

HARMONIC MAPS AND KALUZA-KLEIN METRICS ON SPHERES

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ABSTRACT. This article studies the harmonicity of vector fields on Riemannian manifolds, viewed as maps in the tangent bundle equipped with a family of Riemannian metrics. Geometric and topological rigidity conditions are obtained, especially for surfaces and vector fields of constant norm, and existence is proved on two-tori. Classifications are given for conformal, quadratic and Killing vector fields on spheres. Finally, the class of metric considered on the tangent bundle is enlarged, permitting new vector fields to become harmonic.

1. Introduction. Though very interesting in many settings, the theory of harmonic maps fails to produce any worthwhile result when applied to vector fields, seen as maps from a Riemannian manifold (M, g) into its tangent bundle TM equipped with its simplest metric, the Sasaki metric. This situation has led researchers to consider constrained problems on the same functional, e.g., harmonic sections and harmonic unit sections. However, recently, new classes of metrics on TM have been shown to allow a richer existence theory and, with respect to adequate metrics, standard vector fields can produce new harmonic maps, for example, a two-parameter family including the Sasaki metric [4], g -natural metrics [1] or an ad-hoc Riemannian metric based on a deformation of the horizontal distribution [17].

The main difficulty here is to strike a balance between the harmonicity of vector fields and the geometric relevance of the metric on TM . In this paper, given a Riemannian manifold (M, g) , we consider on TM Riemannian metrics in the intersection of the largest known class of metrics on tangent bundles, i.e., g -natural metrics and Kaluza-Klein metrics, as commonly defined on principal bundles (cf. [19]).

Recall that, at a point $(p, e) \in TM$, the tangent space $T_{(p,e)}TM$ splits into its horizontal and vertical spaces [8]:

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$$(1) \quad T_{(p,e)}TM = H_{(p,e)} \oplus V_{(p,e)},$$

where $V_{(p,e)}$ is the kernel of the differential of the canonical projection $\pi : TM \rightarrow M$ and $H_{(p,e)}$ is the kernel of the connecting map

$$K_{(p,e)} = K : T_{(p,e)}TM \longrightarrow T_pM, \quad K(V) = d(\exp_p \circ R_{-e} \circ \tau)(V),$$

where $\tau : U \subset TM \rightarrow T_pM$ sends a vector $v \in T_qM$, with $(q, v) \in U$, U being an open neighborhood of (p, e) in TM , by parallel transport along the unique geodesic from q to p , to a vector in T_pM and the map R_{-e} is simply the translation by $-e$ in T_pM . One can check that $H_{(p,e)} \cap V_{(p,e)} = \{0\}$ and $H_{(p,e)} \oplus V_{(p,e)} = T_{(p,e)}TM$. Any vector in $T_{(p,e)}TM$ can be decomposed into its horizontal and vertical parts and any vector $X \in T_pM$ admits a horizontal lift $X^h \in H_{(p,e)}$ and a vertical lift $X^v \in V_{(p,e)}$ defined by

$$K_{(p,e)}(X^v) = X, \quad d\pi_{(p,e)}(X^h) = X.$$

Metrics on TM can therefore be characterized by their values on horizontal and vertical lifts.

Definition 1.1 [3]. Let (M, g) be a Riemannian manifold. A metric G on TM will be called g -natural if, at the point $(p, e) \in TM$, it has the form

$$\begin{aligned} G(X^h, Y^h) &= A(|e|^2)g(X, Y) + D(|e|^2)g(X, e)g(e, Y); \\ G(X^h, Y^v) &= E(|e|^2)g(X, Y) + F(|e|^2)g(X, e)g(e, Y); \\ G(X^v, Y^v) &= B(|e|^2)g(X, Y) + C(|e|^2)g(X, e)g(e, Y), \end{aligned}$$

where A, B, C, D, E and F are real C^2 -functions of $|e|^2$. Conditions are needed on A, B, C and D to ensure that G is positive definite.

Remark 1.1. To have $\pi : TM \rightarrow M$ conformal submersion, we need $D = 0$, and to have the horizontal and the vertical distributions orthogonal one to the other, we need $E = F = 0$.

The class of metrics we study in this article sits inside g -natural metrics but retains the geometric properties of the tangent bundle.

Definition 1.2. Let (M, g) be a Riemannian manifold, a metric G on TM will be called Kaluza-Klein if, at the point $(p, e) \in TM$, it takes the form

$$\begin{aligned} G(X^h, Y^h) &= A(|e|^2)g(X, Y); \\ G(X^h, Y^v) &= 0; \\ G(X^v, Y^v) &= B(|e|^2)g(X, Y) + C(|e|^2)g(X, e)g(e, Y), \end{aligned}$$

where A , B and C are real functions of $|e|^2$. The functions A will be assumed strictly positive and B and C such that G is positive definite, i.e., $B(t) > 0$ and $B(t) + tC(t) > 0$ for all $t \geq 0$.

Standard computations with the Koszul formula or general results for g -natural metrics from [3] give the expression of the Levi-Civita connection.

Proposition 1.1. *Let G be a Kaluza-Klein metric on TM . Then the corresponding Levi-Civita connection $\bar{\nabla}$ is characterized, at $(p, e) \in TM$, by*

$$\begin{aligned} \bar{\nabla}_{X^h} Y^h &= (\nabla_X Y)^h - \frac{A'}{B + |e|^2 C} g(X, Y) e^v - \frac{1}{2} (R(X, Y) e)^v; \\ \bar{\nabla}_{X^h} Y^v &= \left(\frac{-B}{2A} R(Y, e) X + \frac{A'}{A} g(Y, e) X \right)^h + (\nabla_X Y)^v; \\ \bar{\nabla}_{X^v} Y^h &= \left(\frac{B}{2A} R(e, X) Y + \frac{A'}{A} g(X, e) Y \right)^h; \\ \bar{\nabla}_{X^v} Y^v &= \frac{B'}{B} (g(X, e) Y^v + g(Y, e) X^v) \\ &\quad + \left(C' - \frac{2B'C}{B} \right) \frac{1}{B + |e|^2 C} g(X, e) g(Y, e) e^v \\ &\quad + \frac{C - B'}{B + |e|^2 C} g(X, Y) e^v, \end{aligned}$$

all functions being evaluated at $|e|^2$, prime denotes derivation and $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$ is the curvature tensor of (M, g) .

Remark 1.2. Note that, since $\bar{\nabla}_{X^v} Y^v$ is vertical, the fibers of TM are totally geodesic.

The geometry of a sub-class of Kaluza-Klein metrics, called generalized Cheeger-Gromoll metrics, is studied in [6].

2. Harmonic maps. The energy of a smooth map $\phi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds is

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 v_g,$$

where $|d\phi|$ is the Hilbert-Schmidt norm of $d\phi$. If M is not compact, $E(\phi)$ is defined over compact subsets. Critical points of this functional are called harmonic maps and are characterized by the vanishing of the tension field [12]:

$$\tau(\phi) = \text{trace } \nabla d\phi = 0.$$

Harmonic maps generalize not only harmonic functions but also geodesics and holomorphic maps between Kähler manifolds. The starting point of the theory is the Eells-Sampson existence result.

Theorem 2.1 [12]. *Let (M, g) and (N, h) be compact manifolds with Riem^N negative. Then in each homotopy class there exists a harmonic map from (M, g) to (N, h) .*

Although it has developed into a rich subject (cf. [9–11]), this theory does not lend itself to the study of vector fields. Since the tangent bundle of an n -dimensional manifold M is itself a $2n$ -dimensional manifold, one can see vector fields as maps from M to TM , and once a Riemannian metric has been chosen on M , equip TM with a Riemannian metric of its own. Given the canonical decomposition (1) of the bitangent space and the isomorphisms between the horizontal and vertical spaces, and the tangent space of M , the simplest possible construction of a Riemannian metric on TM is the Sasaki metric:

$$\begin{aligned} g_{\text{Sasaki}}(X^h, Y^h) &= g(X, Y); \\ g_{\text{Sasaki}}(X^h, Y^v) &= 0; \\ g_{\text{Sasaki}}(X^v, Y^v) &= g(X, Y), \end{aligned}$$

for all vectors $X, Y \in T_p M$, $(p, e) \in TM$ and $p \in M$. Unfortunately, elementary computations based on Proposition 1.1 show that, for a

vector field $\sigma : M \rightarrow TM$, the vertical part of $\tau(\sigma)$ is

$$\nabla^* \nabla \sigma = 0,$$

and a mere integration by parts implies that σ must be parallel, with all the topological obstructions that this implies (cf. [13, 16]).

An alternative metric on TM was proposed by Cheeger and Gromoll [7] and explicated by Tricerri and Musso [15]

$$\begin{aligned} g_{\text{CG}}(X^h, Y^h) &= g(X, Y); \\ g_{\text{CG}}(X^h, Y^v) &= 0; \\ g_{\text{CG}}(X^v, Y^v) &= \omega(e)(g(X, Y) + g(X, e)g(Y, e)), \end{aligned}$$

where $\omega(e) = 1/(1 + |e|^2)$.

While this metric proved useful for other problems, it carries the same rigidity as the Sasaki metric and, when M is compact, no non-parallel harmonic section or map can exist for this metric [4, 17].

