ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 3, 2012

THE MACKEY MACHINE FOR CROSSED PRODUCTS BY REGULAR GROUPOIDS. II.

GEOFF GOEHLE

ABSTRACT. We prove that given a regular groupoid G whose isotropy subgroupoid S has a Haar system, along with a dynamical system (A, G, α), there is an action of G on the spectrum of $A \rtimes S$ such that the spectrum of $A \rtimes G$ is homeomorphic to the orbit space of this action via induction. In addition, we give a strengthening of these results in the case where the crossed product is a groupoid algebra.

Introduction. This paper continues the development of the Mackey machine for groupoid crossed products which was started in [**6**]. In the first paper of this series we constructed an induction process for groupoid crossed products and proved that for crossed products by regular groupoids every irreducible representation of $A \rtimes G$ is induced from a representation of a "stabilizer" crossed product $A(u) \rtimes S_u$.

In this work we realize our ultimate goal of identifying the space of irreducible representations of certain crossed products by exhibiting a natural action of G on the spectrum $(A \rtimes S)^\wedge$, showing that induction defines a map from the spectrum of $A \rtimes S$ onto the spectrum of $A \rtimes G$, and then proving that this map factors to a homeomorphism between the orbit space $(A \rtimes S)^\wedge/G$ and $(A \rtimes G)^\wedge$. This identification theorem is a partial generalization of work done by Williams for transformation group C^*-algebras [**16**] and results of Echterhoff for transformation groupoids [**3**, Theorem 1]. Furthermore, it is also related to work done by Orloff Clark on groupoid C^*-algebras [**10, 11**]. An outline of the paper is roughly as follows. Section 1 covers some basic crossed product theory, as well as a few facts concerning crossed products by groupoid group bundles. Section 2 contains the main result of the paper. The proof is quite technical and has been broken up into four subsections. We finish with Section 3 which strengthens the results of Section 2 in the context of groupoid algebras.

2010 AMS *Mathematics subject classification.* Primary 47L65, 47A67.
Received by the editors on August 12, 2009, and in revised form on October 13, 2009.

DOI:10.1216/RMJ-2012-42-3-873

Before we begin in earnest we should first make some remarks about our hypotheses. In order to work with the crossed product $A \rtimes S$ we must assume that S has a Haar system. It is worth pointing out that this is equivalent to assuming that the stabilizer subgroups S_u vary continuously with respect to the Fell topology in S [**13**]. It should also be noted that, to a large extent, the results of this paper are contained, with more detail and a great deal of background material in the author's thesis [**5**]. Lastly, the author would like to thank the referee for their comments and for bringing [**3**] to our attention.

1. Preliminaries. We will be using the same notation and terminology as in [**6**]. In particular, we will let G denote a second countable, locally compact Hausdorff groupoid with a Haar system λ. Given an element $u \in G^{(0)}$ of the unit space of G, we will use $S_u = \{\gamma \in G : s(\gamma) = r(\gamma) = u\}$ to denote the stabilizer, or isotropy, subgroup of G over u. We use $S = \{\gamma \in G : s(\gamma) = r(\gamma)\}$ to denote the stabilizer, or isotropy, subgroupoid of G formed by bundling together all of the stabilizer subgroups. We will let A denote a separable $C_0(G^{(0)})$-algebra, and we will let $\mathcal{A}$ be its associated usc-bundle. Given A and G as above, we let α denote an action of G on A as defined in [**9**, Definition 4.1] and call (A, G, α) a groupoid dynamical system. We construct the groupoid crossed product $A \rtimes_\alpha G$ as a universal completion of the algebra of compactly supported sections $\Gamma_c(G, r^*\mathcal{A})$ in the usual fashion.

One important aspect of groupoid dynamical systems is that, given (A, G, α), there is a natural action of G on the spectrum of A induced by α.

Proposition 1.1. *If (A, G, α) is a groupoid dynamical system then there is a continuous action of G on $\widehat{A}$ given by $\gamma \cdot \pi = \pi \circ \alpha_\gamma^{-1}$.*

Proof. Since A is a $C_0(G^{(0)})$-algebra, it follows from [**17**, Proposition C.5] that there is a continuous map $r : \widehat{A} \to G^{(0)}$. Furthermore, we view $\widehat{A}$ as being fibered over $G^{(0)}$ so that if $\pi \in \widehat{A}$ with $r(\pi) = u$ then we can factor π to a representation π' of $A(u)$. Given $\gamma \in G$, we know $\alpha_\gamma : A(s(\gamma)) \to A(r(\gamma))$ so that if $r(\pi) = s(\gamma)$ we can define $\gamma \cdot \pi \in \widehat{A}$ by $\gamma \cdot \pi(a) = \pi'(\alpha_\gamma^{-1}(a(r(\gamma))))$. Of course, when we factor $\gamma \cdot \pi$ to $A(r(\gamma))$, we get $(\gamma \cdot \pi)' = \pi' \circ \alpha_\gamma^{-1}$ as desired. The difficult part in proving that this defines a groupoid action is showing that it is continuous.

Suppose $\gamma_i \to \gamma$ and $\pi_i \to \pi$ such that $s(\gamma_i) = r(\pi_i)$ for all i and $s(\gamma) = r(\pi)$. Let $O_J = \{\rho \in \widehat{A} : J \not\subset \ker \rho\}$ be an open set in $\widehat{A}$ containing $\gamma \cdot \pi$. Suppose, to the contrary, that $\gamma_i \cdot \pi_i$ is not eventually in O_J. By passing to a subnet and relabeling, we can assume $\gamma_i \cdot \pi_i \notin O_J$ for all i. Fix $a \in J$ and choose $b \in A$ such that $b(s(\gamma)) = \alpha_\gamma^{-1}(a(r(\gamma)))$. Since the action is continuous, $\alpha_{\gamma_i}^{-1}(a(r(\gamma_i))) \to b(s(\gamma))$. Since the norm is upper-semicontinuous, the set $\{a \in \mathcal{A} : \|a\| < \varepsilon\}$ is open for all $\varepsilon > 0$. Because $\alpha_{\gamma_i}^{-1}(a(r(\gamma_i))) - b(s(\gamma_i)) \to 0$, we eventually have $\|\alpha_{\gamma_i}^{-1}(a(r(\gamma_i))) - b(s(\gamma_i))\| < \varepsilon$ for all $\varepsilon > 0$. Hence, $\|\alpha_{\gamma_i}^{-1}(a(r(\gamma_i))) - b(s(\gamma_i))\| \to 0$. Next, $\gamma_i \cdot \pi_i \notin O_J$ for all i so that $\gamma_i \cdot \pi_i(a) = \pi'(\alpha_{\gamma_i}^{-1}(a(r(\gamma_i)))) = 0$ for all i. Thus,

$$\begin{aligned} \|\pi_i(b)\| &= \|\pi_i'(b(s(\gamma_i)) - \alpha_{\gamma_i}^{-1}(a(r(\gamma_i))))\| \\ &\le \|b(s(\gamma_i)) - \alpha_{\gamma_i}^{-1}(a(r(\gamma_i)))\| \to 0. \end{aligned} \tag{1}$$

It is shown in [**12**, Lemma A.30] that the map $\pi \mapsto \|\pi(b)\|$ is lower-semicontinuous on $\widehat{A}$. In other words, given $\varepsilon \ge 0$, the set $\{\rho \in \widehat{A} : \|\rho(b)\| \le \varepsilon\}$ is closed. Thus (1) implies that eventually $\pi_i \in \{\rho \in \widehat{A} : \|\rho(b)\| \le \varepsilon\}$. Therefore, the fact that $\pi_i \to \pi$ implies $\|\pi(b)\| \le \varepsilon$. This is true for all $\varepsilon > 0$ so that

$$0 = \pi(b) = \pi'(b(s(\gamma))) = \pi'(\alpha_\gamma^{-1}(a(r(\gamma)))) = \gamma \cdot \pi(a).$$

This is a contradiction since $a \in J$ was arbitrary and we assumed that $\gamma \cdot \pi \in O_J$. □

1.1. Bundle crossed products. An important class of groupoids are those for which the range and source map are identical. Such a space is called a (groupoid) group bundle, and we will use p to denote both the range and the source. The premier example of a groupoid group bundle is the stabilizer subgroupoid S of a groupoid G. The reason this class of groupoids is important for what follows is that crossed products by group bundles have extra structure.

Proposition 1.2. *Suppose (A, S, α) is a groupoid dynamical system, and S is a group bundle. Then $A \rtimes_\alpha S$ is a $C_0(S^{(0)})$-algebra with the* *action defined for $\phi \in C_0(S^{(0)})$ and $f \in \Gamma_c(S, p^*\mathcal{A})$ by $\phi \cdot f(s) := \phi(p(s))f(s)$. Furthermore, the restriction map from $\Gamma_c(S, p^*\mathcal{A})$ to $C_c(S_u, A(u))$ factors to an isomorphism of $A \rtimes_\alpha S(u)$ onto $A(u) \rtimes_{\alpha|_{S_u}} S_u$.*

Proof. Given $\phi \in C_0(S^{(0)})$ and $f \in \Gamma_c(S, p^*\mathcal{A})$, define $\Phi(\phi)f = \phi \cdot f$ as in the statement of the proposition. It is easy to see that $\Phi(\phi)f \in \Gamma_c(S, p^*\mathcal{A})$ and that $\Phi(\phi)$ is linear as a function on $\Gamma_c(S, p^*\mathcal{A})$. We need to extend $\Phi(\phi)$ to an element of the multiplier algebra. First, simple calculations show that, on $\Gamma_c(S, p^*\mathcal{A})$, $\Phi(\phi)$ is $A \rtimes S$-linear and is adjointable with adjoint $\Phi(\overline{\phi})$.

