \frac{(2+q)}{[n]\,q^7} \sum_{k=0}^{\infty} \left(\frac{[k]}{[n]\,q^{k-2}}\right)^2 S_{n,k}(q;qx) \\
&\quad + \frac{(1+q)}{[n]^2 q^5} \sum_{k=0}^{\infty} \frac{[k]}{[n]\,q^{k-2}} S_{n,k}(q;qx) \\
&= \frac{1}{q^9}\left(\frac{q^4}{[n]^2} x + \left(2q^4+q^3\right) \frac{x^2}{[n]} + q^3 x^3 \right) \\
&\quad + \frac{(2+q)}{[n]\,q^7}\left(q^3 x^2 + \frac{q^3}{[n]} x \right) + \frac{(1+q)\,q^2}{[n]^2 q^5} x
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{q^5 [n]^2} x + \frac{(2q+1)}{q^6 [n]} x^2 + \frac{1}{q^6} x^3 \\
&\quad + \frac{(2+q)}{[n] q^4} x^2 + \frac{(2+q)}{[n]^2 q^4} x + \frac{(1+q)}{[n]^2 q^3} x \\
&= \frac{1}{q^6} x^3 + \frac{(1+2q+2q^2+q^3)}{q^6 [n]} x^2 + \frac{(1+2q+2q^2+q^3)}{q^5 [n]^2} x \\
&= \frac{1}{q^6} x^3 + \frac{(1+q)(1+q+q^2)}{[n] q^6} x^2 + \frac{(1+q)(1+q+q^2)}{[n]^2 q^5} x
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{n,q}(t^4;x) &= \frac{1}{[n]^4 q^6} \\
&\quad \sum_{k=0}^{\infty} \frac{[k+3][k+2][k+1][k]}{q^{4k}} S_{n,k}(q;qx) \\
&= \frac{1}{[n]^4 q^6} \sum_{k=0}^{\infty} \frac{[k]^4 + q^k \left(3+2q+q^2\right)[k]^3 + q^{2k}\left(3+4q+3q^2+q^3\right)[k]^2}{q^{4k}} \\
&\qquad + \frac{q^{3k}\left(1+2q+2q^2+q^3\right)[k]}{q^{4k}} S_{n,k}(q;qx) \\
&= \frac{1}{q^{14}} \sum_{k=0}^{\infty} \left(\frac{[k]}{[n] q^{k-2}} \right)^4 S_{n,k}(q;qx) \\
&\quad + \frac{\left(3+2q+q^2\right)}{[n] q^{12}} \sum_{k=0}^{\infty} \left(\frac{[k]}{[n] q^{k-2}} \right)^3 S_{n,k}(q;qx) \\
&\quad + \frac{\left(3+4q+3q^2+q^3\right)}{[n]^2 q^{10}} \sum_{k=0}^{\infty} \left(\frac{[k]}{[n] q^{k-2}} \right)^2 S_{n,k}(q;qx) \\
&\quad + \frac{\left(1+2q+2q^2+q^3\right)}{[n]^3 q^8} \sum_{k=0}^{\infty} \frac{[k]}{[n] q^{k-2}} S_{n,k}(q;qx) \\
&= \frac{1}{q^{14}} \left(\frac{q^5}{[n]^3} x + \left(3q^3+3q^2+q\right) \frac{q^2}{[n]^2} x^2 \right. \\
&\qquad \left. + \left(3q+2+\frac{1}{q}\right) \frac{q^3}{[n]} x^3 + q^2 x^4 \right) \\
&\quad + \frac{\left(3+2q+q^2\right)}{[n] q^{12}} \left(\frac{q^4}{[n]^2} x + \left(2q^2+q\right) \frac{q^2}{[n]} x^2 + q^3 x^3 \right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\left(3+4q+3q^2+q^3\right)}{[n]^2 q^{10}}\left(q^3x^2+\frac{q^3}{[n]}x\right)\\
&+\frac{\left(1+2q+2q^2+q^3\right)q^2}{[n]^3 q^8}x\\
=&\ \frac{1}{q^{12}}x^4+\frac{1+2q+3q^2+(3+2q+q^2)q^3}{[n]\,q^{12}}x^3\\
&+\tfrac{1+3q+3q^2+(3+2q+q^2)(2q+1)q^2+(3+4q+3q^2+q^3)q^4}{[n]^2q^{11}}x^2\\
&+\tfrac{1+(3+2q+q^2)q+(3+4q+3q^2+q^3)q^2+(1+2q+2q^2+q^3)q^3}{[n]^3q^9}x\\
=&\ \frac{1}{q^{12}}x^4+\frac{(1+q)\,(1+q^2)(1+q+q^2)}{[n]\,q^{12}}x^3\\
&+\frac{(1+q)\,(1+q^2)(1+q+q^2)^2}{[n]^2\,q^{11}}x^2\\
&+\frac{(1+q)^2\,(1+q^2)(1+q+q^2)}{[n]^3\,q^9}x. \qquad \square
\end{aligned}
$$