In a first attempt to relax existence conditions, this metric was generalized in [4] by introducing a two-parameter family of metrics, which includes g_{Sasaki} and g_{CG}

$$\begin{aligned} g_{m,r}(X^h, Y^h) &= g(X, Y); \\ g_{m,r}(X^h, Y^v) &= 0; \\ g_{m,r}(X^v, Y^v) &= \omega^m(e)(g(X, Y) + rg(X, e)g(Y, e)). \end{aligned}$$

Depending on the choice of (m, r) , r positive to ensure positive definiteness, one can obtain new harmonic maps from vector fields, e.g., the Hopf vector field from $(\mathbf{S}^3, g_{\text{can}})$ into $(T\mathbf{S}^3, g_{2,0})$. However, in some cases, for example \mathbf{S}^2 , rigidity persists and wider classes of metrics on TM are now investigated [1].

Vector fields allow a richer situation than the general case of maps, since the energy functional can define several variational problems. First, harmonic maps, that is, critical points of E with respect to all possible variations of the map; second, harmonic sections, i.e., critical points of E only with respect to variations through vector fields; and, finally, topology permitting, unit harmonic sections, when variations are restricted to unit vector fields. These problems clearly sit one

inside the other, and when the canonical projection $\pi : TM \rightarrow M$ is a Riemannian submersion, their associated Euler-Lagrange equations can be deduced from the tension field. Indeed, the characterizing equation of harmonic sections is precisely the vertical part of the tension field (hence the same rigidity for harmonic maps and sections for the Sasaki and Cheeger-Gromoll metrics), and, for unit harmonic sections, it is the proportionality of the vertical part of the tension field and the section itself.

Note that C.M. Wood extended in [19] Eells-Sampson's flow technique to deduce a similar existence result for sections into a fiber bundle equipped with a Kaluza-Klein type metric (which will be our case), but the homotopy of the space of vector fields is trivial.

Proposition 2.1. *Let (M, g) be an m -dimensional Riemannian manifold and G a Kaluza-Klein metric on TM . The tension field of a vector field $\sigma : (M, g) \rightarrow (TM, G)$ is given by $\tau(\sigma) = [\tau^h(\sigma)]^h + [\tau^v(\sigma)]^v$, where*

$$\begin{aligned}\tau^h(\sigma) &= \frac{-B}{A} R(\nabla_{e_i} \sigma, \sigma) e_i + \frac{2A'}{A} g(\nabla_{e_i} \sigma, \sigma) e_i; \\ \tau^v(\sigma) &= -\nabla^* \nabla \sigma + \frac{2B'}{B} \nabla_{X(\sigma)} \sigma + \frac{1}{B + |\sigma|^2 C} \\ &\quad \times \left(-mA' + \left(C' - \frac{2B'C}{B} \right) |X(\sigma)|^2 + (C - B') |\nabla \sigma|^2 \right) \sigma,\end{aligned}$$

where $\{e_i\}_{i=1, \dots, m}$ is a local orthonormal frame of (M, g) , R is its Riemann curvature tensor and $X(\sigma) = \text{grad } |\sigma|^2/2$.

Proof. Let $\sigma : M \rightarrow TM$ be a vector field. Then $d\sigma : TM \rightarrow TTM$ and, from the definition of the horizontal and vertical lifts,

$$d\sigma(X) = X^h + (\nabla_X \sigma)^v.$$

Combined with Proposition 1.1, this yields that, in a local orthonormal frame $\{e_i\}_{i=1, \dots, m}$,

$$\begin{aligned}
\tau(\sigma) &= \sum_{i=1}^m \bar{\nabla}_{d\sigma(e_i)} d\sigma(e_i) - d\sigma(\nabla_{e_i} e_i) \\
&= \sum_{i=1}^m (\nabla_{e_i} e_i)^h - \frac{A'}{B + |\sigma|^2 C} \sigma^v + (\nabla_{e_i} \nabla_{e_i} \sigma)^v \\
&\quad + \left(\frac{-B}{2A} R(\nabla_{e_i} \sigma, \sigma) e_i + \frac{A'}{A} g(\nabla_{e_i} \sigma, \sigma) e_i \right)^h \\
&\quad + \left(\frac{A'}{A} g(\nabla_{e_i} \sigma, \sigma) e_i + \frac{B}{2A} R(\sigma, \nabla_{e_i} \sigma) e_i \right)^h \\
&\quad + \frac{2B'}{B} g(\nabla_{e_i} \sigma, \sigma) (\nabla_{e_i} \sigma)^v + \left(C' - \frac{2B'C}{B} \right) \frac{g^2(\nabla_{e_i} \sigma, \sigma)}{B + |\sigma|^2 C} \sigma^v \\
&\quad + \frac{C - B'}{B + |\sigma|^2 C} g(\nabla_{e_i} \sigma, \nabla_{e_i} \sigma) \sigma^v - (\nabla_{e_i} e_i)^h - (\nabla_{\nabla_{e_i} e_i} \sigma)^v;
\end{aligned}$$

hence, the expression for $\tau(\sigma)$. \square

Remark 2.1. i) Parallel vector fields are harmonic maps if and only if $A' = 0$. Therefore, with the exception of the last section, we will assume A to be a constant function and harmonicity will then be independent of the value of A , so we can choose it to be equal to 1.

ii) With this choice, $A = 1$, the horizontal part satisfies $\tau^h(f\sigma) = f\tau^h(\sigma)$, for any function f . Moreover, harmonic sections are characterized by $\tau^v(\sigma) = 0$ and $\tau^h(\sigma)$ is independent of B and C .

iii) The canonical projection $\pi : (TM, G) \rightarrow (M, g)$ becomes a harmonic morphism, i.e., pulls back harmonic maps onto harmonic maps.

3. Rigidity conditions. As, when $A \equiv 1$, the horizontal part of $\tau(\sigma)$ depends only upon the geometry of (M, g) and not upon the choice of the functions B and C , we can find obstructions to the existence of non-trivial harmonic maps (i.e., $\tau(\sigma) = 0$).

Proposition 3.1. *If $A \equiv 1$ and (M, g) is a Riemannian manifold of constant sectional curvature κ , then*

$$\tau^h(\sigma) = -\kappa B(\nabla_\sigma \sigma - (\operatorname{div} \sigma)\sigma).$$

If M is compact with $\kappa \neq 0$ and σ is a harmonic map of constant length k , then $\operatorname{div} \sigma = \nabla_\sigma \sigma = 0$. Moreover,

- (1) if $|\nabla \sigma|^2 \leq \kappa k^2$, then σ is a Killing vector field;
- (2) if $-2\kappa k^2 \geq |L_\sigma g|^2$, then σ is parallel.

Proof. When (M, g) has constant sectional curvature, the expression of $\tau^h(\sigma)$ is a direct consequence of Proposition 2.1. If σ is a harmonic map of constant norm k , taking the inner product of the equality $(\operatorname{div} \sigma)\sigma = \nabla_\sigma \sigma$, gives $\sigma(k^2/2) = (\operatorname{div} \sigma)k^2$, so that $\operatorname{div} \sigma = 0$, hence $\nabla_\sigma \sigma = 0$.

If M is compact, we can use the Yano formula

$$\int_M \langle \nabla^* \nabla \sigma, \sigma \rangle - \operatorname{Ricci}(\sigma, \sigma) - \frac{1}{2} |L_\sigma g|^2 + (\operatorname{div} \sigma)^2 v_g = 0,$$

to deduce that

$$\int_M |\nabla \sigma|^2 - \kappa k^2 - \frac{1}{2} |L_\sigma g|^2 v_g = 0,$$

and obtain the second part of the proposition. \square

On surfaces, a topological obstruction appears.

Proposition 3.2. *Let (M^2, g) be a Riemannian surface with Gaussian curvature K_g . If a vector field is a harmonic map from M^2 to TM^2 equipped with a Kaluza-Klein metric with $A \equiv 1$, then it must vanish on the set $\Omega = \{x \in M^2 : K_g(x) \neq 0\}$.*

In particular, no non-zero vector field of \mathbf{S}^2 can be a harmonic map, whatever the metric chosen on \mathbf{S}^2 .

Proof. Let g be a Riemannian metric on M^2 , and let K_g be its Gaussian curvature. First recall that, if $\{X, Y\}$ is a local orthonormal

frame on M^2 , then

$$\begin{aligned}
 \operatorname{div} [\operatorname{div} (X)X - \nabla_X X] &= \langle \nabla_X (\langle \nabla_Y X, Y \rangle X - \nabla_X X), X \rangle \\
 &\quad + \langle \nabla_Y (\langle \nabla_Y X, Y \rangle X - \nabla_X X), Y \rangle \\
 &= \langle \nabla_X \nabla_Y X, Y \rangle + \langle \nabla_Y X, \nabla_X Y \rangle - \langle \nabla_X \nabla_X X, X \rangle \\
 &\quad + \langle \nabla_Y X, Y \rangle^2 - \langle \nabla_Y \nabla_X X, Y \rangle \\
 &= -[\langle \nabla_Y \nabla_X X, Y \rangle - \langle \nabla_X \nabla_Y X, Y \rangle - \langle \nabla_{\nabla_Y X} X, Y \rangle] \\
 &\quad - \langle \nabla_X \nabla_X X, X \rangle,
 \end{aligned}$$

but

$$\begin{aligned}
 -\langle \nabla_X \nabla_X X, X \rangle &= |\nabla_X X|^2 \\
 &= \langle \nabla_X X, Y \rangle^2 \\
 &= -\langle \nabla_X Y, X \rangle \langle \nabla_X X, Y \rangle \\
 &= -\langle \nabla_{\nabla_X Y} X, Y \rangle.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \operatorname{div} [\operatorname{div} (X)X - \nabla_X X] &= -[\langle \nabla_Y \nabla_X X, Y \rangle - \langle \nabla_X \nabla_Y X, Y \rangle \\
 (2) \quad &\quad - \langle \nabla_{\nabla_Y X} X, Y \rangle - \langle \nabla_{\nabla_X Y} X, Y \rangle] \\
 &= -K_g.
 \end{aligned}$$