Now extend Φ to the unitization $C_0(S^{(0)})^1$ by setting $\Phi(\phi + \lambda 1)f = \Phi(\phi)f + \lambda f$. An elementary computation shows that Φ preserves the operations on $C_0(S^{(0)})^1$. Suppose $\phi \in C_0(G^{(0)})$ and $f \in \Gamma_c(S, p^*\mathcal{A})$. In order to show $\Phi(\phi)$ is bounded it will suffice to show that $\langle \phi \cdot f, \phi \cdot f\rangle \leq \|\phi\|_\infty^2 \langle f, f\rangle$ where $\langle \cdot, \cdot \rangle$ is the usual inner product when $A \rtimes S$ is viewed as an $A \rtimes S$-module. However, this is equivalent to proving

$$0 \leq \|\phi\|_\infty^2 \langle f, f\rangle - \langle \Phi(\phi)f, \Phi(\phi)f\rangle = \langle \Phi(\|\phi\|_\infty^2 1 - \overline{\phi}\phi)f, f\rangle.$$

Since general C^*-algebraic nonsense assures us that $\|\phi\|_\infty^2 1 - \overline{\phi}\phi$ is positive in $C_0(S^{(0)})^1$, it follows that there is some $\xi \in C_0(S^{(0)})^1$ such that $\xi^*\xi = \|\phi\|_\infty^2 1 - \overline{\phi}\phi$. We now compute

$$\langle \Phi(\|\phi\|_\infty^2 1 - \overline{\phi}\phi)f, f\rangle = \langle \Phi(\xi^*)\Phi(\xi)f, f\rangle = \langle \Phi(\xi)f, \Phi(\xi)f\rangle \geq 0.$$

Hence, $\Phi(\phi)$ is bounded and extends to a multiplier on $A \rtimes S$. Furthermore, simple calculations show that Φ is a nondegenerate homomorphism from $C_0(S^{(0)})$ into the center of the multiplier algebra of $A \rtimes S$. Thus $A \rtimes S$ is a $C_0(S^{(0)})$-algebra.

Let us now address the second part of the proposition. Fix $u \in S^{(0)}$, and recall that $A \rtimes S(u) = A \rtimes S/I_u$ where

$$I_u = \overline{\operatorname{span}}\,\{\phi \cdot a : \phi \in C_0(S^{(0)}), a \in A \rtimes S, \phi(u) = 0\}.$$

Next, observe that S acts trivially on its unit space so that $\{u\}$ is a closed S-invariant subset in $S^{(0)}$ and $O = S^{(0)} \setminus \{u\}$ is an open S-invariant subset. It follows from [**6**, Theorem 3.3] that restriction factors to an isomorphism from $A \rtimes S/\mathrm{Ex}\,(O)$ onto $A(u) \rtimes S_u$. Thus we will be done if we can show that $I_u = \mathrm{Ex}\,(O) = \{f \in \Gamma_c(S, p^*\mathcal{A}) : \operatorname{supp} f \subset S \setminus S_u\}$. Given $f \in \mathrm{Ex}\,(O)$ let $\phi \in C_c(S^{(0)})$ be zero on u and one on $p(\operatorname{supp} f)$. Then $\phi \cdot f = f \in I_u$ and $I_u \subset \mathrm{Ex}\,(O)$. Now suppose $f \in I_u$. Given $\varepsilon > 0$, the set $K = \{s : \|f(s)\| \geq \varepsilon\}$ is a compact subset of $\operatorname{supp} f$ and as such we can find $\phi \in C_c(S^{(0)})$ such that ϕ is one on $p(K)$, zero on a neighborhood of u and $0 \leq \phi \leq 1$. It follows quickly that $\phi \cdot f \in \mathrm{Ex}\,(O)$ and that $\|\phi \cdot f - f\| < \varepsilon$. Since ε was arbitrary, this is enough to show that $\mathrm{Ex}\,(O) \subset I_u$. $\square$

Remark 1.3. One important consequence of Proposition 1.2 is that the irreducible representations of $A \rtimes S$ are well behaved. To elaborate, [**17**, Proposition C.6] states that, as a set, the spectrum $(A \rtimes S)^\wedge$ can be identified with the disjoint union $\coprod_{u \in S_u} (A(u) \rtimes S_u)^\wedge$. In other words, every irreducible representation of the crossed product $A \rtimes S$ is lifted from an irreducible covariant representation of the group crossed product $A(u) \rtimes S_u$ for some $u \in S^{(0)}$ via restriction on $\Gamma_c(S, p^*\mathcal{A})$. This fact is at the heart of the analysis in Section 2.

We finish this section with a technical lemma. Recall that, given a $C_0(X)$-algebra A with associated usc-bundle $\mathcal{A}$ and a locally compact Hausdorff subset $Y \subset X$, we define $A(Y) := \Gamma_0(Y, \mathcal{A})$.

Lemma 1.4. *Suppose (A, S, α) is a groupoid dynamical system, S is a group bundle and C is a closed subset of $S^{(0)}$. Then $A \rtimes_\alpha S(C)$ and $A(C) \rtimes_\alpha S|_C$ are isomorphic as $C_0(C)$-algebras.*

Proof. Since the action of S on its unit space is trivial, both C and $U = S^{(0)} \setminus C$ are S-invariant subsets. It follows from [**6**, Theorem 3.3] that restriction factors to an isomorphism $\overline{\rho}_1$ of $A \rtimes S/\mathrm{Ex}\,(U)$ onto $A(C) \rtimes S|_C$. Now let

$$I_C = \overline{\operatorname{span}}\,\{\phi \cdot f : \phi \in C_0(S^{(0)}), f \in \Gamma_c(S, p^*\mathcal{A}), \phi(C) = 0\}.$$

It follows from some basic $C_0(X)$-algebra theory that the restriction map $\rho_2 : A \rtimes S \to A \rtimes S(C)$, where we view both spaces as section algebras of the usc-bundle associated to $A \rtimes S$, factors to an isomorphism $\overline{\rho}_2 : A \rtimes S/I_C \to A \rtimes S(C)$. Similar to the previous proposition, an approximation argument shows that $I_C = \mathrm{Ex}\,(U)$, and therefore we may form the isomorphism $\rho = \overline{\rho}_2 \circ \overline{\rho}_1^{-1}$ of $A(C) \rtimes S|_C$ onto $A \rtimes S(C)$. The fact that ρ is $C_0(C)$-linear then follows from a straightforward calculation. □

2. Groupoid crossed products. As mentioned in the introduction, we aim to identify the spectrum of groupoid crossed products via induction and the stabilizer subgroupoid. The key to this construction is the following map, which we will eventually factor to a homeomorphism.

Remark 2.1. We say that a groupoid G is *regular* if it satisfies one of the equivalent conditions of the Mackey-Glimm dichotomy [**13**]. In particular, G is regular whenever $G^{(0)}/G$ is T_0 or almost Hausdorff.

Proposition 2.2. *Suppose (A, G, α) is a groupoid dynamical system, that G is regular, and that the isotropy groupoid S has a Haar system. Then $\Phi : (A \rtimes S)^\wedge \to (A \rtimes G)^\wedge$ given by $\Phi(R) = \mathrm{Ind}_S^G R$ is a continuous surjection.*

Recall that $A \rtimes S$ is a $C_0(G^{(0)})$-algebra and that restriction factors to an isomorphism of $A \rtimes S(u)$ with $A(u) \rtimes S_u$. The main difficulty is showing that induction respects this fibering.

Lemma 2.3. *Suppose (A, G, α) is a groupoid dynamical system and that the stabilizer subgroupoid S has a Haar system. Given $u \in G^{(0)}$ and a representation R of $A(u) \rtimes S_u$, let $\rho : A \rtimes S \to A(u) \rtimes S_u$ be given on $\Gamma_c(S, p^*\mathcal{A})$ by restriction. Then $Ind_{S_u}^G R$ is naturally equivalent to $Ind_S^G(R \circ \rho)$.*

Proof. The proof of this lemma is relatively straightforward so we shall limit ourselves to sketching an outline. Fix $u \in G^{(0)}$ and suppose R is a representation of $A(u) \rtimes S_u$ on $\mathcal{H}$. Recall from [**6**, Theorem 2.1] that $\mathrm{Ind}_{S_u}^G R$ acts on the Hilbert tensor product $\mathcal{Z}_{S_u}^G \otimes_{A(u) \rtimes S_u} \mathcal{H}$ where $\mathcal{Z}_{S_u}^G$ is a Hilbert $A(u) \rtimes S_u$-module. Furthermore, recall that $\mathcal{Z}_{S_u}^G$ is a completion of $C_c(G_u, A(u))$. Similarly $\mathrm{Ind}_S^G(R \circ \rho)$ acts on

$\mathcal{Z}_S^G \otimes_{A \rtimes S} \mathcal{H}$ where the Hilbert $A \rtimes S$-module $\mathcal{Z}_S^G$ is a completion of $\Gamma_c(G, s^*\mathcal{A})$. Let $\pi : \Gamma_c(G, s^*\mathcal{A}) \to C_c(G_u, A(u))$ be given by restriction. We now define $U : \Gamma_c(G, s^*\mathcal{A}) \odot \mathcal{H} \to C_c(G_u, A(u)) \odot \mathcal{H}$ on elementary tensors by $U(f \otimes h) = \pi(f) \otimes h$. It then follows from some relatively painless calculations that U is isometric and extends to a unitary map from $\mathcal{Z}_S^G \otimes_{A \rtimes S} \mathcal{H}$ onto $\mathcal{Z}_{S_u}^G \otimes_{A(u) \rtimes S_u} \mathcal{H}$ which intertwines $\mathrm{Ind}_{S_u}^G R$ and $\mathrm{Ind}_S^G(R \circ \rho)$. $\square$

Remark 2.4. In light of how natural the unitary intertwining $\mathrm{Ind}_{S_u}^G R$ and $\mathrm{Ind}_S^G(R \circ \rho)$ is, we shall often confuse the two. Furthermore, since every irreducible representation of $A \rtimes S$ is lifted from a fiber via restriction, we will feel free to use the notation $\mathrm{Ind}_S^G R$ even when R is an irreducible representation of $A(u) \rtimes S_u$ and will interpret $\mathrm{Ind}_S^G R$ as either $\mathrm{Ind}_{S_u}^G R$ or $\mathrm{Ind}_S^G(R \circ \rho)$ as we see fit. We trust the reader will forgive the author for these abuses.