Theorem 4. *Let $q_n \in (0,1)$. Then the sequence $\{\mathcal{P}_{n,q_n}(f)\}$ converges to f uniformly on $[0,A]$ for each $f \in C_2^*[0,\infty)$ if and only if $\lim_{n\to\infty} q_n = 1$.*

Proof. The proof is similar to that of Theorem 2 [**9**]. $\square$

Lemma 7. *Assume that $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$. For every $x \in [0,\infty)$, there hold*

$$
\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n}\mathcal{P}_{n,q_n}((t-x)^2;x)=2(1-a)x^2+2x,\\
&\lim_{n\to\infty}[n]_{q_n}^2\mathcal{P}_{n,q_n}((t-x)^4;x)=12x^2+24(1-a)x^3+12(1-a)^2x^4.
\end{aligned}
$$

Proof. First, we have

$$
\begin{aligned}
\lim_{n\to\infty}[n]_{q_n}\mathcal{P}_{n,q_n}((t-x)^2;x)&=\lim_{n\to\infty}[n]_{q_n}\left\{\left(\frac{1}{q_n^2}-1\right)x^2+\frac{(1+q_n)}{q_n^2\,[n]_{q_n}}x\right\}\\
&=\lim_{n\to\infty}\left(\frac{(1-q_n^n)(1+q_n)}{q_n^2}x^2+\frac{(1+q_n)}{q_n^2}x\right)\\
&=2\,(1-a)\,x^2+2x.
\end{aligned}
$$

In order to calculate the second limit, we need the expression for $\mathcal{P}_{n,q_n}((t-x)^4;x):$

$$\begin{aligned}
&\mathcal{P}_{n,q_n}\left((t-x)^4;x\right)\\
&\quad=\mathcal{P}_{n,q_n}(t^4;x)-4x\mathcal{P}_{n,q_n}(t^3;x)\\
&\quad\quad+6x^2\mathcal{P}_{n,q_n}(t^2;x)-4x^3\mathcal{P}_{n,q_n}(t;x)+x^4\\
&\quad=\frac{1}{q_n^{12}}x^4+\frac{[2]_{q_n}[3]_{q_n}(1+q_n^2)}{[n]_{q_n}q_n^{12}}x^3\\
&\quad\quad+\frac{[2]_{q_n}[3]_{q_n}^2(1+q_n^2)}{[n]_{q_n}^2q_n^{11}}x^2+\frac{[2]_{q_n}^2[3]_{q_n}(1+q_n^2)}{[n]_{q_n}^3q_n^9}x\\
&\quad\quad-4x\left\{\frac{1}{q_n^6}x^3+\frac{[2]_{q_n}[3]_{q_n}}{[n]_{q_n}q_n^6}x^2+\frac{[2]_{q_n}[3]_{q_n}}{[n]_{q_n}^2q_n^5}x\right\}\\
&\quad\quad+6x^2\left\{\frac{1}{q_n^2}x^2+\frac{[2]_{q_n}}{q_n^2[n]_{q_n}}x\right\}-3x^4\\
&\quad=\frac{(1-4q_n^6+6q_n^{10}-3q_n^{12})}{q_n^{12}}x^4\\
&\quad\quad+\left\{\frac{[2]_{q_n}[3]_{q_n}(1+q_n^2)-4[2]_{q_n}[3]_{q_n}q_n^6+6q_n^{10}[2]_{q_n}}{q_n^{12}[n]_{q_n}}\right\}x^3\\
&\quad\quad+\left\{\frac{[2]_{q_n}[3]_{q_n}^2(1+q_n^2)-4q_n^6[2]_{q_n}[3]_{q_n}}{q_n^{11}[n]_{q_n}^2}\right\}x^2+\frac{[2]_{q_n}^2[3]_{q_n}(1+q_n^2)}{[n]_{q_n}^3q_n^9}x\\
&\quad=\frac{\left(1+2q_n^2+3q_n^4-3q_n^8\right)(1-q_n^n)^2(q_n+1)^2}{q_n^{12}[n]_{q_n}^2}x^4\\
&\quad\quad+\left\{\frac{(q_n^n-1)}{1}\right.\\
&\quad\quad\left.\times\frac{(q_n+1)\left(2q_n^7-4q_n^2-5q_n^3-6q_n^4-6q_n^5-2q_n^6-2q_n+6q_n^8+6q_n^9-1\right)}{q_n^{12}[n]_{q_n}^2}\right\}x^3\\
&\quad\quad+\left\{\frac{[2]_{q_n}[3]_{q_n}^2(1+q_n^2)-4q_n^6[2]_{q_n}[3]_{q_n}}{q_n^{11}[n]_{q_n}^2}\right\}x^2\\
&\quad\quad+\frac{[2]_{q_n}^2[3]_{q_n}(1+q_n^2)}{[n]_{q_n}^3q_n^9}x.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \lim \left[n\right]_{q_n}^2 \mathcal{P}_{n,q_n}\left(\left(t-x\right)^4; x\right)$$
$$= \lim_{n\to\infty} \frac{(1-q_n^n)^2}{(1-q_n)^2} \left\{ \frac{\left(2q_n^2+3q_n^4-3q_n^8+1\right)\left(q_n-1\right)^2\left(q_n+1\right)^2}{q_n^{12}} x^4 \right.$$
$$+ \left(\frac{\left(q_n-1\right)\left(q_n+1\right)}{1} \right.$$
$$\left. \times \frac{\left(2q_n^7-4q_n^2-5q_n^3-6q_n^4-6q_n^5-2q_n^6-2q_n+6q_n^8+6q_n^9-1\right)}{q_n^{12}\left[n\right]_{q_n}} \right) x^3$$
$$+ \left(\frac{\left(q_n+1\right)\left(q_n+2q_n^2+q_n^3+q_n^4-4q_n^6+1\right)\left(q_n+q_n^2+1\right)}{q_n^{11}\left[n\right]_{q_n}^2} \right) x^2$$
$$\left. + \left(\frac{(1+q_n)^2(1+q_n^2)(1+q_n+q_n^2)}{q_n^9\left[n\right]_{q_n}^3} \right) x \right\}$$
$$= 12\left(1-a\right)^2 x^4 + 24\left(1-a\right)x^3 + 12x^2. \qquad \square$$