Thus, if a vector field $\sigma : (M^2, g) \rightarrow (TM^2, G)$ is a harmonic map, then necessarily the horizontal part of its tension field must vanish:

$$K_g(\operatorname{div}(\sigma)\sigma - \nabla_\sigma \sigma) = 0.$$

Assume σ is non-zero, and let U be an open subset of $\Omega \subset M^2$ where σ does not vanish. First, observe that, on U , we have

$$\begin{aligned}
 \operatorname{div} \left(\frac{\sigma}{|\sigma|} \right) \frac{\sigma}{|\sigma|} - \nabla_{\sigma/|\sigma|} \frac{\sigma}{|\sigma|} &= \left(\left(\frac{1}{|\sigma|} \right) \operatorname{div} \sigma + \sigma \left(\frac{1}{|\sigma|} \right) \right) \frac{\sigma}{|\sigma|} \\
 &\quad - \frac{1}{|\sigma|^2} \nabla_\sigma \sigma - \frac{1}{|\sigma|} \sigma \left(\frac{1}{|\sigma|} \right) \sigma \\
 &= \frac{1}{|\sigma|^2} \left(\operatorname{div}(\sigma)\sigma - \nabla_\sigma \sigma \right).
 \end{aligned}$$

Thus, formula (2) applied to $\sigma/|\sigma|$ contradicts the hypothesis. \square

When M is compact, the Divergence theorem and the Kato inequality put constraints on the vanishing of the vertical portion of the tension field.

Proposition 3.3. *Let (M, g) be a compact Riemannian manifold and $\sigma : (M, g) \rightarrow (TM, G)$, where G is a Kaluza-Klein metric with $A \equiv 1$, a harmonic section, i.e., $\tau^v(\sigma) = 0$. If $2B' + |\sigma|^2 C' \geq 0$ and $-B + |\sigma|^2 B' + |\sigma|^4 C' \leq 0$, then the norm of σ is constant. If, moreover, $B + |\sigma|^2 B'$ does not vanish, then σ must be parallel.*

Proof. If $\sigma : (M, g) \rightarrow (TM, G)$ is a harmonic section, then

$$\begin{aligned} & \nabla^* \nabla \sigma - \frac{2B'}{B} \nabla_{X(\sigma)} \sigma \\ &= \frac{1}{B + |\sigma|^2 C} \left(-mA' + (C' - \frac{2B'C}{B}) |X(\sigma)|^2 + (C - B') |\nabla \sigma|^2 \right) \sigma, \end{aligned}$$

and taking the inner-product with σ implies

$$(3) \quad \Delta \frac{|\sigma|^2}{2} = \frac{1}{B + |\sigma|^2 C} \left(-(B + |\sigma|^2 B') |\nabla \sigma|^2 + (2B' + |\sigma|^2 C') |X(\sigma)|^2 \right).$$

If $2B' + |\sigma|^2 C'$ is positive, then we can use the inequality $|X(\sigma)|^2 \leq |\nabla \sigma|^2 |\sigma|^2$ to obtain

$$\Delta \frac{|\sigma|^2}{2} \leq \frac{1}{B + |\sigma|^2 C} (-B + |\sigma|^2 B' + |\sigma|^4 C') |\nabla \sigma|^2,$$

and the second condition of the proposition implies that

$$\Delta |\sigma|^2 \leq 0;$$

therefore, $|\sigma|$ is constant and the left-hand side of equation (3) vanishes. \square

Proposition 3.4. *Let (M, g) be a complete Riemannian manifold with positive Ricci curvature. If $\sigma : (M, g) \rightarrow (TM, G)$, where G is a Kaluza-Klein metric with $A \equiv 1$, is a harmonic map and $B + |\sigma|^2 B' = 2B' + |\sigma|^2 C' = 0$, then σ has constant norm.*

Proof. Clearly, from the previous proof, $\Delta|\sigma|^2 = 0$ and, by [21], we conclude that $|\sigma|$ is constant. \square

Example 3.1. (1) The functions $B(t) = C(t) = K > 0$ or $B(t) = (2t + 1)/(t + 1)$ and $C(t) = K > 0$ are examples of functions satisfying the conditions of Proposition 3.3.

(2) If the norm of the section σ is bounded away from zero, the functions $B(t) = K/t$ and $C(t) = -(K/t^2) + L$, with $K > 0$ and $L > 0$, satisfy the conditions of Proposition 3.4 and can be extended over the whole of \mathbf{R} .

4. Constant norm. Vector fields of constant norm usually do not exist, but, when they do, they provide particularly interesting examples and are often linked to other geometric structures. Building on the formulas of Proposition 2.1, we can rule out their harmonicity for some combinations of the functions B and C .

Proposition 4.1. *If a vector field $\sigma : (M, g) \rightarrow (TM, G)$ of constant norm k is harmonic (either as a section or a map), from a Riemannian manifold into its tangent bundle equipped with a Kaluza-Klein metric (with $A \equiv 1$) then either σ is parallel or*

$$B(k^2) + k^2 B'(k^2) = 0.$$

Proof. If σ has constant norm k , then the vertical part of its tension field becomes

$$(4) \quad \nabla^* \nabla \sigma = \frac{C - B'}{B + |\sigma|^2 C} |\nabla \sigma|^2 \sigma,$$

and taking the inner-product with σ yields

$$|\nabla \sigma|^2 = \frac{C - B'}{B + |\sigma|^2 C} k^2 |\nabla \sigma|^2.$$

If σ is not parallel, then

$$B(k^2) + k^2 B'(k^2) = 0. \quad \square$$

This proposition, which can also be derived from [1, formula (4.23)], leads to a rewriting of the condition of harmonic section.

Proposition 4.2. *If $\sigma : M \rightarrow TM$ is a vector field of constant norm k , then $\sigma : (M, g) \rightarrow (TM, G)$ is a harmonic section (i.e., $\tau^v(\sigma) = 0$), where G is a Kaluza-Klein metric for the functions $A \equiv 1$ and $B(t) = Ke^{-t/k^2}$ ($K > 0$) and any choice for C , if and only if the unit section σ/k is a unit harmonic section for the Sasaki metric on TM .*

Proof. If σ is harmonic but not parallel, by the previous proposition, function B must satisfy

$$B(k^2) + k^2 B'(k^2) = 0,$$

which is satisfied for $B(t) = Ke^{-t/k^2}$ (though this is far from the only possibility), and equation (4) becomes

$$\nabla^* \nabla \sigma = \frac{1}{k^2} |\nabla \sigma|^2 \sigma,$$

that is, $\alpha = \sigma/k$ is a unit harmonic section with respect to the Sasaki metric. \square

Proposition 4.3. *The characteristic vector field of a Sasakian manifold (M, g) is a harmonic section into (TM, G) , where G is a Kaluza-Klein metric with $A \equiv 1$ and $B(t) = Ke^{-t}$.*

Remark 4.1. (1) As we have already remarked, when $A \equiv 1$, the vanishing of the horizontal part of the tension field is independent of the choice of the Kaluza-Klein metric on TM . For example, the Hopf vector fields on \mathbf{S}^{2n+1} or some unit Killing vector fields on $\widetilde{SL_2(\mathbf{R})}$ or the Heisenberg three-space, turn out to also be harmonic maps (cf. [5]).

(2) The last two propositions can also be obtained as consequences of [1, Theorem 10 and Theorem 13].

5. The two-torus. Having established in Proposition 4.2 a link between unit harmonic sections for the Sasaki metric and harmonic

sections for some Kaluza-Klein metrics, we can exploit results from Wiegink [18] to obtain a first existence result.

Theorem 5.1. *For any metric g on the two-torus \mathbf{T}^2 , there exist a unitary vector field σ and a Kaluza-Klein metric G , with $A \equiv 1$, such that $\sigma : (\mathbf{T}^2, g) \rightarrow (T\mathbf{T}^2, G)$ is a harmonic section. Moreover, σ is a harmonic map if and only if the Gaussian curvature of (\mathbf{T}^2, g) vanishes.*

Proof. Let σ be a unit vector field on \mathbf{T}^2 and u a function on \mathbf{T}^2 , put $\tilde{g} = e^{-u}g$ and $\tilde{\sigma} = e^{-u}\sigma$. Then $|\tilde{\sigma}|_{\tilde{g}}^2 = 1$ and the energies of these two vector fields are (up to a constant)

$$E(\sigma) = \frac{1}{2} \int_M B(|\sigma|_g^2) |\nabla \sigma|_g^2 v_g, \quad E(\tilde{\sigma}) = \frac{1}{2} \int_M B(|\sigma|_{\tilde{g}}^2) |\tilde{\nabla} \tilde{\sigma}|_{\tilde{g}}^2 v_{\tilde{g}},$$

since $g(\nabla_{e_i} \sigma, \sigma) = \tilde{g}(\tilde{\nabla}_{\tilde{e}_i} \tilde{\sigma}, \tilde{\sigma}) = 0$, where $\{e_i\}$ and $\{\tilde{e}_i\}$ are orthonormal frames with respect to g and \tilde{g} . Note that $v_{\tilde{g}} = e^{2u}v_g$ and

$$|\tilde{\nabla} \sigma|_{\tilde{g}}^2 = e^{-2u} (|\nabla \sigma|_g^2 + |\text{grad}_g u|_g^2 + 2((\text{div } \sigma)\sigma - \nabla_\sigma \sigma)(u)).$$

Therefore,

$$\begin{aligned} (5) \quad E(\tilde{\sigma}) - E(\sigma) &= \frac{1}{2} \int_M B(1) (|\text{grad}_g u|_g^2 + 2((\text{div } \sigma)\sigma - \nabla_\sigma \sigma)(u)) v_g, \end{aligned}$$

but

$$\begin{aligned} \text{div } (u((\text{div } \sigma)\sigma - \nabla_\sigma \sigma)) &= u \text{div } ((\text{div } \sigma)\sigma - \nabla_\sigma \sigma) + ((\text{div } \sigma)\sigma - \nabla_\sigma \sigma)(u) \\ &= -uK_g + ((\text{div } \sigma)\sigma - \nabla_\sigma \sigma)(u), \end{aligned}$$

since

$$\text{div } ((\text{div } \sigma)\sigma - \nabla_\sigma \sigma) = -K_g,$$

as we saw in the proof of Proposition 3.2. Besides, by the Divergence theorem, the second integral of (5) vanishes and we conclude that the

quantity $E(\tilde{\sigma}) - E(\sigma)$ depends only upon the function u , not upon the section σ .