The advantage of viewing the induction as occurring on S is that induction from a fixed algebra is a continuous process.

Proof of Proposition 2.2. As noted above, every irreducible representation of $A \rtimes S$ is of the form $R \circ \rho$ where R is an irreducible representation of $A(u) \rtimes S_u$ for some $u \in G^{(0)}$ and ρ is the canonical extension of the restriction map. Since G is regular, we know from [**8**, Proposition 4.13] that $\mathrm{Ind}_S^G R$ is irreducible. Thus Φ is well defined. The surjectivity follows immediately from [**6**, Theorem 4.1], and the continuity follows from the fact that Rieffel induction is a continuous process [**12**, Corollary 3.35]. $\square$

2.1. Groupoid actions. The goal of this section is to lay groundwork for establishing the equivalence relation on $(A \rtimes S)^\wedge$ induced by Φ.

Proposition 2.5. *Suppose G is a locally compact groupoid and that the isotropy subgroupoid S has a Haar system. Then there is a continuous homomorphism ω from G to $\mathbf{R}^+$ such that for all $f \in C_c(S)$*

$$\int_S f(s)d\beta^{r(\gamma)}(s) = \omega(\gamma) \int_S f(\gamma s \gamma^{-1})\, d\beta^{s(\gamma)}(s). \tag{2}$$

Furthermore, given $s \in S$, we have $\omega(s) = \Delta^u(s)^{-1}$ where Δ^u is the modular function for the group S_u.

Proof. By and large this is proved in the same way as [**15**, Lemma 4.1]. The only difference is that the stabilizer subgroupoid S may not be abelian and that, rather than being S-invariant, $\omega(s) = \Delta^u(s)^{-1}$ for all $s \in S_u$. This is shown by the following calculation for $s \in S_u$ and $f \in C_c(S)$

$$\begin{aligned}\omega(s)^{-1}\int_S f(t)d\beta^u(t) &= \int_S f(sts^{-1})\,d\beta^u(t)\\ &= \int_S f(ts^{-1})\,d\beta^u(t)\\ &= \Delta^u(s)\int f(t)\,d\beta^u(t).\end{aligned}$$

Since the remainder of the proof is identical to that of [**15**, Lemma 4.1], we will not reproduce it here. □

Next we demonstrate the following construction which, although we only make use of it indirectly, is interesting in its own right.

Proposition 2.6. *Suppose (A, G, α) is a groupoid dynamical system and that the isotropy subgroupoid S has a Haar system. Then there is an action δ of G on $A \rtimes_\alpha S$ defined by the collection $\{\delta_\gamma\}_{\gamma\in G}$ where, for $f \in C_c(S_{s(\gamma)}, A(s(\gamma)))$,*

$$\delta_\gamma(f)(s) = \omega(\gamma)^{-1}\alpha_\gamma(f(\gamma^{-1}s\gamma)). \tag{3}$$

Proof. It is easy enough to show that $\delta_\gamma : A(s(\gamma)) \rtimes S_{s(\gamma)} \to A(r(\gamma)) \rtimes S_{r(\gamma)}$ is a well-defined isomorphism and that δ respects the groupoid operations on G. The difficult part is in proving that the action is continuous. Suppose $\mathcal{E}$ is the usc-bundle associated to the $C_0(G^{(0)})$-algebra $A \rtimes S$. Given $\gamma_n \to \gamma_0$ in G and $a_n \to a$ in $\mathcal{E}$ such that $s(\gamma_n) = p(a_n) = u_n$ for all $n \geq 0$, we must show that $\delta_{\gamma_n}(a_n) \to \delta_{\gamma_0}(a_0)$. Fix $\varepsilon > 0$, and let $v_n = r(\gamma_n)$ for all $n \geq 0$. First, choose $b \in A \rtimes S$ such that $b(u_0) = a_0$. Next, using the fact that $\Gamma_c(S, p^*\mathcal{A})$ is dense in

$A \rtimes S$, we can choose $f \in \Gamma_c(S, p^*\mathcal{A})$ such that $\|f(u) - b(u)\| < \varepsilon/2$ for all $u \in G^{(0)}$. Recall that $f(u)$, the image of f in $A(u) \rtimes S_u$, is exactly the restriction of f to S_u. We now make the following

Claim. *If $f \in \Gamma_c(S, p^*\mathcal{A})$ and $\gamma_n \to \gamma_0$ as above, then $\delta_{\gamma_n}(f(u_n)) \to \delta_{\gamma_0}(f(u_0))$.*

Proof of Claim. First, suppose $v_n = v_0$ infinitely often. Then we can pass to a subsequence, relabel and assume $v_n = v_0$ for all $n \geq 0$. Now suppose we can pass to another subsequence such that for each $n > 0$ there exists s_n with

$$\|\delta_{\gamma_n}(f(u_n))(s_n) - \delta_{\gamma_0}(f(u_0))(s_n)\| \geq \varepsilon > 0. \tag{4}$$

If this is to hold, we must either have $\gamma_n^{-1} s_n \gamma_n \in \operatorname{supp} f$ infinitely often or $\gamma_0^{-1} s_n \gamma_0 \in \operatorname{supp} f$ infinitely often. In either case we may pass to a subsequence, multiply by the appropriate groupoid elements, and find s_0 such that $s_n \to s_0$. However, we then have $f(\gamma_n^{-1} s_n \gamma_n) \to f(\gamma_0^{-1} s_0 \gamma_0)$ and $f(\gamma_0^{-1} s_n \gamma_0) \to f(\gamma_0^{-1} s_0 \gamma_0)$. Since both ω and α are continuous, it follows that $\delta_{\gamma_n}(f(u_n))(s_n)$ and $\delta_{\gamma_0}(f(u_0))(s_n)$ both converge to $\delta_{\gamma_0}(f(u_0))(s_0)$ and this contradicts (4). It follows quickly that $\delta_{\gamma_n}(f(u_n)) \to \delta_{\gamma_0}(f(u_0))$ with respect to the inductive limit topology, and thus in $A(v_0) \rtimes S_{v_0} \subset \mathcal{E}$.

Next, suppose that we may remove an initial segment, and assume that $v_n \neq v_0$ for all $n > 0$. We may also pass to a subsequence, relabel and assume that $v_n \neq v_m$ for all $n \neq m$. Let $K = \{v_n\}_{n=0}^{\infty}$. Then $C = S|_K = p^{-1}(K)$ is closed in S, and we can define ι on C by $\iota(s) = n$ if and only if $p(s) = v_n$. Some simple computations then show that the function $F(s) = \delta_{\gamma_{\iota(s)}}(f(\iota(s)))(s)$ is continuous and compactly supported on C. Thus $F \in \Gamma_c(C, p^*\mathcal{A}) \subset A(K) \rtimes S|_K$. It follows from Lemma 1.4 that $A(K) \rtimes S|_K$ is isomorphic to the restriction $A \rtimes S(K)$. In particular, we may view F as a continuous section of $\mathcal{E}$ on K, where we recall that $F(v_n)$ denotes the restriction of F to S_{v_n}. Since F is continuous, we must have $F(v_n) \to F(v_0)$. However, we clearly constructed F so that $F(v_n) = \delta_{\gamma_n}(f(u_n))$ for all $n \geq 0$ and this proves our claim. □

Thus, $\delta_{\gamma_n}(f(u_n)) \to \delta_{\gamma_0}(f(u_0))$. Since both $a_n \to a_0$ and $b(u_n) \to a_0$, it follows that $\|a_n - b(u_n)\| \to 0$ so that eventually

$$\|\delta_{\gamma_n}(f(u_n)) - \delta_{\gamma_n}(a_n)\| \leq \|f(u_n) - b(u_n)\| + \|b(u_n) - a_n\| < \varepsilon.$$

Since $\|\delta_{\gamma_0}(f(u_0)) - \delta_{\gamma_0}(a_0)\| = \|f(u_0) - b(u_0)\| < \varepsilon$ by construction, it now follows from [**17**, Proposition C.20] that $\delta_{\gamma_n}(a_n) \to \delta_{\gamma_0}(a_0)$, and we are done. $\square$

The following corollary will eventually form our foundation for the equivalence classes determined by Φ.