Theorem 5. *Assume that $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$. For any $f \in C_2^*[0,\infty)$ such that $f', f'' \in C_2^*[0,\infty)$, the following equality holds*

$$\lim_{n\to\infty} \lim \left[n\right]_{q_n} \left(\mathcal{P}_{n,q_n}(f;x) - f(x)\right) = \left((1-a)x^2 + x\right) f''(x)$$

uniformly on any $[0, A]$, $A > 0$.

Proof. Let $f, f', f'' \in C_2^*[0,\infty)$ and $x \in [0,\infty)$ be fixed. By the Taylor formula we may write

$$\begin{aligned} f(t) &= f(x) + f'(x)(t-x) \\ &\quad + \frac{1}{2}f''(x)(t-x)^2 + r(t;x)(t-x)^2, \end{aligned} \tag{4.1}$$

where $r(t;x)$ is the Peano form of the remainder, $r(\cdot;x) \in C_2^*[0,\infty)$ and $\lim_{t\to x} r(t;x) = 0$. Applying $\mathcal{P}_{n,q_n}$ to (4.1) we obtain

$$\left[n\right]_{q_n} \left(\mathcal{P}_{n,q_n}(f;x) - f(x)\right)$$
$$= \frac{1}{2} f''(x) \left[n\right]_{q_n} \mathcal{P}_{n,q_n}\left(\left(t-x\right)^2; x\right)$$
$$+ \left[n\right]_{q_n} \mathcal{P}_{n,q_n}\left(r\left(t;x\right)\left(t-x\right)^2; x\right).$$

By the Cauchy-Schwartz inequality, we have

(4.2)
$$\mathcal{P}_{n,q_n}\left(r\left(t;x\right)\left(t-x\right)^2;x\right) \leq \sqrt{\mathcal{P}_{n,q_n}\left(r^2\left(t;x\right);x\right)}\sqrt{\mathcal{P}_{n,q_n}\left(\left(t-x\right)^4;x\right)}.$$

Observe that $r^2(x;x)=0$ and $r^2(\cdot;x)\in C_2^*[0,\infty)$. Then it follows from Theorem 4 that

$$\lim_{n\to\infty}\mathcal{P}_{n,q_n}\left(r^2\left(t;x\right);x\right)=r^2\left(x;x\right)=0 \tag{4.3}$$

uniformly with respect to $x\in[0,A]$. Now from (4.2), (4.3) and Lemma 7 we immediately get

$$\lim_{n\to\infty}[n]_{q_n}\mathcal{P}_{n,q_n}\left(r\left(t;x\right)\left(t-x\right)^2;x\right)=0.$$

Then we get the following

$$\lim_{n\to\infty}[n]_{q_n}(\mathcal{P}_{n,q_n}(f;x)-f(x))=\lim_{n\to\infty}\left(\frac{1}{2}f''(x)\,[n]_{q_n}\mathcal{P}_{n,q_n}\left((t-x)^2;x\right)\right.$$
$$\left.+\,[n]_{q_n}\mathcal{P}_{n,q_n}\left(r\left(t;x\right)\left(t-x\right)^2;x\right)\right)=\left((1-a)x^2+x\right)f''(x). \qquad \square$$

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