Now, given a Riemannian metric g on \mathbf{T}^2 , choose a flat metric \tilde{g} conformal to g . This is always possible by [14], and, for this flat metric, take a parallel vector field $\tilde{\sigma}$. Then $\tilde{\sigma}$ is not only a harmonic section but also an absolute minimizer of the energy functional. From our relation on the energies of $\tilde{\sigma}$ and σ , we deduce that σ must be a unit harmonic section (since equation (5) holds only for unit sections), and we know that this implies that σ is a harmonic section from (\mathbf{T}^2, g) into $(T\mathbf{T}^2, G)$, with the function $B(t) = Ke^{-t}$ and any function C . \square

Proposition 5.1. *Let ξ be a Killing vector field on (\mathbf{T}^2, g) . Then $\sigma = \xi/|\xi|$ is a harmonic section from (\mathbf{T}^2, g) into $(T\mathbf{T}^2, G)$ (or from an open subset of \mathbf{T}^2 if ξ vanishes at some points), where G is Kaluza-Klein metric with the functions $A \equiv 1$ and $B(t) = Ke^{-t}$ and any function C .*

Proof. Recall that the condition on σ is $\nabla^*\nabla\sigma = |\nabla\sigma|^2\sigma$. Since

$$\begin{aligned}\nabla\frac{1}{|\xi|} &= -\frac{\nabla|\xi|}{|\xi|^2}; & \Delta\frac{1}{|\xi|} &= -\frac{\Delta|\xi|}{|\xi|^2} - 2\frac{|\nabla|\xi||^2}{|\xi|^3}; \\ \nabla^*\nabla\sigma &= K_g\frac{\xi}{|\xi|} + \Delta\left(\frac{1}{|\xi|}\right)\xi - 2\nabla_{\nabla(1/|\xi|)}\xi \\ &= K_g\frac{\xi}{|\xi|} + \left(-\frac{\Delta|\xi|}{|\xi|^2} - 2\frac{|\nabla|\xi||^2}{|\xi|^3}\right)\xi + \frac{2}{|\xi|^2}\nabla_{\nabla|\xi|}\xi;\end{aligned}$$

and

$$|\nabla\sigma|^2 = \frac{|\nabla\xi|^2}{|\xi|^2} + \frac{|\nabla|\xi||^2}{|\xi|^2} - \frac{2}{|\xi|^3}\langle\xi, \nabla_{\nabla|\xi|}\xi\rangle,$$

we have

$$\begin{aligned}\nabla^*\nabla\sigma - |\nabla\sigma|^2\sigma &= K_g\frac{\xi}{|\xi|} - \frac{\Delta|\xi|}{|\xi|^2}\xi - 2\frac{|\nabla|\xi||^2}{|\xi|^3}\xi + \frac{2}{|\xi|^2}\nabla_{\nabla|\xi|}\xi \\ &\quad - \frac{|\nabla\xi|^2}{|\xi|^2}\frac{\xi}{|\xi|} - \frac{|\nabla|\xi||^2}{|\xi|^2}\frac{\xi}{|\xi|} + \frac{2}{|\xi|^3}\langle\xi, \nabla_{\nabla|\xi|}\xi\rangle\frac{\xi}{|\xi|} \\ &= -2\frac{|\nabla|\xi||^2}{|\xi|^3}\xi + \frac{2}{|\xi|^2}\nabla_{\nabla|\xi|}\xi\end{aligned}$$

$$\begin{aligned}
& -2 \frac{|\nabla|\xi||^2}{|\xi|^2} \frac{\xi}{|\xi|} + \frac{2}{|\xi|^3} \langle \xi, \nabla_{\nabla|\xi|}\xi \rangle \frac{\xi}{|\xi|} \\
& = \frac{2}{|\xi|^3} (\nabla_{\nabla(|\xi|^2/2)}\xi - |\nabla|\xi||^2 \xi),
\end{aligned}$$

since

$$K_g|\xi|^2 = \langle \nabla^* \nabla \xi, \xi \rangle = |\nabla \xi|^2 + \Delta \frac{|\xi|^2}{2},$$

and

$$\frac{2}{|\xi|^3} \langle \xi, \nabla_{\nabla|\xi|}\xi \rangle \frac{\xi}{|\xi|} = \frac{2}{|\xi|^3} (\nabla|\xi|) \left(\frac{|\xi|^2}{2} \right) \frac{\xi}{|\xi|} = \frac{2}{|\xi|^3} |\nabla|\xi||^2 \xi.$$

Finally, since ξ is Killing, $\langle \nabla_\xi \xi, \xi \rangle = 0$ and $\nabla(|\xi|^2/2) = -\nabla_\xi \xi$, so $\nabla_{\nabla_\xi \xi} \xi$ is collinear with ξ , that is, $-\nabla_{\nabla_\xi \xi} \xi = f\xi$, but

$$\langle -\nabla_{\nabla_\xi \xi} \xi, \xi \rangle = |\nabla_\xi \xi|^2 = \left| \nabla \frac{|\xi|^2}{2} \right|^2 = |\xi|^2 |\nabla|\xi||^2;$$

hence, if ξ is Killing, then σ is harmonic. \square

Remark 5.1. This proposition remains true on any surface.

6. Spherical vector fields. The easiest scheme to construct vector fields on a Riemannian manifold is to take the gradient of a non-constant function. In the case of spheres, the simplest choice is a linear map given by the ambient inner-product. Unfortunately, these conformal vector fields cannot be harmonic sections (or a fortiori harmonic maps) for a Kaluza-Klein metric on $T\mathbf{S}^n$.

Proposition 6.1. *Let $a \in \mathbf{R}^{n+1} \setminus \{0\}$, $n \geq 2$, and define the function*

$$\lambda : \mathbf{S}^n \longrightarrow \mathbf{R}, \quad x \longmapsto \lambda(x) = \langle a, x \rangle.$$

Then the vector field $\sigma = \text{grad}^{\mathbf{S}^n} \lambda$ is never a harmonic section (or harmonic map) for any Kaluza-Klein metric on $T\mathbf{S}^n$ with $A \equiv 1$.

Proof. If $\sigma = \text{grad}^{\mathbf{S}^n} \lambda$ with $\lambda(x) = \langle a, x \rangle$, then standard computations show that ([20])

$$\begin{aligned}
\nabla^* \nabla \sigma &= \sigma; & X(\sigma) &= -\lambda \sigma; & \nabla_{X(\sigma)} \sigma &= \lambda^2 \sigma; \\
|\sigma|^2 &= |a|^2 - \lambda^2; & |X(\sigma)|^2 &= \lambda^2(|a|^2 - \lambda^2); & |\nabla \sigma|^2 &= n\lambda^2.
\end{aligned}$$

From these terms, one can easily check that $\tau^v(\sigma) = 0$ if and only if

$$B - \lambda^2(2 - n)B' = \lambda^2(|a|^2 - \lambda^2)C' + [n\lambda^2 - (|a|^2 - \lambda^2)]C.$$

But since there always exists a point $p \in \mathbf{S}^n$ such that $\lambda(p) = 0$, at this point the equation becomes

$$B(|a|^2) = -|a|^2 C(|a|^2),$$

which is impossible since this would contradict the positive definiteness of metric G . \square

While linear maps never yield harmonic maps on \mathbf{S}^n , quadratic functions restrained to the sphere can have harmonic gradient, depending upon the parity of the dimension.

Theorem 6.1. *Let λ be a quadratic form restricted to the n -sphere ($n \geq 2$) and $\sigma = \text{grad}^{\mathbf{S}^n} \lambda / 2$ its associated vector field.*

(i) *If n is even, then σ can never be a harmonic section from $(\mathbf{S}^n, g_{\text{can}})$ into $T\mathbf{S}^n$ with a Kaluza-Klein metric (with $A \equiv 1$).*

(ii) *If n is odd, σ is harmonic if and only if λ has exactly two distinct eigenvalues $a_1 > a_2$ of multiplicity $(n + 1)/2$ and $n > 3$.*

In this case, possible choices for the Kaluza-Klein metric (with $A \equiv 1$) on $T\mathbf{S}^n$ are, setting $\mu = a_1 - a_2$:

- *For $n = 5$, $C = 0$ and $B(t) = K e^{(-8t)/\mu^2}$, $K > 0$;*
- *For $n > 5$, $C = 0$ and (a prolongation of)*

$$B(t) = K \left[\frac{n-3}{2} \mu^2 - (n-5)t \right]^{(n+3)/(n-5)}, \quad K > 0,$$

for $t \in [0, (\mu^2/4)]$.