Corollary 2.7. *Suppose (A, G, α) is a groupoid dynamical system and that the stabilizer subgroupoid has a Haar system. Then the action δ induces an action of G on $(A \rtimes S)^\wedge$ given by $\delta \cdot R = R \circ \delta_\gamma^{-1}$. Furthermore, if $R = \pi \rtimes U$ then $\gamma \cdot R = \rho \rtimes V$ where*

$$\rho(a) = \pi(\alpha_\gamma^{-1}(a)), \quad \textit{and} \quad V_s = U_{\gamma^{-1}s\gamma}. \tag{5}$$

Proof. The fact that the action exists follows immediately from Proposition 1.1. Calculating that ρ and V are given as above is accomplished by composing the canonical injections of $A(r(\gamma))$ and $S_{r(\gamma)}$ into $M(A(r(\gamma)) \rtimes S_{r(\gamma)})$ with $\gamma \cdot R$. $\square$

Remark 2.8. We have omitted many of the calculations in these proofs for brevity. However, enterprising readers wishing to verify the above computations should make note of the fact that, if Δ^u is the modular function for S_u, then

$$\Delta^{s(\gamma)}(s) = \Delta^{r(\gamma)}(\gamma s \gamma^{-1}) \quad \text{for } \gamma \in G. \tag{6}$$

2.2. Equivalent representations. The primary obstacle in working with induced representations is that they are not very concrete. The purpose of this section is to describe a selection of concrete representations which are equivalent to $\operatorname{Ind}_S^G R$ for a given R. This material is at least inspired by [**15**], when it doesn't copy it directly. We begin by citing the following

Lemma 2.9 [**10**, Lemma 3.2]. *Let G be a locally compact Hausdorff groupoid. Suppose $u \in G^{(0)}$, that A is a subgroup of S_u and that β is a Haar measure on A. Then the following hold.*

(a) *The formula*

$$Q(f)([\gamma]) = \int_A f(\gamma s)\, d\beta(s)$$

defines a surjection from $C_c(G)$ *onto* $C_c(G_u/A)$.

(b) *There is a non-negative continuous function* b *on* G_u *such that, for any compact set* $K \subset G_u$*, the support of* b *and* KA *have compact intersection and for all* $\gamma \in G_u$

$$\int_A b(\gamma s)\, d\beta(s) = 1. \tag{7}$$

The function b in Lemma 2.9 is the normalization of a function b' which satisfies all of the conditions of (b) except for (7). This function is guaranteed to exist by [**2**, Lemma 1]. Furthermore, [**4**] also proves that b' is positive, continuous, and b' is not zero on any entire equivalence class. We now define

$$\rho(\gamma) = \int_A b'(\gamma s)\Delta(s)^{-1} d\beta(s) \tag{8}$$

for $\gamma \in G_u$ where Δ is the modular function for A. Notice that $\rho(\gamma) > 0$ for all γ because the modular function is strictly greater than zero and b' is positive and not zero on any entire equivalence class. An important property of ρ is that for $\gamma \in G_u$ and $s \in A$

$$\begin{aligned}\rho(\gamma s) &= \int_A b'(\gamma s t)\Delta(t)^{-1} d\beta(t) = \int_A b'(\gamma t)\Delta(s)\Delta(t)^{-1} d\beta(t)\\ &= \Delta(s)\rho(\gamma).\end{aligned} \tag{9}$$

We can now cite the following

Lemma 2.10 [**10**, Lemma 3.3]. *There is a Radon measure* σ *on* G_u/A *such that*

$$\int_G f(\gamma)\rho(\gamma)\, d\lambda_u(\gamma) = \int_{G_u/A} \int_A f(\gamma s)\, d\beta(s)\, d\sigma([\gamma]) \tag{10}$$

for all $f \in C_c(G_u)$.

Remark 2.11. It is not particularly difficult to show that σ has full support on G_u/A.

Suppose (A, G, α) is a groupoid dynamical system with stabilizer subgroupoid S. For all $u \in S^{(0)}$, let β^u be a Haar measure on S_u. Using Lemma 2.10, for each $u \in G^{(0)}$, there exists a Radon measure σ^u with full support on G_u/S_u and an associated continuous strictly positive function ρ^u on G_u such that

$$\int_G f(\gamma)\rho^u(\gamma)\,d\lambda_u(\gamma) = \int_{G_u/S_u}\int_S f(\gamma s)\,d\beta^u(s)\,d\sigma^u([\gamma]).$$

For the rest of this section whenever we have (A, G, α) and S as above we will let $\sigma = \{\sigma^u\}$ and $\rho = \{\rho^u\}$ be defined in this way. Next, we construct a Hilbert space which we will use for one of our equivalent representations.

Lemma 2.12. *Fix $u \in G^{(0)}$, and suppose $R = \pi \rtimes U$ is a covariant representation of $A(u) \rtimes S_u$ on a separable Hilbert space $\mathcal{H}$. Let $\mathcal{V}_u$ be the set of Borel functions $\phi : G_u \to \mathcal{H}$ such that $\phi(\gamma s) = U_s^*\phi(\gamma)$ for all $\gamma \in G_u$ and $s \in S_u$. Define*

$$\mathcal{L}^2_U(G_u, \mathcal{H}, \sigma^u) = \left\{\phi \in \mathcal{V}_u : \int_{G_u/S_u} \|\phi(\gamma)\|^2 d\sigma^u([\gamma]) < \infty\right\},$$

and let $L^2_U(G_u, \mathcal{H}, \sigma^u)$ be the quotient of $\mathcal{L}^2_U(G_u, \mathcal{H}, \sigma^u)$ where we identify functions which agree almost everywhere. Then $L^2_U(G_u, \mathcal{H}, \sigma^u)$ is a Hilbert space with the inner product

$$(\phi, \psi) := \int_{G_u/S_u} (\phi(\gamma), \psi(\gamma))\,d\sigma^u([\gamma]).$$

Since the proof of this lemma is, by and large, straightforward, we will omit it here for brevity. Similar arguments can be found in [**17**, page 290] or in [**5**, Lemma 6.36]. Using this Hilbert space we have the following

Proposition 2.13. *Suppose (A, G, α) is a groupoid dynamical system and that the stabilizer subgroupoid S has a Haar system. Fix*

$u \in G^{(0)}$, and let $R = \pi \rtimes U$ be a covariant representation of $A(u) \rtimes S_u$ acting on the separable Hilbert space $\mathcal{H}$. Then $\mathrm{Ind}_{S_u}^{G} R$ is equivalent to the representation T^R on $L^2_U(G_u, \mathcal{H}, \sigma^u)$ defined for $f \in \Gamma_c(G, r^*\mathcal{A})$ and $\phi \in \mathcal{L}^2_U(G_u, \mathcal{H}, \sigma^u)$ by

$$(11)\quad T^R(f)\phi(\gamma) = \int_G \pi(\alpha_\gamma^{-1}(f(\gamma\eta^{-1})))\phi(\eta)\rho^u(\eta)^{1/2}\rho^u(\gamma)^{-1/2}d\lambda_u(\eta).$$

Proof. First recall that $\mathrm{Ind}_{S_u}^{G} R$ acts on the Hilbert space $\mathcal{Z}_{S_u}^{G} \otimes_{A(u)\rtimes S_u} \mathcal{H}$ where $\mathcal{Z}_{S_u}^{G}$ is the completion of the pre-Hilbert $A(u) \rtimes S_u$-module $C_c(G_u, A(u))$. Define $V : C_c(G_u, A(u)) \odot \mathcal{H} \to L^2_U(G_u, \mathcal{H}, \sigma^u)$ on elementary tensors by

$$(12)\qquad V(z \otimes h)(\gamma) = \int_S U_s\pi(z(\gamma s))h\rho^u(\gamma s)^{-1/2}\, d\beta^u(s).$$

It is not difficult to prove that $V(z\otimes h)$ is an element of $\mathcal{L}^2_U(G_u, \mathcal{H}, \sigma^u)$. Furthermore, simple computations show that V is isometric and extends to an isometry from $\mathcal{Z}_{S_u}^{G} \otimes_{A(u)\rtimes S_u} \mathcal{H}$ into $L^2_U(G_u, \mathcal{H}, \sigma^u)$. In order to show that V is a unitary, it will suffice to show that, given $\phi \in \mathcal{L}^2_U(G_u, \mathcal{H}, \sigma^u)$ such that $(V(z\otimes h), \phi) = 0$ for all $z \in C_c(G_u, A(u))$ and $h \in \mathcal{H}$, then ϕ is zero λ_u-almost everywhere. We have

$$(13)\qquad \begin{aligned} 0 = (V(z \otimes h), \phi) &= \int_{G_u/S_u} (V(z\otimes h)(\gamma), \phi(\gamma))d\sigma^u([\gamma]) \\ &= \int_{G_u/S_u}\int_S (U_s\pi(z(\gamma s))h, \phi(\gamma))\rho^u(\gamma s)^{-1/2}d\beta^u(s)\, d\sigma^u([\gamma]) \\ &= \int_{G_u/S_u}\int_S (\pi(z(\gamma s))h, \phi(\gamma s))\rho^u(\gamma s)^{-1/2}d\beta^u(s)\, d\sigma^u([\gamma]) \\ &= \int_G (((\pi\circ z)\otimes h)(\gamma), \phi(\gamma))\rho^u(\gamma)^{1/2}d\lambda_u(\gamma) \end{aligned}$$

where $(\pi \circ z) \otimes h$ denotes the function $\gamma \mapsto \pi(z(\gamma))h$. Now suppose $K \subset G_u$ is compact, and let $\phi|_K$ be the function obtained by letting ϕ be zero off K. If $g \in C_c(G_u)$ is one on K, then by Lemma 2.9,

$$F([\gamma]) = \int_S g(\gamma s)\rho^u(\gamma s)^{-1}d\beta^u(s)$$

defines an element of $C_c(G_u/S_u)$. We observe that

$$\begin{aligned}\int_G \|\phi|_K(\gamma)\|^2 d\lambda_u(\gamma)& \\ &\leq \int_G g(\gamma)\|\phi(\gamma)\|^2 d\lambda_u(\gamma) \\ &= \int_{G_u/H_u} \|\phi(\gamma)\|^2 \int_{S_u} g(\gamma s)\rho^u(\gamma s)^{-1} d\beta^u(s)\, d\sigma^u([\gamma]) \\ &\leq \|\phi\|^2 \|F\|_\infty .\end{aligned}$$

Thus $\phi|_K \in L^2(G_u, \mathcal{H})$. Next, given $z \in C_c(G_u, A(u))$ such that $\operatorname{supp} z \subset K$, we conclude from (13) that

$$\begin{aligned} 0 &= \int_G (((\pi \circ z) \otimes h)(\gamma), \phi(\gamma))\rho^u(\gamma)^{1/2} d\lambda_u(\gamma) \\ &= ((\pi \circ z) \otimes h, \phi(\rho^u)^{1/2})_{L^2(K,\mathcal{H},\lambda_u)}. \end{aligned} \tag{14}$$