Proof. Let λ be the restriction to \mathbf{S}^n of a quadratic form on \mathbf{R}^{n+1} . Since it is symmetric, the matrix of λ can be diagonalized into a matrix M and the vector field $\sigma = \nabla^{\mathbf{S}^n} \lambda / 2$ can be written

$$\sigma(x) = M(x) - \lambda(x)x,$$

for all $x \in \mathbf{S}^n$.

Straightforward computations show that (cf. [5])

$$\begin{aligned}\nabla^* \nabla \sigma &= (n+3)\sigma; & \nabla_{X(\sigma)} \sigma &= (\sigma_3 - 3\lambda\sigma_2) + (4\lambda^2 - \lambda_2)\sigma; \\ |\sigma|^2 &= \lambda_2 - \lambda^2; & X(\sigma) &= \sigma_2 - 2\lambda\sigma;\end{aligned}$$

$$|\nabla \sigma|^2 = |M|^2 - 2\lambda_2 - 2\lambda \operatorname{trace} M + (n+3)\lambda^2$$

where, for any $k \in \mathbf{N}^*$,

$$\lambda_k(x) = M^k(x) \cdot x, \quad \text{and} \quad \sigma_k(x) = \frac{1}{2} \nabla \lambda_k = M^k(x) - \lambda_k(x)x,$$

where \cdot denotes the Euclidean scalar product in \mathbf{R}^{n+1} , and thus $\lambda_1 = \lambda$, $\sigma_1 = \sigma$. Inspection of $\tau^v(\sigma)$ quickly reveals that a necessary condition for σ to be a harmonic section is $\nabla_{X(\sigma)} \sigma$ parallel to σ , which implies that M has exactly two distinct eigenvalues (cf. [5, Lemma 4.1]). Since a quadratic form with a single eigenvalue restricts to a constant function on \mathbf{S}^n , we can assume the first eigenvalue of M , say μ , to be strictly positive (of multiplicity p), and the other one to be zero. In this case, if $x = x_\mu + x_0$ is the decomposition into eigenvectors, then ([5])

$$\sigma_2 = \mu\sigma; \quad \lambda(x) = \mu|x_\mu|^2; \quad \lambda_2 = \mu^2|x_\mu|^2,$$

and

$$\sigma_3 - 3\lambda\sigma_2 = (\mu^2 - 3\lambda\mu)\sigma = \mu^2(1 - 3|x_\mu|^2)\sigma.$$

Hence,

$$\begin{aligned}\nabla_{X(\sigma)} \sigma &= (\mu^2 - 4\mu^2|x_\mu|^2(1 - |x_\mu|^2))\sigma; & |\sigma|^2 &= \mu^2|x_\mu|^2(1 - |x_\mu|^2); \\ |X(\sigma)|^2 &= \mu^2(1 - 4|x_\mu|^2(1 - |x_\mu|^2))|\sigma|^2; \\ |\nabla \sigma|^2 &= p\mu^2 - 2\mu^2(p+1)|x_\mu|^2 + (n+3)\mu^2|x_\mu|^4.\end{aligned}$$

Therefore, $\tau^v(\sigma) = 0$ if and only if

$$\begin{aligned}(6) \quad & (n+3)B - [2(\mu^2 - 4|\sigma|^2) \\ & \quad - (p - 2(p+1)|x_\mu|^2 + (n+3)|x_\mu|^4)\mu^2]B' \\ & = |\sigma|^2(\mu^2 - 4|\sigma|^2)C' + [(p - 2(p+1)|x_\mu|^2 + (n+3)|x_\mu|^4)\mu^2 \\ & \quad - (n+3)|\sigma|^2]C.\end{aligned}$$

To a point $x = x_\mu + x_0 \in \mathbf{S}^n$, we associate the circle made up of points $\tilde{x} = \tilde{x}_\mu + \tilde{x}_0 \in \mathbf{S}^n$ such that $|\tilde{x}_0|^2 = |x_\mu|^2$, or equivalently $|\tilde{x}_\mu|^2 = 1 - |x_\mu|^2$. Note that $|\sigma_{\tilde{x}}|^2 = |\sigma_x|^2$. We evaluate equation (6) at x and \tilde{x} , and subtract to obtain

$$(n+1-2p)(2|x_\mu|^2-1)(B'-C)=0, \quad \text{for all } x \in \mathbf{S}^n.$$

If $2p \neq n+1$, by continuity, this implies that $B' = C$. But then, equation (6) would become

$$(n+3)B + [-2\mu^2 + (n+1)|\sigma|^2]C = |\sigma|^2(\mu^2 - 4|\sigma|^2)B'',$$

and at points where $|\sigma|^2 = \mu^2/4$, this is

$$B(\mu^2/4) + (\mu^2/4)C(\mu^2/4) = 0,$$

which contradicts the positive definiteness of G .

Therefore, necessarily $p = (n+1)/2$, which forces n to be odd and equation (6) becomes

$$\begin{aligned} (7) \quad (n+3)B + \left[\frac{n-3}{2}\mu^2 - (n-5)|\sigma|^2 \right] B' \\ = |\sigma|^2(\mu^2 - 4|\sigma|^2)C' + \left[\frac{n+1}{2}|\mu|^2 - 2(n+3)|\sigma|^2 \right] C. \end{aligned}$$

Three cases appear: $n=3$, $n=5$ and $n>5$.

For $n=3$, equation (7) becomes

$$6B + 2|\sigma|^2 B' = |\sigma|^2(\mu^2 - 4|\sigma|^2)C' + 2(\mu^2 - 6|\sigma|^2)C,$$

which can be written as an ordinary differential equation in the variable $t = |\sigma|^2 \in [0, \mu^2/4]$:

$$3B(t) + tB'(t) = \frac{t}{2}(\mu^2 - 4t)C'(t) + (\mu^2 - 6t)C(t),$$

with $t \in [0, \mu^2/4]$. The homogeneous equation $3B(t) + tB'(t) = 0$ has $B(t) = K/t^3$ ($K \in \mathbf{R}$) as a solution, but it is not defined at $t=0$. If $B(t) = K(t)/t^3$ is a solution, then function K must satisfy

$$K'(t) = t \left[\frac{t^2}{2}(\mu^2 - 4t)C(t) \right]',$$

with $K(0) = 0$. Therefore,

$$\begin{aligned} \frac{t^2}{2}(\mu^2 - 4t)C(t) &= \int_0^t \frac{K'(s)}{s} ds \\ &= \frac{K(t)}{t} + \int_0^t \frac{K(s)}{s^2} ds \\ &\geq \frac{K(t)}{t} = t^2 B(t); \end{aligned}$$

hence $(\mu^2 - 4t)C(t) \geq 2B(t)$, which, for $t = \mu^2/4$, contradicts B non-negative.

If $n = 5$, equation (7) can also be written as an ordinary differential equation in $t = |\sigma|^2$

$$(8) \quad 8B + \mu^2 B' = t(\mu^2 - 4t)C' + (3\mu^2 - 16t)C,$$

which is satisfied for $C = 0$ and $B(t) = Ke^{(-8t)/\mu^2}$ ($K > 0$). For more solutions, start with $B(t) = K(t)e^{(-8t)/\mu^2}$ and then K must satisfy

$$K'(t) = \frac{1}{\mu^2}(t(\mu^2 - 4t)C' + (3\mu^2 - 16t)C)e^{(8t)/\mu^2},$$

for $t \in [0, (\mu^2/4)]$, and, for any C positive (to ensure that the metric is Riemannian), we can construct a positive solution to (8).

For $n > 5$, if we take $C = 0$, the ODE becomes

$$(n+3)B = \left[(n-5)t - \frac{n-3}{2}\mu^2\right]B',$$

so we have the solution

$$B(t) = K \left[\frac{n-3}{2}\mu^2 - (n-5)t \right]^{(n+3)/(n-5)}, \quad K > 0,$$

for $t \in [0, (\mu^2/4)]$ and we prolong B to the whole of \mathbf{R} by continuity (keeping B strictly positive). To obtain more solutions, put $B(t) = K(t)f(t)$, where $f(t) = [(n-3/2)\mu^2 - (n-5)t]^{(n+3)/(n-5)}$. Then equation (7) gives the condition

$$\begin{aligned} K'(t) &= \left[t(\mu^2 - 4t)C' + \left(\frac{n+1}{2}\mu^2 - 2(n+3)t \right) C \right] \\ &\quad \times \left[\frac{n-3}{2}\mu^2 - (n-5)t \right]^{(-2(n-1))/(n-5)}, \end{aligned}$$

for $t \in [0, (\mu^2/4)]$, and we can find a primitive to construct a positive solution on $[0, (\mu^2/4)]$ and extend it to \mathbf{R}^+ . \square

Killing vector fields are one of the most geometrically meaningful types of sections of the tangent bundle. While we can again observe different behavior according to the parity of the dimension, harmonic Killing vector fields will exist in all cases, except dimension two.

Definition 6.1. Let ξ be a Killing vector field on \mathbf{S}^n . Since the set of Killing vector fields is a Lie algebra isomorphic to $\mathfrak{so}(n+1)$, there exists an antisymmetric matrix A such that $\xi(x) = Ax$. We define its invariant sub-space to be

$$F_\xi = \ker A.$$

Recall that, from the diagonalization process of matrix A over the complex numbers, there exists a $Q \in O(n+1)$ such that

$${}^tQAQ = \text{diag}(\theta_2 J, \theta_4 J, \dots, \theta_{2s} J, 0, \dots, 0),$$

where $\theta_2, \theta_4, \dots, \theta_{2s}$ are non-zero real numbers, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $2s = (n+1) - \dim F_\xi$. Therefore, for any Killing vector field on \mathbf{S}^n , a choice of orthonormal coordinates (x_1, \dots, x_{n+1}) exists on \mathbf{R}^{n+1} such that

$$\xi = \sum_{i=1}^s \theta_{2i} \sigma_{2i},$$

where σ_{2i} is the Killing vector field given by rotation in the (x_{2i-1}, x_{2i}) -plane, i.e.,

$$\sigma_{2i}(x_1, \dots, x_{n+1}) = (0, \dots, 0, -x_{2i}, x_{2i-1}, 0, \dots, 0).$$

Theorem 6.2. *Let ξ be a Killing vector field on \mathbf{S}^{2p} . Then the dimension of its invariant sub-space is $\dim F_\xi = 2k + 1$, with $0 \leq k \leq p - 1$, and ξ is a harmonic section into $T\mathbf{S}^{2p}$ equipped with a Kaluza-Klein metric G (with $A \equiv 1$) if and only if*

- (1) $k \neq p - 1$, i.e., $\dim F_\xi$ is not maximal;

(2) $\xi = \lambda(-x_2, x_1, \dots, -x_{2p-2k}, x_{2p-2k-1}, 0, \dots, 0)$, λ a non-zero constant.

One possible choice for the functions defining the metric G is $C = 0$ and $B(t) = Ke^{-(2p-1)t/(2\lambda^2(p-1-k))}$, $K > 0$.

Proof. Let ξ be a Killing vector field on the even-dimensional sphere \mathbf{S}^{2p} ; then its invariant sub-space F_ξ has dimension $2k+1$ with $0 \leq k \leq p-1$. In suitable coordinates, the Killing vector field ξ can be written $\sum_{i=1}^{p-k} \theta_{2i} \sigma_{2i}$, where the θ_{2i} 's are non-zero constants and σ_{2i} is the Killing vector field given by rotation in the (x_{2i-1}, x_{2i}) -plane. For Killing vector fields on \mathbf{S}^{2p} , the equation of harmonic sections is

$$(9) \quad (2p-1)\xi + \frac{2B'}{B} \nabla_{\nabla_\xi \xi} \xi \\ = \frac{1}{B + |\xi|^2 C} \left[(C - B') |\nabla \xi|^2 + \left(C' - \frac{2B'C}{B} \right) |\nabla_\xi \xi|^2 \right] \xi,$$

since $\nabla_\xi \xi = -\nabla|\xi|^2/2$ and $\nabla^* \nabla \xi = (2p-1)\xi$.

First assume that $\theta_{2i} = \lambda$ for all $i = 1, \dots, p-k$. Then $\xi = \lambda\sigma$, with $\sigma = \sum_{i=1}^{p-k} \sigma_{2i}$, and direct computations show that $|\sigma|^2 = \sum_{i=1}^{2p-2k} x_i^2$ and

$$\nabla_\sigma \sigma = ((|\sigma|^2 - 1)x_1, \dots, (|\sigma|^2 - 1)x_{2p-2k}, |\sigma|^2 x_{2p-2k+1}, \dots, |\sigma|^2 x_{2p+1}),$$

so

$$|\nabla_\sigma \sigma|^2 = |\sigma|^2(1 - |\sigma|^2) \quad \text{and} \quad \nabla_{\nabla_\sigma \sigma} \sigma = (|\sigma|^2 - 1)\sigma.$$

Moreover, since

$$|\nabla \sigma|^2 = (2p-1)|\sigma|^2 - (1/2)\Delta|\sigma|^2$$

and

$$\begin{aligned} \Delta|\sigma|^2 &= \Delta \mathbf{R}^{2n+2} |\sigma|^2 + 2(1+2p)|\sigma|^2 \\ &= -4(p-k) + 2(1+2p)|\sigma|^2, \end{aligned}$$

we deduce that $|\nabla\sigma|^2 = 2(p - k - |\sigma|^2)$. Hence

$$\begin{aligned}\nabla_{\nabla_\xi\xi}\xi &= \lambda^3(|\sigma|^2 - 1)\sigma; & |\nabla_\xi\xi|^2 &= \lambda^4|\sigma|^2(1 - |\sigma|^2); \\ |\nabla\xi|^2 &= 2\lambda^2(p - k - |\sigma|^2),\end{aligned}$$

so equation (9) becomes

$$\begin{aligned}(10) \quad 2\lambda^2(p - k - 1)B' + (2p - 1)B \\ = \lambda^2[2(p - k) - (2p + 1)|\sigma|^2]C + \lambda^4|\sigma|^2(1 - |\sigma|^2)C' .\end{aligned}$$

If $k \neq p - 1$, then we can choose $C = 0$ and $B(t) = Ke^{-(2p-1)t/(2\lambda^2(p-1-k))}$, $K > 0$. For any positive function C , one can find a positive solution of (10) over the interval $[0, \lambda^2]$ and extend over to \mathbf{R}^+ , to obtain a suitable function B .

If $k = p - 1$, equation (10) becomes

$$(2p - 1)B = \lambda^4|\sigma|^2(1 - |\sigma|^2)C' + \lambda^2(2 - (2p + 1)|\sigma|^2)C,$$

and $\xi = \lambda(-x_2, x_1, 0, \dots, 0)$. Evaluated at the point $p_1 = (1, 0, \dots, 0)$, this equation becomes

$$(2p - 1)[B(|\xi(p_1)|^2) + |\xi(p_1)|^2 C(|\xi(p_1)|^2)] = 0,$$

which contradicts the fact that G is Riemannian.

Suppose now that there exist i and j such that $\theta_{2i} \neq \theta_{2j}$. Then

$$\xi = (-\theta_2 x_2, \theta_2 x_1, \dots, -\theta_{2p-2k} x_{2p-2k}, \theta_{2p-2k} x_{2p-2k-1}, 0, \dots, 0),$$

and straightforward computations show that $\nabla_{\sigma_{2i}}\sigma_{2j} = 0$ when $i \neq j$ and

$$\begin{aligned}\nabla_{\sigma_{2i}}\sigma_{2i} &= (|\sigma_{2i}|^2 x_1, \dots, |\sigma_{2i}|^2 x_{2i-2}, \\ &(|\sigma_{2i}|^2 - 1)x_{2i-1}, (|\sigma_{2i}|^2 - 1)x_{2i}, |\sigma_{2i}|^2 x_{2i+1}, \dots, |\sigma_{2i}|^2 x_{2p+1}),\end{aligned}$$

so

$$\begin{aligned}\nabla_\xi\xi &= \sum_{i=1}^{p-k} \theta_{2i}^2 \nabla_{\sigma_{2i}}\sigma_{2i} \\ &= ((|\xi|^2 - \theta_2^2)x_1, \dots, (|\xi|^2 - \theta_{2p-2k}^2)x_{2p-2k}, \\ &\quad |\xi|^2 x_{2p+1-2k}, \dots, |\xi|^2 x_{2p+1}),\end{aligned}$$

and

$$|\nabla_\xi \xi|^2 = -|\xi|^4 + \sum_{i=1}^{p-k} \theta_{2i}^4 |\sigma_{2i}|^2.$$

From the expression of $\nabla_\xi \xi$, one can also deduce that

$$\begin{aligned} \nabla_{\nabla_\xi \xi} \xi &= \theta_2(|\xi|^2 - \theta_2^2) \sigma_2 + \theta_4(|\xi|^2 - \theta_4^2) \sigma_4 + \cdots \\ &\quad + \theta_{2p-2k}(|\xi|^2 - \theta_{2p-2k}^2) \sigma_{2p-2k} \\ &= \sum_{i=1}^{p-k} \theta_{2i}(|\xi|^2 - \theta_{2i}^2) \sigma_{2i}. \end{aligned}$$

For a Killing vector field, $|\nabla \xi|^2 = (2p-1)|\xi|^2 - (1/2)\Delta|\xi|^2$ and $\Delta|\xi|^2 = \sum_{i=1}^{p-k} \theta_{2i}^2 \Delta|\sigma_{2i}|^2$, but, since $|\sigma_{2i}|^2$ is a homogeneous polynomial,

$$\begin{aligned} \Delta|\sigma_{2i}|^2 &= \Delta \mathbf{R}^{2p+1} |\sigma_{2i}|^2 + 2(1+2p)|\sigma_{2i}|^2 \\ &= -4 + 2(1+2p)|\sigma_{2i}|^2; \end{aligned}$$

hence,

$$\Delta|\xi|^2 = -4 \sum_{i=1}^{p-k} \theta_{2i}^2 + 2(2p+1)|\xi|^2,$$

and

$$|\nabla \xi|^2 = 2 \left(\sum_{i=1}^{p-k} \theta_{2i}^2 - |\xi|^2 \right).$$

Therefore, equation (9) becomes

$$\begin{aligned} (2p-1) \sum_{i=1}^{p-k} \theta_{2i} \sigma_{2i} - \frac{2B'}{B} \sum_{i=1}^{p-k} \theta_{2i} (\theta_{2i}^2 - |\xi|^2) \sigma_{2i} \\ = \frac{\sum_{i=1}^{p-k} \theta_{2i} \sigma_{2i}}{B + |\xi|^2 C} \left[(C - B') 2 \left(\sum_{l=1}^{p-k} \theta_{2l}^2 - |\xi|^2 \right) \right. \\ \left. + \left(C' - \frac{2B'C}{B} \right) \left(\sum_{l=1}^{p-k} \theta_{2l}^4 |\sigma_{2l}|^2 - |\xi|^4 \right) \right]. \end{aligned}$$