Because ρ^u is strictly positive, it follows that $\phi|_K$ will be zero λ_u-almost everywhere if we can show that elements of the form $(\pi \circ z) \otimes h$ span a dense set in $L^2(K, \mathcal{H}, \lambda_u)$. However, we can restrict ourselves even further and work with elementary tensors of the form

$$f \otimes (\pi(a)h) = ((f \otimes a) \circ \pi) \otimes h$$

where $f \in C_c(K)$, $a \in A(u)$ and $h \in \mathcal{H}$. However, using nondegeneracy, it is fairly clear that these elements span a dense set in $L^2(K, \mathcal{H}, \lambda_u)$. Thus $\phi|_K$ is zero λ_u-almost everywhere. Since K was arbitrary and G_u is σ-compact, the result follows. Hence V is a unitary and as such we can define the representation $T^R := V \operatorname{Ind}_{S_u}^G R V^*$. The fact that T^R is given by (11) is the result of a slightly messy computation. ◻

Next, because G_u is second countable, we can find a Borel cross section $c : G_u/S_u \to G_u$, and this allows us to define a Borel map $\delta : G_u \to S_u$ such that $\gamma = c([\gamma])\delta(\gamma)$. We will need these maps in order to find a representation equivalent to T^R which acts on $L^2(G_u/S_u, \mathcal{H}, \sigma^u)$.

Proposition 2.14. *Suppose (A, G, α) is a groupoid dynamical system with stabilizer subgroupoid S. Fix $u \in G^{(0)}$, let $R = \pi \rtimes U$ be a* *representation of $A(u) \rtimes S_u$ on the separable Hilbert space $\mathcal{H}$ and let δ be as above. Then T^R and $\operatorname{Ind}_{S_u}^G R$ are equivalent to the representation N^R on $L^2(G_u/S_u, \mathcal{H}, \sigma^u)$ given by*

$$(15)\quad N^R(f)(\phi)([\gamma]) = \int_G U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(f(\eta)))U^*_{\delta(\eta^{-1}\gamma)}\phi([\eta^{-1}\gamma])\cdots \\ \cdots\rho^u(\eta^{-1}\gamma)^{1/2}\rho^u(\gamma)^{-1/2}d\lambda^{r(\gamma)}(\eta).$$

Proof. Define $W : L^2_U(G_u, \mathcal{H}, \sigma^u) \to L^2(G_u/S_u, \mathcal{H}, \sigma^u)$ by $W(\phi)([\gamma]) = \phi(c([\gamma]))$ where c is the Borel cross section described previously. It follows from a brief computation that W is a unitary and as such we can use it to define the representation $N^R = WT^RW^*$. The fact that N^R is given by (15) follows from another computation. □

Remark 2.15. Before we move forward, we need some more measure theoretic trickery. Observe that, because G_u is second countable, the range map factors to a Borel isomorphism between G_u/S_u and $G \cdot u$. We use this isomorphism to push the measure σ^u forward to a measure σ^u_* on $G \cdot u$. It is clear that, by identifying $L^2(G_u/S_u, \mathcal{H}, \sigma^u)$ and $L^2(G \cdot u, \mathcal{H}, \sigma^u_*)$, we can view N^R as a representation on the latter space. It is easy to see that, in this case, the action of N^R is given by

$$N^R(f)(\phi)(\gamma \cdot u) = \int_G U_{\delta(\gamma)}\pi(\alpha_\gamma^{-1}(f(\eta)))U^*_{\delta(\eta^{-1}\gamma)}\phi(\eta^{-1}\gamma \cdot u)\cdots \\ \cdots\rho^u(\eta^{-1}\gamma)^{1/2}\rho^u(\gamma)^{-1/2}d\lambda^{r(\gamma)}(\eta).$$

Since this identification is fairly natural, we won't make much of a fuss about it.

The reason we went through the effort to build N^R is that, as the next lemma demonstrates, it interfaces nicely with the multiplication representation of $C^b(G \cdot u)$ on $L^2(G \cdot u, \mathcal{H})$. We will be able to take advantage of this later on.

Lemma 2.16. *Suppose (A, G, α) is a groupoid dynamical system with stabilizer subgroupoid S. Let $u \in G^{(0)}$ and $R = \pi \rtimes U$ be a* *representation of* $A(u) \rtimes S_u$*. Consider the representation of* $C_0(G^{(0)})$ *on* $L^2(G \cdot u, \mathcal{H}, \sigma_*^u)$ *defined via*

$$N^u(f)\phi(v) = f(v)\phi(v).$$

Furthermore, given $f \in C_0(G^{(0)})$ *and* $g \in \Gamma_c(G, r^*\mathcal{A})$*, define* $f \cdot g(\gamma) := f(r(\gamma))g(\gamma)$*. Then* $N^u(f)N^R(g) = N^R(f \cdot g)$ *for all* $f \in C_0(G^{(0)})$ *and* $g \in \Gamma_c(G, r^*\mathcal{A})$*.*

Proof. The representation N^u is nothing more than the restriction map sending $C_0(G^{(0)})$ to $C^b(G \cdot u)$ composed with the usual multiplication representation of $C^b(G \cdot u)$ on $L^2(G \cdot u, \mathcal{H})$. It is easy to see that, if f and g are as above, then $f \cdot g \in \Gamma_c(G, r^*\mathcal{A})$. The last statement follows from a computation. □

We can now prove the following proposition, which tells us that the equivalence classes on $A \rtimes S$ induced by Φ are exactly the orbits of the G action described in Corollary 2.7.

Proposition 2.17. *Suppose* (A, G, α) *is a groupoid dynamical system and that the stabilizer subgroupoid* S *has a Haar system. Fix* $u \in G^{(0)}$*, and let* R *be an irreducible representation of* $A(u) \rtimes S_u$ *on a separable Hilbert space* $\mathcal{H}$*. Then* $\Phi(R)$ *is equivalent to* $\Phi(\gamma \cdot R)$ *for all* $\gamma \in G_u$*. Furthermore, if* G *is regular and* L *and* R *are irreducible representations of* $A(u) \rtimes S_u$ *and* $A(v) \rtimes S_v$*, respectively, then* $\Phi(L)$ *is equivalent to* $\Phi(R)$ *if and only if there exists a* $\gamma \in G_u$ *such that* $\gamma \cdot L$ *is equivalent to* R*.*

Proof. Let $R = \pi \rtimes U$ be as above, and recall that $\gamma \cdot R = \rho \rtimes V$ is given by Corollary 2.7. It follows from Proposition 2.13 that it suffices to show that T^R and $T^{\gamma \cdot R}$ are equivalent. Suppose $u = s(\gamma)$, $v = r(\gamma)$, and define $W : L^2_U(G_u, \mathcal{H}, \sigma^u) \to L^2_V(G_v, \mathcal{H}, \sigma^v)$ by

$$W(\phi)(\eta) = \omega(\gamma)^{1/2}\rho^u(\eta\gamma)^{1/2}\rho^v(\eta)^{-1/2}f(\eta\gamma) \quad \text{for } \eta \in G_v.$$

The fact that W is a unitary which intertwines T^R and $T^{\gamma \cdot R}$ now follows from a relatively straightforward series of computations.

Remark 2.18. Those readers wishing to verify these calculations should make note of the fact that, for $\gamma \in G$ as above,

(16)
$$\int_{G_v/S_v} \phi([\eta\gamma])\omega(\gamma)\rho^u(\eta\gamma)\rho^v(\eta)^{-1} d\sigma^v([\eta]) = \int_{G_u/S_u} \phi([\eta])\, d\sigma^u([\eta]).$$

Moving on, suppose G is regular and that we are given L and R as in the second half of the proposition. If $\Phi(L)$ is equivalent to $\Phi(R)$, then it follows from Proposition 2.14 that N^R is equivalent to N^L. Let W be the intertwining unitary, and let N^u and N^v be as in Lemma 2.16. We compute

$$\begin{aligned} WN^v(f)N^R(g)h &= WN^R(f\cdot g)h = N^L(f\cdot g)Wh \\ &= N^u(f)N^L(g)Wh = N^u(f)WN^R(g)h. \end{aligned}$$

Since N^R is nondegenerate, this implies that N^v is unitarily equivalent to N^u. However, if $G\cdot u \cap G\cdot v = \varnothing$, then [**16**, Lemma 4.15] implies that N^u and N^v can have no equivalent subrepresentations. Hence $G\cdot u = G\cdot v$, and there exists a γ such that $v = \gamma\cdot u$. Then R and $\gamma\cdot L$ are both irreducible representations of $A(v)\rtimes S_v$, and we assumed that $\Phi(R)$ is equivalent to $\Phi(L)$, which is in turn equivalent to $\Phi(\gamma\cdot L)$ by the above. It then follows from [**6**, Proposition 4.13] that R is equivalent to $\gamma\cdot L$, and we are done. □

2.3. Restriction to the stabilizers. Now that we know which representations have the same image under Φ, it is time to show that Φ is open. The key construction is a restriction process from $A\rtimes G$ to $A\rtimes S$. This is defined using the following map.

Proposition 2.19. *Suppose that (A, G, α) is a groupoid dynamical system and the stabilizer subgroupoid S has a Haar system. Then there is a nondegenerate homomorphism $M : A\rtimes S \to M(A\rtimes G)$ such that*

$$M(f)g(\gamma) = \int_S f(s)\alpha_s(g(s^{-1}\gamma))\, d\beta^{r(\gamma)}(s) \tag{17}$$

for $f \in \Gamma_c(S, p^*\mathcal{A})$ and $g \in \Gamma_c(G, r^*\mathcal{A})$.