From the independence of the vectors σ_{2i} , we deduce that

$$(2p-1) - \frac{2B'}{B}(\theta_{2i}^2 - |\xi|^2) = \frac{1}{B + |\xi|^2 C} \left[(C - B')2 \left(\sum_{l=1}^{p-k} \theta_{2l}^2 - |\xi|^2 \right) + \left(C' - \frac{2B'C}{B} \right) \left(\sum_{l=1}^{p-k} \theta_{2l}^4 |\sigma_{2l}|^2 - |\xi|^4 \right) \right],$$

for all $i = 1, \dots, p-k$,

and the right-hand side of the equation is independent of i , so necessarily

$$\frac{2B'}{B}\theta_{2i}^2 = \frac{2B'}{B}\theta_{2j}^2, \quad \text{for all } i, j = 1, \dots, p-k.$$

Since there exist i and j such that $\theta_{2i} \neq \theta_{2j}$, then $B'(|\xi|^2) = 0$ and the equation becomes

$$(2p-1)B = \left[\sum_{l=1}^{p-k} \theta_{2l}^4 |\sigma_{2l}|^2 - |\xi|^4 \right] C' + \left[2 \sum_{l=1}^{p-k} \theta_{2l}^2 - (2p+1)|\xi|^2 \right] C.$$

Let $p_1 = (1, 0, \dots, 0)$. Then $\xi(p_1) = \theta_2 \sigma_2(p_1)$, $|\sigma_2(p_1)|^2 = 1$ and $|\xi(p_1)|^2 = \theta_2^2$, and the condition becomes

$$(2p-1)B(\theta_2^2) = \left(2 \sum_{l=1}^{p-k} \theta_{2l}^2 - (2p+1)\theta_2^2 \right) C(\theta_2^2),$$

which is equivalent to

$$(2p-1) (B(\theta_2^2) + \theta_2^2 C(\theta_2^2)) = \left(2 \sum_{l=2}^{p-k} \theta_{2l}^2 \right) C(\theta_2^2).$$

Since G is positive definite, $B(\theta_2^2) + \theta_2^2 C(\theta_2^2)$ is strictly positive; hence, $C(\theta_2^2)$ is also strictly positive. Therefore, $2 \sum_{l=1}^{p-k} \theta_{2l}^2 - (2p+1)\theta_2^2 > 0$ (recall that B is a non-negative function). Similarly, testing the equation at the point $p_j = (0, \dots, 0, 1, 0, \dots, 0)$ (1 at the $2j$ -th position) yields

$$2 \sum_{l=1}^{p-k} \theta_{2l}^2 - (2p+1)\theta_{2j}^2 > 0, \quad \text{for all } j = 1, \dots, p-k,$$

and, summing up these inequalities, we obtain

$$-(2k+1) \sum_{l=1}^{p-k} \theta_{2l}^2 > 0,$$

which is clearly impossible. Hence, if ξ is a harmonic section, then $\theta_{2i} = \theta_{2j}$ for all $i, j = 1, \dots, p-k$. \square

Remark 6.1. Since, when $p = 1$, k can only take the value 0, there exists no harmonic Killing vector field on \mathbf{S}^2 , whatever the Kaluza-Klein metric (with $A \equiv 1$) on $T\mathbf{S}^2$.

The harmonic sections obtained in Proposition 6.2 cannot be harmonic maps because, although they are certainly divergence free, their norms cannot be constant, unless null, as Killing vector fields on even-spheres must vanish at some points.

On the subject of odd-dimensional spheres, the situation is slightly different.

Theorem 6.3. *Let ξ be a Killing vector field on \mathbf{S}^{2p+1} . Then the dimension of its invariant sub-space is $\dim F_\xi = 2k$, with $0 \leq k \leq p$, and ξ is a harmonic section into $T\mathbf{S}^{2p+1}$ equipped with a Kaluza-Klein metric G , with $A \equiv 1$, if and only if*

- (1) $k \neq p$, i.e., $\dim F_\xi$ is not maximal;
- (2) $\xi = \lambda(-x_2, x_1, \dots, -x_{2p-2k+2}, x_{2p-2k+1}, 0, \dots, 0)$.

If $k = 0$, there is no condition on function C (except that the metric G must be Riemannian) while B must satisfy

$$\lambda^2 B'(\lambda^2) + B(\lambda^2) = 0.$$

For example, $B(t) = Ke^{-t/\lambda^2}$ or $B(t) = K(1+t)^{-(1+(1/\lambda^2))}$, $K > 0$, are solutions. Moreover ξ will be a harmonic map.

If $k \neq 0$, one solution is given by $C = 0$ and $B(t) = Ke^{-pt/(\lambda^2(p-k))}$, $K > 0$. However, as in the even-dimensional case, ξ will not be a harmonic map because it does not have constant norm.

Proof. Let ξ be a Killing vector field on the sphere \mathbf{S}^{2p+1} . Then its invariant sub-space F_ξ has dimension $2k$ with $0 \leq k \leq p$. For Killing vector fields on \mathbf{S}^{2p+1} , the equation of harmonic sections is

$$(11) \quad (2p)\xi + \frac{2B'}{B}\nabla_{\nabla_\xi\xi}\xi = \frac{1}{B+|\xi|^2C} \left[(C-B')|\nabla\xi|^2 + (C' - \frac{2B'C}{B})|\nabla_\xi\xi|^2 \right] \xi.$$

First, assume that all the coefficients θ_{2i} are equal to λ . Then $\xi = \lambda\sigma$, where $\sigma = \sum_{i=1}^{p-k} \sigma_{2i}$, and

$$\begin{aligned} \nabla_{\nabla_\xi\xi}\xi &= \lambda^3(|\sigma|^2 - 1)\sigma; & |\nabla_\xi\xi|^2 &= \lambda^4|\sigma|^2(1 - |\sigma|^2); \\ |\nabla\xi|^2 &= 2\lambda^2(p+1-k - |\sigma|^2), \end{aligned}$$

and equation (11) becomes

$$(12) \quad \begin{aligned} \lambda^2(p-k)B' + pB &= \lambda^2[(p+1-k) - (p+1)|\sigma|^2]C \\ &+ \frac{\lambda^4}{2}|\sigma|^2(1 - |\sigma|^2)C'. \end{aligned}$$

In the special case when $\dim F_\xi = 0$, then $\xi = \lambda\sigma$, where $\sigma(x) = (-x_2, x_1, \dots, -x_{2p+2}, x_{2p+1})$, and ξ has constant norm $|\lambda|$. Equation (12) then simplifies to

$$\lambda^2 B'(\lambda^2) + B(\lambda^2) = 0,$$

so we can choose $B(t) = Ke^{-t/\lambda^2}$, $K > 0$, or $B(t) = K(1+t)^{-(1+(1/\lambda^2))}$, and C to be any function (as long as the resulting metric G is positive definite). Since ξ has constant norm, it turns out to also be a harmonic map.

If $\dim F_\xi = 2k$ with $k = 1, \dots, p-1$, then we have the solution $C = 0$ and $B(t) = Ke^{-pt/(\lambda^2(p-k))}$, $K > 0$. As in the even-dimensional case, to any positive function C , one can find a positive function B solving (12).

If $\dim F_\xi = 2p$, then the equation becomes

$$pB = \lambda^2[1 - (p+1)|\sigma|^2]C + \frac{\lambda^4}{2}|\sigma|^2(1 - |\sigma|^2)C',$$

and $\xi = \lambda(-x_2, x_1, 0, \dots, 0)$. We evaluate this equation at the point $p_1 = (1, 0, \dots, 0)$ to obtain the condition

$$pB(\lambda^2) = -p\lambda^2 C(\lambda^2),$$

which contradicts the positive definiteness of the metric G .

In case there exist coefficients i and j such that $\theta_{2i} \neq \theta_{2j}$, $\dim F_\xi$ must be different from $2p$ and the decomposition

$$\xi = \sum_{i=1}^{p+1-k} \theta_{2i} \sigma_{2i},$$

leads to

$$\begin{aligned} |\nabla_\xi \xi|^2 &= -|\xi|^4 + \sum_{i=1}^{p+1-k} \theta_{2i}^4 |\sigma_{2i}|^2; \\ \nabla_{\nabla_\xi \xi} \xi &= \sum_{i=1}^{p+1-k} \theta_{2i} (|\xi|^2 - \theta_{2i}^2) \sigma_{2i}; \\ |\nabla \xi|^2 &= 2 \left(\sum_{i=1}^{p+1-k} \theta_{2i}^2 - |\xi|^2 \right). \end{aligned}$$

Replacing in equation (11) and using the independence of the Killing vector fields σ_{2i} , yields

$$\begin{aligned} 2p\theta_{2i} + \frac{2B'}{B} \theta_{2i} (|\xi|^2 - \theta_{2i}^2) &= \frac{1}{B + |\xi|^2 C} \left[(C - B') |\nabla \xi|^2 \right. \\ &\quad \left. + \left(C' - 2 \frac{B'C}{B} \right) |\nabla_\xi \xi|^2 \right] \theta_{2i}; \end{aligned}$$

therefore,

$$\frac{2B'}{B} \theta_{2i}^2 = \frac{2B'}{B} \theta_{2j}^2, \quad \text{for all } i, j = 1, \dots, p+1-k;$$

hence, the necessary condition $B'(|\xi|^2) = 0$. Under this condition, equation (11) becomes

$$2pB = \left[-|\xi|^4 + \sum_{l=1}^{p+1-k} \theta_{2l}^4 |\sigma_{2l}|^2 \right] C' + \left[2 \sum_{l=1}^{p+1-k} \theta_{2l}^2 - 2(p+1) |\xi|^2 \right] C;$$

evaluated at the point $p_1 = (1, 0, \dots, 0)$, this gives

$$2pB(\theta_2^2) = \left[2 \sum_{l=1}^{p+1-k} \theta_{2l}^2 - 2(p+1)\theta_2^2 \right] C(\theta_2^2),$$

which is equivalent to

$$2p(B(\theta_2^2) + \theta_2^2 C(\theta_2^2)) = \left[2 \sum_{l=2}^{p+1-k} \theta_{2l}^2 \right] C(\theta_2^2),$$

and this implies that $C(\theta_2^2)$ is strictly positive and therefore, since B is a strictly positive function,

$$\sum_{l=1}^{p+1-k} \theta_{2l}^2 - (p+1)\theta_2^2 > 0.$$

Similarly, evaluating the equation at the point $p_j = (0, \dots, 0, 1, 0, \dots, 0)$ (1 at the $2j$ -th position) yields

$$\sum_{l=1}^{p+1-k} \theta_{2l}^2 - (p+1)\theta_{2j}^2 > 0, \quad \text{for all } j = 1, \dots, p+1-k,$$

and, summing up these inequalities, we obtain

$$-k \sum_{l=1}^{p+1-k} \theta_{2l}^2 > 0,$$

which is clearly impossible. Hence, ξ cannot be a harmonic section. \square

7. A larger class of metrics. We can render harmonic some of the vector fields considered previously, if we enlarge our class of metrics on the tangent bundle by dropping the condition $A \equiv 1$. Then the conditions for harmonic sections and harmonic maps are given by Proposition 2.1.

Proposition 7.1. *Let $a \in \mathbf{R}^{n+1} \setminus \{0\}$ and $\lambda : \mathbf{S}^n \rightarrow \mathbf{R}$, $\lambda(x) = \langle a, x \rangle$. Then the conformal gradient field $\sigma = \text{grad}^{\mathbf{S}^n} \lambda$ is a harmonic section for*

(1) $n = 2$, $C = 0$, $A(t) = B + A_0$ ($A_0 \in \mathbf{R}^+ \setminus \{0\}$) and $B(t) = Ke^{-t/2}$, $K > 0$, and σ is also a harmonic map.

(2) $n > 2$, $C = 0$, $A(t) = B(t) + A_0$ ($A_0 \in \mathbf{R}^+ \setminus \{0\}$) and $B(t) = K[n + (n - 2)|a|^2 - (n - 2)t]^{1/(n-2)}$, $K > 0$; but σ is not a harmonic map.

Proof. With the notations of the proof of Proposition 6.1, we see that a conformal gradient field σ will be a harmonic section if

$$B - \lambda^2(2 - n)B' + nA' = \lambda^2(|a|^2 - \lambda^2)C' + [n\lambda^2 - (|a|^2 - \lambda^2)]C,$$

and if we choose $C = 0$ and $A = B + A_0$ ($A_0 \in \mathbf{R}^+ \setminus \{0\}$), the condition becomes

$$B(t) + [(n - 2)|a|^2 + n - (n - 2)t]B'(t) = 0,$$

with $t = |\sigma|^2 = |a|^2 - \lambda^2$.

If $n = 2$, $B(t) = Ke^{-t/2}$, $K > 0$, is a solution; and if $n > 2$, $B(t) = K[n + (n - 2)|a|^2 - (n - 2)t]^{1/(n-2)}$, $K > 0$, is a solution, for $t \in [0, |a|^2]$, which can be extended to \mathbf{R}^+ .

For these solutions the horizontal part of the tension field for a conformal vector field is

$$-(n - 1)\lambda B\sigma = 2\lambda B'\sigma,$$

and this is only satisfied in the case $n = 2$. \square

Remark 7.1. A similar result can be found in [2].

Proposition 7.1. *For any Killing vector field ξ on \mathbf{S}^n , $n \geq 2$, there exists a Kaluza-Klein metric on $T\mathbf{S}^n$ making ξ into a harmonic section. If $n = 2$, then ξ is also a harmonic map but not if $n = 2p$, $p > 1$. If $n = 2p + 1$, ξ will not be a harmonic map unless it has constant norm (cf. Proposition 6.3).*

Proof. Let $\xi : \mathbf{S}^{2p} \rightarrow T\mathbf{S}^{2p}$ be a Killing vector field. As in Proposition 6.2, we call F_ξ its invariant sub-space and $2k + 1$ its dimension. Up to a change of coordinates, $\sigma = \sum_{i=1}^{p-k} \theta_{2i} \sigma_{2i}$, where the θ_{2i} 's are non-zero constants and σ_{2i} is the Killing vector field given

by rotation in the (x_{2i-1}, x_{2i}) -plane. For Killing vector fields on \mathbf{S}^{2p} , the equation of harmonic sections, with respect to the new metric G , is

$$(13) \quad (2p-1)\xi + \frac{2B'}{B}\nabla_{\nabla_\xi\xi}\xi \\ = \frac{1}{B+|\xi|^2C}\left[(C-B')|\nabla\xi|^2 + \left(C' - \frac{2B'C}{B}\right)|\nabla_\xi\xi|^2 - 2pA'\right]\xi.$$

First assume that the coefficients θ_{2i} are all equal to λ . If $k \neq p-1$, we know, from Proposition 6.2, a Kaluza-Klein metric makes ξ a harmonic section. If $k = p-1$ (this is the general case on \mathbf{S}^2), equation (13) becomes

$$(2p-1)B + 2pA' = \lambda^4|\sigma|^2(1-|\sigma|^2)C' + \lambda^2(2-(2p+1)|\sigma|^2)C,$$

where $\xi = \lambda\sigma$ (as in Proposition 6.2). Choosing $C = 0$ and $A = B + A_0$ ($A_0 \in \mathbf{R}^+ \setminus \{0\}$), we obtain the solution $B(t) = Ke^{((1/2p)-1)t}$, $K > 0$.

In the more general case where there exist coefficients i and j such that $\theta_{2i} \neq \theta_{2j}$, by the same arguments as in Proposition 6.2, we show that $B(t) = B_0$ ($B_0 \in \mathbf{R}^+ \setminus \{0\}$) and equation (13) becomes

$$(2p-1)B_0 + 2pA' = |\nabla_\xi\xi|^2C' - \frac{1}{2}\Delta|\xi|^2C,$$

which admits the positive solution $A(t) = (-1 + 1/2p)B_0t + A_0$ for $t \in [0, \sup|\xi|^2]$ and continuously prolonged over \mathbf{R}^+ .

Similar arguments apply to the odd-dimensional case.

REFERENCES

1. M. Abbassi, G. Calvaruso and D. Perrone, *Harmonic sections of tangent bundles equipped with Riemannian g -natural metrics*, Quart. J. Math. **6** (2011), 259–288.
2. ———, *Some examples of harmonic maps for g -natural metrics*, Ann. Math. Blaise Pascal **16** (2009), 305–320.
3. M. Abbassi and M. Sarik, *On natural metrics on tangent bundles of Riemannian manifolds*, Arch. Math. (Brno) **41** (2005), 71–92.
4. M. Benyounes, E. Loubeau and C.M. Wood, *Harmonic sections of Riemannian vector bundles and metrics of Cheeger-Gromoll type*, Diff. Geom. Appl. **25** (2007), 322–334.
5. ———, *Harmonic maps and sections on spheres*, arXiv:math/0703060v1 [math.DG].

6. M. Benyounes, E. Loubeau and C.M. Wood, *The geometry of generalized Cheeger-Gromoll metrics*, Tokyo J. Math. **32** (2009), 287–312.
7. J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. Math. **96** (1972), 413–433.
8. P. Dombrowski, *On the geometry of the tangent bundle*, J. reine und angew. Math. **210** (1962), 73–88.
9. J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978), 1–68.
10. ———, *Another report on harmonic maps*, Bull. London Math. Soc. **20** (1988), 385–524.
11. ———, *Selected topics on harmonic maps*, Amer. Math. Soc., Providence, RI, 1983.
12. J. Eells and J.H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.
13. T. Ishihara, *Harmonic sections of tangent bundles*, J. Math. Tokushima Univ. **13** (1979), 23–27.
14. J. Kazdan and F.W. Warner, *Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures*, Ann. Math. **101** (1975), 317–331.
15. E. Musso and F. Tricerri, *Riemannian metrics on tangent bundles*, Ann. Mat. Pura Appl. **150** (1988), 1–19.
16. O. Nouhaud, *Applications harmoniques d'une variété Riemannienne dans son fibré tangent*, C.R. Acad. Sci. Paris **284** (1977), 815–818.
17. C. Oniciuc, *The tangent bundle and harmonicity*, An. Stiint. Univ. Al. I. Cuza Mat. **43** (1997), 151–172.
18. G. Wiegink, *Total bending of vector fields on Riemannian manifolds*, Math. Ann. **303** (1995), 325–344.
19. C.M. Wood, *An existence theorem for harmonic sections*, Manuscr. Math. **68** (1990), 69–75.
20. Y.L. Xin, *Some results on stable harmonic maps*, Duke Math. J. **47** (1980), 609–613.
21. S.T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

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