Proof. Since M is basically defined via convolution, it is easy to show that $M(f)g$ is a continuous compactly supported section. Some lengthy computations, which we omit for brevity, show that, for $f \in \Gamma_c(G, r^*\mathcal{A})$ and $g, h \in \Gamma_c(S, p^*\mathcal{A})$,

$$M(f)(g*h) = M(f)g*h, \quad \text{and} \quad (M(f)g)^* * h = g^* * (M(f^*)h). \tag{18}$$

The challenging part is proving the following lemma. However, since the proof is long and unenlightening, it has be relegated to the end of the section.

Lemma 2.20. *The set of functions of the form $M(f)g$ with $f \in \Gamma_c(S, p^*\mathcal{A})$ and $g \in \Gamma_c(G, r^*\mathcal{A})$ is dense in $\Gamma_c(G, r^*\mathcal{A})$ with respect to the inductive limit topology.*

Now, we want to show that $M(f)$ is bounded so that it extends to a multiplier on $A \rtimes G$. Let ρ be a state on $A \rtimes G$, and define an inner product on $A \rtimes G$ via $(f, g)_\rho = \rho(\langle f, g\rangle)$ where we give $A \rtimes G$ its usual inner-product as an $A \rtimes G$-module. Let $\mathcal{H}_\rho$ be the Hilbert space completion of $A \rtimes G$ with respect to this pre-inner product. We would like to apply the disintegration theorem [**9**, Theorem 7.8] when $\mathcal{H}_0$ is the image of $\Gamma_c(G, r^*\mathcal{A})$ in $\mathcal{H}_\rho$. Define π on $\mathcal{H}_0$ by

$$\pi(f)g = M(f)g$$

for $f \in \Gamma_c(S, p^*\mathcal{A})$ and $g \in \Gamma_c(G, r^*\mathcal{A})$. It is easy to show that $\pi(f)$ is well defined and that π is a homomorphism from $\Gamma_c(S, p^*\mathcal{A})$ to the algebra of linear operators on $\mathcal{H}_0$. It follows from Lemma 2.20 that elements of the form $\pi(f)g$ are dense in $\mathcal{H}_\rho$. Fix $g, h \in \Gamma_c(G, r^*\mathcal{A})$. We would like to see that $f \mapsto (\pi(f)g, h)_\rho$ is continuous with respect to the inductive limit topology. It suffices to see that the map $f \mapsto M(f)g$ is continuous with respect to the inductive limit topology, and this is not hard to prove. Finally, the fact that $(\pi(f)g, h)_\rho = (g, \pi(f^*)h)_\rho$ follows immediately from the fact that $(M(f)g)^* * h = g^* * (M(f^*)h)$. Thus the disintegration theorem implies that π extends to a representation of $A \rtimes G$. In particular, we have

$$\rho(\langle M(f)g, M(f)g\rangle) = (\pi(f)g, \pi(f)g)_\rho \leq \|f\|^2 (g, g)_\rho \leq \|f\|^2 \|g\|^2.$$

By choosing ρ such that $\rho(\langle M(f)g, M(f)g\rangle) = \|M(f)g\|^2$, we conclude that $\|M(f)g\| \leq \|f\|\|g\|$. Thus, $M(f)$ is bounded, and it follows from (18) that $M(f)$ is $A \rtimes G$-linear and adjointable with adjoint $M(f^*)$. Hence, $M(f)$ extends to a multiplier on $A \rtimes G$. What's more, $\|M(f)\| \leq \|f\|$ so that M extends to all of $A \rtimes S$. It is then easy to show that M is a homomorphism on a dense subspace so that it must be a homomorphism everywhere. Finally, the fact that M is nondegenerate follows from Lemma 2.20. $\square$

The point is that nondegenerate maps into multiplier algebras yield continuous restriction processes through the usual general nonsense [**12**], as stated in the following

Corollary 2.21. *Suppose* (A, G, α) *is a groupoid dynamical system and that the stabilizer subgroupoid* S *has a Haar system. Then, there exists a restriction map* $\mathrm{Res}_M : \mathcal{I}(A \rtimes G) \to \mathcal{I}(A \rtimes S)$ *such that* Res_M *is continuous and is characterized by* $\mathrm{Res}_M(\ker R) = \ker \overline{R} \circ M$ *for all representations* R *of* $A \rtimes G$.

This next lemma demonstrates the relationship between induction and this restriction process.

Lemma 2.22. *Suppose* (A, G, α) *is a groupoid dynamical system and that the stabilizer subgroupoid* S *has a Haar system. Then, given* $u \in G^{(0)}$ *and an irreducible representation* R *of* $A(u) \rtimes S_u$*, we have*

$$\mathrm{Res}_M \ker \mathrm{Ind}_{S_u}^G R = \bigcap_{\gamma \in G_u} \ker(\gamma \cdot R). \tag{19}$$

Proof. Suppose $R = \pi \rtimes U$ is as above. Recall from Proposition 2.14 that $\mathrm{Ind}_{S_u}^G R$ is equivalent to N^R, and let $Q = \overline{N^R} \circ M$ so that $\mathrm{Res}_M \ker \mathrm{Ind}_{S_u}^G R = \ker Q$. Now, given $f \in A \rtimes S$, it is straightforward to show that the collection $\{c([\gamma]) \cdot R(f)\}$ is a Borel field of operators on the trivial bundle $G_u/S_u \times \mathcal{H}$ and that we can form the direct integral representation $\int_{G_u/S_u}^{\oplus} c([\gamma]) \cdot R \, d\sigma^u([\gamma])$. It then follows from a fairly hideous computation that $Q = \int_{G_u/S_u}^{\oplus} c([\gamma]) \cdot R \, d\sigma^u([\gamma])$. Hence, for

$f \in A \rtimes S$ and $\phi \in \mathcal{L}^2(G_u/S_u, \mathcal{H}, \sigma^u)$, we have

$$Q(f)\phi([\gamma]) = (c([\gamma]) \cdot R)(f)\phi([\gamma]). \tag{20}$$

Now suppose $f \in A \rtimes S$ and $Q(f) = 0$. Let $\{g_i\} \in C_c(G_u/S_u)$ be a countable set of functions which separates points, and let h_j be a countable basis for $\mathcal{H}$. For each g_i and h_j, (20) implies

$$(c([\gamma]) \cdot R)(f)(g_i \otimes h_j)([\gamma]) = g_i([\gamma])(c([\gamma]) \cdot R)(f)h_j = 0 \tag{21}$$

for all $[\gamma] \notin N_{ij}$ where N_{ij} is a σ^u-null set. Let $N = \cup_{ij} N_{ij}$ and observe that, given $[\gamma] \notin N$ (21) holds for all i and j. In particular, we can pick g_i so that $g_i([\gamma]) \neq 0$ and conclude that $(c([\gamma]) \cdot R)(f) = 0$. Thus, $(c([\gamma]) \cdot R)(f) = 0$ for all $[\gamma] \notin N$. It then follows from (10) that $(c([\gamma]) \cdot R)(f) = 0$ for λ_u-almost every $\gamma \in G_u$.

Next, suppose $s \in S_u$. An elementary computation shows that R and $s \cdot R$ are unitarily equivalent. In particular, $\gamma \cdot R = c([\gamma]) \cdot (\delta(\gamma) \cdot R) \cong c([\gamma]) \cdot R$, and therefore the previous paragraph implies that $\gamma \cdot R(f) = 0$ for λ_u-almost all γ. Since G acts continuously on $(A \rtimes S)^\wedge$, the map $\gamma \mapsto \gamma \cdot R(f)$ is continuous. Furthermore, $\operatorname{supp} \lambda_u = G_u$ and $\gamma \cdot R(f) = 0$ for λ_u-almost every $\gamma \in G_u$ so that we must have $\gamma \cdot R(f) = 0$ for all $\gamma \in G_u$. Hence, $\ker Q \subset \cap_{\gamma \in G_u} \ker(\gamma \cdot R)$. The other inclusion is straightforward. □

We conclude the section with the afore promised proof of Lemma 2.20.

Proof of Lemma 2.20. Fix $\varepsilon > 0$ and $g \in \Gamma_c(G, r^*\mathcal{A})$. Let $K = r(\operatorname{supp} g)$, and choose some fixed open neighborhood U of K in S. We make the following claim.

Claim. *There is a relatively compact open neighborhood O of K in S such that $O \subset U$ and for all $\gamma \in G$ and $s \in O$*

$$\|\alpha_s(g(s^{-1}\gamma)) - g(\gamma)\| < \varepsilon/2. \tag{22}$$

Proof of Claim. Suppose not. Then, for every relatively compact neighborhood $W \subset U$ of K, there exist $\gamma_W \in G$ and $s_W \in W$ such that

$$\|\alpha_{s_W}(g(s_W^{-1}\gamma_W)) - g(\gamma_W)\| \geq \varepsilon/2. \tag{23}$$

When we order W by reverse inclusion, the sets $\{\gamma_W\}$ and $\{s_W\}$ form nets in G and S, respectively. In order for (23) to hold, we must either have $s_W^{-1}\gamma_W \in \operatorname{supp} g$ or $\gamma_W \in \operatorname{supp} g$ for each W. In either case we have $r(\gamma_W) \in K$ and, since W is a neighborhood of K, $\gamma_W \in W \operatorname{supp} g \subset \overline{U} \operatorname{supp} g$. Furthermore, $s_W \in W \subset \overline{U}$ for all W. Since $\overline{U}$ and $\overline{U} \operatorname{supp} g$ are compact, we can pass to a subnet twice, relabel and find $s \in S$ and $\gamma \in G$ such that $s_W \to s$ and $\gamma_W \to \gamma$. However, s_W is eventually in every neighborhood of K, so that we must have $s \in K \subset G^{(0)}$. This implies that $s_W^{-1}\gamma_W \to \gamma_W$. Using the continuity of the action, this contradicts (23). □

Let O be the open set from above, and choose $f \in C_c(S)^+$ such that $\operatorname{supp} f \subset O$ and that $\int_S f(s)\beta^u(s) = 1$ for all $u \in K$. Next, let $\{a_l\}$ be an approximate identity for A. We make the following claim.

Claim. *There exists an l_0 such that*

$$\|a_{l_0}(r(\gamma))\alpha_s(g(s^{-1}\gamma)) - \alpha_s(g(s^{-1}\gamma))\| < \varepsilon/2 \tag{24}$$

for all $s \in \operatorname{supp} f$ and $\gamma \in G$.

Proof of Claim. Suppose not. Then, for each l, there exist $\gamma_l \in G$ and $s_l \in \operatorname{supp} f$ such that

$$\|a_l(r(\gamma_l))\alpha_{s_l}(g(s_l^{-1}\gamma_l)) - \alpha_{s_l}(g(s_l^{-1}\gamma_l))\| \geq \varepsilon/2. \tag{25}$$

In order for (25) to hold, we must have $s_l^{-1}\gamma_l \in \operatorname{supp} g$ for all l. But then $\gamma_l \in (\operatorname{supp} f)^{-1} \operatorname{supp} g$. Since both this set and $\operatorname{supp} f$ are compact, we can pass through two subnets, relabel and find $\gamma \in G$ and $s \in S$ such that $\gamma_l \to \gamma$ and $s_l \to s$. However, we now have $\alpha_{s_l}(g(s_l^{-1}\gamma_l)) \to \alpha_s(g(s^{-1}\gamma))$. Choose $b \in A$ such that $b(r(\gamma)) = \alpha_s(g(s^{-1}\gamma))$. Then $a_l b \to b$. Since $\alpha_{s_l}(g(s_l^{-1}\gamma_l)) \to b(r(\gamma))$ and $b(r(\gamma_l)) \to b(r(\gamma))$, we must have $\|\alpha_{s_l}(g(s_l^{-1}\gamma_l)) - b(r(\gamma_l))\| \to 0$. Putting everything together, it follows that, eventually,

$$\begin{aligned}\|a_l(r(\gamma_l))\alpha_{s_l}(g(s_l^{-1}\gamma_l)) - \alpha_{s_l}(g(s_l^{-1}\gamma_l))\| &\leq 2\|\alpha_{s_l}(g(s_l^{-1}\gamma_l)) - b(r(\gamma_l))\| \\ &\quad + \|a_l b - b\| < \varepsilon/2\end{aligned}$$

and this contradicts (25). □

Consider $f\otimes a_{l_0} \in \Gamma_c(S, p^*\mathcal{A})$. First observe that $\operatorname{supp} f\otimes a_{l_0} \subset U$ and that U was chosen independently of ε. Next, given $\gamma \in G$ if $r(\gamma) \notin K$ then $g(s\gamma) = 0$ for all $s \in S_{r(\gamma)}$ so that, in particular,

$$M(f \otimes a_{l_0})g(\gamma) - g(\gamma) = \int_S f(s)a_{l_0}(r(\gamma))\alpha_s(g(s^{-1}\gamma))\, d\beta^{r(\gamma)}(s) = 0.$$

If $r(\gamma) \in K$, then

$$\begin{aligned}
&\|M(f \otimes a_{l_0})g(\gamma) - g(\gamma)\| \\
&= \left\| \int_S f(s)a_{l_0}(r(\gamma))\alpha_s(g(s^{-1}\gamma))d\beta^{r(\gamma)}(s) - \int_S f(s)\, d\beta^{r(\gamma)}(s)g(\gamma)\right\| \\
&\leq \int_S f(s)\|a_{l_0}(r(\gamma))\alpha_s(g(s^{-1}\gamma)) - g(\gamma)\|\, d\beta^{r(\gamma)}(s) \\
&\leq \int_S f(s)\|a_{l_0}(r(\gamma))\alpha_s(g(s^{-1}\gamma)) - \alpha_s(g(s^{-1}\gamma))\|\, d\beta^{r(\gamma)}(s) \\
&\quad + \int_S f(s)\|\alpha_s(g(s^{-1}\gamma)) - g(\gamma)\|\, d\beta^{r(\gamma)}(s) \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Hence, $\|M(f\otimes a_{l_0})g - g\|_\infty < \varepsilon$. This suffices to show that elements of the form $M(f)g$ are dense in $\Gamma_c(G, r^*\mathcal{A})$ with respect to the inductive limit topology. ◻

2.4. Identifying the spectrum. We have now acquired everything we need to identify the spectrum of $A \rtimes G$ and prove the main result of the paper.

Theorem 2.23. *Suppose that (A, G, α) is a groupoid dynamical system and that the isotropy subgroupoid S has a Haar system. If G is regular, then $\Phi : (A \rtimes S)^\wedge \to (A \rtimes G)^\wedge$ defined by $\Phi(R) = \operatorname{Ind}_S^G R$ is open and factors to a homeomorphism from $(A\rtimes S)^\wedge/G$ onto $(A\rtimes G)^\wedge$.*

Proof. It follows from Proposition 2.2 that Φ is a continuous surjection and from Proposition 2.17 that Φ factors to a bijection on $(A \rtimes S)^\wedge/G$. All that remains is to show that Φ is open. Suppose $\Phi(R_i) \to \Phi(R)$ so that, almost by definition, $\ker\Phi(R_i) \to \ker\Phi(R)$.

Using Corollary 2.21, we know that Res_M is continuous, and therefore

$$\begin{aligned}\operatorname{Res}_M \ker \Phi(R_i) &= \operatorname{Res}_M \ker \operatorname{Ind}_S^G R_i \to \operatorname{Res}_M \ker \Phi(R)\\ &= \operatorname{Res}_M \ker \operatorname{Ind}_S^G R.\end{aligned}$$

Let $u = \sigma(R)$ and $u_i = \sigma(R_i)$ for all i where $\sigma : (A \rtimes S)^\wedge \to G^{(0)}$ is the usual map arising from the $C_0(G^{(0)})$-action on $A \rtimes S$. Using the identifications made in Remark 2.4, as well as Lemma 2.22, we have

$$\begin{aligned}\operatorname{Res}_M \ker \operatorname{Ind}_S^G R &= \bigcap_{\gamma \in G_u} \ker(\gamma \cdot R), \quad \text{and}\\ \operatorname{Res}_M \ker \operatorname{Ind}_S^G R_i &= \bigcap_{\gamma \in G_{u_i}} \ker(\gamma \cdot R_i) \quad \text{for all } i.\end{aligned}$$

It follows from the definition of the Jacobson topology that the closed sets associated to $\operatorname{Res}_M \ker \operatorname{Ind}_S^G R$ and $\operatorname{Res}_M \ker \operatorname{Ind}_S^G R_i$ are

$$F = \overline{\{\ker \gamma \cdot R : \gamma \in G_u\}}, \quad \text{and} \quad F_i = \overline{\{\ker \gamma \cdot R_i : \gamma \in G_{u_i}\}},$$

respectively. Since $\ker R \in F$, it follows from [**17**, Lemma 8.38] that, after passing to a subnet and relabeling, there exists a $P_i \in F_i$ such that $P_i \to \ker R$.

Let $\mathcal{U}$ be a neighborhood basis of $\ker R$. For each $U \in \mathcal{U}$ there exists an i_0 such that $i \geq i_0$ implies $P_i \in U$. We let $M := \{(U, i) : U \in \mathcal{U}, P_i \in U\}$ and direct M by decreasing U and increasing i. Then M is a subnet of i such that $P_{(U,i)} \in U$ for all $(U, i) \in M$. Use this fact to find for each $(U, i) \in M$ some $\gamma_{(U,i)} \in G_{u_i}$ such that $\ker \gamma_{(U,i)} \cdot R_i \in U$. Next, given any $U_0 \in \mathcal{U}$, choose i_0 so that $P_{i_0} \in U$ and $(U_0, i_0) \in M$. If $(U, i) \in M$ such that $(U_0, i_0) \leq (U, i)$ then $\ker \gamma_{(U,i)} \cdot R_i \in U \subset U_0$. Thus, $\ker \gamma_{(U,i)} \cdot R_i \to \ker R$, and therefore $\gamma_{(U,i)} \cdot R_i \to R$. This suffices to show that Φ is open. □

Remark 2.24. If there is a problem with Theorem 2.23, it is that $(A \rtimes S)^\wedge$ can be just as mysterious as $(A \rtimes G)^\wedge$. For instance, if A has Hausdorff spectrum (and is separable) then each fiber $A(u)$ can be identified with the compacts. In this case, $A(u) \rtimes S_u$ is relatively well understood [**17**, Section 7.3] and in particular is isomorphic to $C^*(S_u, \overline{\omega}_u)$ where $[\omega_u]$ is the Mackey obstruction for $\alpha|_{S_u}$. However, even if the stabilizers vary continuously, the collection $\{\omega_u\}$ may be poorly behaved, and identifying the total space topology of $(A \rtimes S)^\wedge$ may be difficult.

The following corollary is immediate and interesting enough to be worth writing down.

Corollary 2.25. *Suppose (A, G, α) is a groupoid dynamical system and that G is a regular principal groupoid. Then $(A \rtimes G)^\wedge$ is homeomorphic to $\widehat{A}/G$.*

3. Groupoid algebras. We can use the machinery developed in Section 2 to prove Theorem 2.23 for certain non-regular groupoid algebras. First, we state the following corollary, which immediately follows from Corollary 2.7.

Corollary 3.1. *Suppose G is a locally compact Hausdorff groupoid and that the stabilizer subgroupoid S has a Haar system. Then there is a continuous action of G on $C^*(S)^\wedge$ given for $\gamma \in G$ and $U \in C^*(S)^\wedge$ by*

$$\gamma \cdot U(s) = U(\gamma^{-1} s \gamma). \tag{26}$$

This action factors to an action of G on $\operatorname{Prim} C^*(S)$.

Next, we note that the main result of [**10**] states that every representation of $C^*(G)$ induced from a stability group is irreducible. Therefore, even when G is not regular, we may induce representations from $C^*(S)^\wedge$ to elements of the spectrum of $C^*(G)$. Furthermore, we obtain the following

Proposition 3.2. *Let G be a locally compact Hausdorff groupoid, and suppose that the isotropy subgroupoid S has a Haar system. Then $\Phi : C^*(S)^\wedge \to C^*(G)^\wedge$ defined by $\Phi(U) = \operatorname{Ind}_S^G U$ is continuous and open as a map onto its range.*

Proof. It follows from the above discussion that Φ maps into $C^*(G)^\wedge$, and the continuity of Φ follows from the general theory of Rieffel induction. All that remains is to show Φ is open. Suppose $\operatorname{Ind} U_i \to \operatorname{Ind} U$ in $C^*(G)^\wedge$. Since Res_M is continuous, it follows that

$$I_i = \operatorname{Res}_M \ker \operatorname{Ind}_S^G U_i \to I = \operatorname{Res}_M \ker \operatorname{Ind}_S^G U.$$

Lemma 2.22 then tells us that

$$I = \bigcap_{\gamma \in G_{\hat{p}(U)}} \ker \gamma \cdot U,$$

and

$$I_i = \bigcap_{\gamma \in G_{\hat{p}(U_i)}} \ker \gamma \cdot U_i \quad \text{for all } i.$$

Hence, the closed sets associated to I and I_i are

$$F = \overline{\{\ker \gamma \cdot U : \gamma \in G_{\hat{p}(U)}\}},$$

and

$$F_i = \overline{\{\ker \gamma \cdot U_i : \gamma \in G_{\hat{p}(U_i)}\}},$$

respectively. Since $\ker U \in F$ it follows from [**17**, Lemma 8.38] that, after passing to a subnet and relabeling, there exists a $P_i \in F_i$ such that $P_i \to \ker U$. It then follows from an argument similar to that at the end of the proof of Theorem 2.23 that we can pass to a subnet and find γ_i such that $\gamma_i \cdot U_i \to U$. This suffices to show that Φ is open onto its image. $\square$

Now, if the stability groups of G are GCR, it follows from [**10**, Theorem 1.1] that $C^*(G)$ is Type I or GCR if and only if G is regular. Since we are extending Theorem 2.23 to non-regular groupoids, this means potentially working with non-Type I C^*-algebras. Thus, we must use the primitive ideal space instead of the spectrum. The following is an immediate consequence of Proposition 3.2, once we extend induction to the primitive ideals in the usual fashion.

Corollary 3.3. *Let G be a locally compact Hausdorff groupoid, and suppose the isotropy subgroupoid S has a Haar system. Then*

$\Psi : \operatorname{Prim} C^*(S) \to \operatorname{Prim} C^*(G)$ *defined by* $\Psi(P) = \operatorname{Ind}_S^G P$ *is continuous and open as a map onto its range.*

We would like to factor Ψ to a homeomorphism and, to do that, we will need to get a handle on the equivalence relation determined by Ψ.

Lemma 3.4. *Let G be a locally compact Hausdorff groupoid, and suppose the isotropy subgroupoid S has a Haar system. Then $\Psi(P) = \Psi(Q)$ if and only if $\overline{G \cdot P} = \overline{G \cdot Q}$.*

Proof. Suppose $U, V \in C^*(S)^\wedge$ such that $P = \ker U$ and $Q = \ker V$. If $\ker \operatorname{Ind}_S^G V = \ker \operatorname{Ind}_S^G U$, then $\operatorname{Res}_M \ker \operatorname{Ind}_S^G V = \operatorname{Res}_M \ker \operatorname{Ind}_S^G U$. However, it now follows from Lemma 2.22 that

$$\bigcap_{\gamma \in G_{\hat{p}(U)}} \gamma \cdot P = \bigcap_{\gamma \in G_{\hat{p}(V)}} \gamma \cdot Q$$

where $\widehat{p}$ is the canonical map from $C^*(S)^\wedge$ onto $S^{(0)}$. This implies that the closed sets in $\operatorname{Prim} C^*(S)$ associated to these ideals must be the same. Hence, $\overline{G \cdot P} = \overline{G \cdot Q}$. The reverse direction follows immediately from the fact that Φ is continuous and G-equivariant. $\square$

At this point we recall from [**9**] that a groupoid is said to be EH-regular if every primitive ideal is induced from an isotropy subgroup. That is, given $P \in \operatorname{Prim} C^*(G)$, there exist $u \in G^{(0)}$ and $Q \in \operatorname{Prim} C^*(S_u)$ such that $P = \operatorname{Ind}_{S_u}^G Q$. Of course, it follows from [**8**, Theorem 4.1] that regular groupoids are EH-regular. In the non-regular case the main result in [**9**, Theorem 2.1] states that, if a groupoid G is amenable in the sense of Renault [**1**], then G is EH-regular. This allows us to give the promised strengthening of Theorem 2.23. First, however, recall that the T_0*-ization* of a topological space X is the quotient space $X^{T_0} := X/\sim$ where $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$.

Theorem 3.5. *Suppose G is a locally compact Hausdorff groupoid and that the stabilizer subgroupoid S has a Haar system. If G is EH-regular, and in particular, if G is either amenable or regular, then the map $\Psi : \operatorname{Prim} C^*(S) \to \operatorname{Prim} C^*(G)$ defined by $\Psi(P) = \operatorname{Ind}_S^G P$ factors to a homeomorphism of* $\operatorname{Prim} C^*(G)$ *with* $(\operatorname{Prim} C^*(S)/G)^{T_0}$.

Proof. It follows from Corollary 3.3 that Ψ is continuous and open. Surjectivity clearly follows from the fact that G is EH-regular. Finally, it is straightforward to show that $\overline{G \cdot P} = \overline{G \cdot Q}$ in $\operatorname{Prim} C^*(S)$ if and only if $\overline{\{G \cdot P\}} = \overline{\{G \cdot Q\}}$ in $\operatorname{Prim} C^*(S)/G$. Thus, it follows from Lemma 3.4 that the factorization of Ψ to $(\operatorname{Prim} C^*(S)/G)^{T_0}$ is injective and is therefore a homeomorphism. □

Remark 3.6. In the case where S is abelian, Theorem 3.5 is particularly concrete because $\operatorname{Prim} C^*(S) = \widehat{S}$ is the dual bundle [**4**] associated to S.

As in Section 2, we get the following corollary, which in this case is a very slight extension of [**11**, Proposition 3.8].

Corollary 3.7. *If G is an* EH-*regular, principal groupoid, then* $\operatorname{Prim} C^*(G)$ *is homeomorphic to* $(G^{(0)}/G)^{T_0}$.

REFERENCES

1. Claire Anantharaman-Delaroche and Jean Renault, *Amenable groupoids*, Monographies de L'Enseignement Mathematique **36**, L'Enseignement Mathematique, 2000.

2. Jacques Dixmier, *C^*-algebras*, North-Holland Math. Library **15**, North-Holland Publishing Co., Amsterdam, 1977.

3. Siegfried Echterhoff, *Regularizations of twisted covariant systems and crossed products with continuous trace*, J. Funct. Anal. **116** (1993), 277–314.

4. Geoff Goehle, *Group bundle duality*, Illinois J. Math. **52** (2008), 951–956.

5. ———, *The Mackey machine for crossed products by regular groupoids.* I., Houston J. Math. **36** (2010), 567–590.

6. ———, *Groupoid crossed products*, 2009, arXiv:0905.4681v1.

7. Marius Ionescu and Dana P. Williams, *The generalized Effros-Hahn conjecture for groupoids*, Indiana Univ. Math. J., in press.

8. ———, *Irreducible representations of groupoid C^*-algebras*, Proc. Amer. Math. Soc. **137** (2009), 1323–1332.

9. Paul S. Muhly and Dana P. Williams, *Renault's equivalence theorem for groupoid crossed products*, New York J. Math. Mono. **3** (2008), 1–83.

10. Lisa Orloff Clark, *CCR and GCR groupoid C^*-algebras*, Indiana Univ. Math. J. **56** (2007), 2087–2110.

11. ———, *Classifying the type of principal groupoid C^*-algebras*, J. Operator Theory **57** (2007), 251–266.

12. Iain Raeburn and Dana P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Math. Surv. Mono. **60**, American Mathematical Society, Providence, RI, 1998.

13. Arlan Ramsay, *The Mackey-Glimm dichotomy for foliations and other polish groupoids*, J. Funct. Anal. **94** (1990), 358–374.

14. Jean Renault, *The ideal structure of groupoid crossed product C^*-algebras*, J. Operator Theory **25** (1991), 3–36.

15. Jean N. Renault, Paul S. Muhly and Dana P. Williams, *Continuous trace groupoid C^*-agebras*, III, Trans. Amer. Math. Soc. **348** (1996), 3621–3641.

16. Dana P. Williams, *The topology on the primitive ideal space of transformation group C^*-algebras and C.C.R. transformation group C^*-algebras*, Trans. Amer. Math. Soc. **266** (1981), 335–359.

17. ———, *Crossed products of C^*-algebras*, Math. Surv. Mono. **134**, American Mathematical Society, Providence, RI, 2007.

Mathematics and Computer Science Department, Stillwell 426, Western Carolina University, Cullowhee, NC 28723
Email address: goehle@gmail.com