ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 1, 2012

ON THE MARKER METHOD FOR CONSTRUCTING FINITARY ISOMORPHISMS

STEPHEN M. SHEA

ABSTRACT. The theory of finitary isomorphisms began with the work of Keane and Smorodinsky in the 1970s. They developed machinery known as the "marker and filler" method to show that any two irreducible equal entropy finite state Markov processes are finitarily isomorphic provided they have the same period. In this paper, we formalize the marker portion of their machinery. In doing so, we define d-equivalence of processes. D-equivalence assigns to pairs of processes a nonnegative integer that quantifies, in a sense, how closely related are the two processes. We prove upper bounds for these quantities among Bernoulli schemes, Markov chains and r-processes.

1. Introduction. After Kolmogorov introduced entropy to dynamical systems in 1958 [**8**], and realized its valuable invariant nature, it was hypothesized that entropy is a complete isomorphism invariant for independent processes (Bernoulli schemes). Ornstein would prove this conjecture in 1969 [**11**]. However, prior to Ornstein's result, mathematicians began trying to construct isomorphisms between various independent processes with the same entropy. Meshalkin was one of these mathematicians, and in 1959, he showed that Bernoulli schemes with non-isomorphic state spaces can be isomorphic [**10**]. His results would later be expanded by Blum and Hanson [**2**]. Meshalkin, however, not only constructed an isomorphism, but a finitary isomorphism. The term finitary isomorphism would later be coined by Keane and Smorodinsky in their annals paper of 1979 [**5**]. Ornstein showed that Bernoulli schemes of the same entropy are measure-theoretically isomorphic, but that the coding may depend on infinite pasts or futures. After Ornstein's result, Keane and Smorodinsky showed that Bernoulli schemes with the same entropy are finitarily isomorphic. They later expanded their result to show that finite state irreducible mixing Markov

2010 AMS *Mathematics subject classification.* Primary 37A35.
Keywords and phrases. Bernoulli scheme, finitary isomorphism, r-process, Markov, d-equivalence.
Received by the editors on March 20, 2009, and in revised form on May 26, 2009.
DOI:10.1216/RMJ-2012-42-1-293

processes are finitarily isomorphic to Bernoulli schemes, provided they have the same entropy [**6**].

In a recent paper [**15**], we have extended the methods of Keane and Smorodinsky to show that entropy is a complete finitary isomorphism invariant for finite state r-processes.

Let us formally define what we mean by a finitary isomorphism and r-processes.

Definition 1.1. Let $(X,\mathcal{U},\mu,T)$ and $(Y,\mathcal{V},\nu,S)$ be two processes. An isomorphism, ϕ from $(X,\mathcal{U},\mu,T)$ to $(Y,\mathcal{V},\nu,S)$ is a bimeasurable equivariant map from a subset of X of measure one to a subset of Y of full measure which takes μ to ν. The isomorphism, ϕ is finitary if for almost every $x \in X$ there exists integers $m \leq n$ such that the zero coordinates of $\phi(x)$ and $\phi(x')$ agree for almost all $x' \in X$ with $x[m,n] = x'[m,n]$, and similarly for ϕ^{-1}. If we drop the requirement that ϕ be invertible, we say ϕ is a finitary factor.

Suppose our process X has alphabet A.

Definition 1.2. We say $a \in A$ is a renewal state of X if the σ-algebras $\mathcal{U}(X_{n+1}, X_{n+2}, \dots)$ and $\mathcal{U}(\dots, X_{n-2}, X_{n-1})$ are conditionally independent given the event $[X_n = a]$. If there exists such an a, we say X is a renewal process (often called a stationary regenerative process).

Definition 1.3. Let $a \in A$ be a renewal state in X. We say $a \in A$ has n-Bernoulli distribution if for some nonnegative integer n, $P[X_{n'} = a \mid X_0 = a] = P[X_{n'} = a]$ for all $n' > n$.

We will say a state has n-Bernoulli distribution when we mean there exists such a finite n. If the precise n is of interest, we will make note, but this is, in general, not the case.

Definition 1.4. An r-process, X, is a renewal process such that a renewal state in X has n-Bernoulli distribution for some natural number n.

We can now define a Markov process to be a process in which every state is a renewal state. We can define a Bernoulli scheme to be a Markov process in which every state has 0-Bernoulli distribution.

Common examples of r-processes include Bernoulli schemes and m-dependent Markov chains. M-dependent Markov chains often occur as finite factors of Bernoulli schemes. For more information on the interplay of the Markov property and m-dependence, consult [**9**].

For Keane and Smorodinsky, the first and most crucial step towards proving two processes are finitarily isomorphic was to establish markers. We formalize this marker method with the definition of d-equivalence.

Definition 1.5. Let $a \in A$. The distribution of the state a is defined as the process $\widehat{X}$ obtained by setting

$$\widehat{X}_n = \begin{cases} 0 & \text{if } X_n \neq a \\ 1 & \text{if } X_n = a. \end{cases}$$

Definition 1.6. Let k be a positive integer. The process $X^{(k)}$ called the k-stringing of X is defined as follows. The state space of $X^{(k)}$ is all allowable sequences of length k in X, and $X_n^{(k)} = (X_n, X_{n+1}, \ldots, X_{n+k-1})$ $(n \in \mathbf{Z})$.

$X^{(k)}$ is often also referred to as the k-block presentation.

Definition 1.7. Two processes X and Y are 0-equivalent if X and Y have the same entropy, and if for some positive integers k and j, there exists a renewal state in $X^{(k)}$ with the same distribution as a renewal state in $Y^{(j)}$.

Definition 1.8. Two processes X and Z are 1-equivalent if there exists a process Y such that both X and Z are 0-equivalent to Y.

Definition 1.9. Let d be a positive integer. Two processes X and Z are d-equivalent if there exist d processes $Y_1, Y_2, \ldots, Y_d$ such that X is 0-equivalent to Y_1, Z is 0-equivalent to Y_d, and Y_i is 0-equivalent to Y_{i+1} for $1 \leq i \leq d-1$.

Two finite state mixing d-equivalent Markov processes are finitarily isomorphic [**6**]. Two finite state d-equivalent r-processes are finitarily isomorphic [**15**].

However, finitary isomorphism does not tell the whole story. With the definition of d-equivalence in hand, new questions arise. For instance, we could ask if, given any two Bernoulli schemes X and Y with the same entropy, a positive integer k exists such that these two Bernoulli schemes are always k-equivalent. We could ask the same question if both processes were Markov or r-processes. We could also ask the same question if we were given one process which is Bernoulli and one which is Markov. In the next few sections we answer these questions and others of the same nature.

2. D-equivalence of Bernoulli schemes.

In this section, we show that any two Bernoulli schemes are 2-equivalent. Before we can prove our result in this section and future results, we need a few more definitions.

Definition 2.1. Let $A' \subseteq A$ be a subset of the set of states of X, and let b be a symbol not belonging to A. We say that the process X', defined by

$$X'_n = \begin{cases} X_n & \text{if } X_n \notin A' \\ b & \text{otherwise} \end{cases}$$

is obtained from X by collapsing A'.

Definition 2.2. Let $b_1, b_2, \dots, b_l$ be symbols not belonging to A, $q_1, q_2, \dots, q_l$ a probability vector, and $a_i \in A$. We say that the process X' is obtained from X by independently splitting a_i according to $q_1, q_2, \dots, q_l$ if X' is defined as follows: The states of X' are $b_1, b_2, \dots, b_l, a_1, \dots, a_{i-l}, a_{i+1}, \dots, a_m$ and if $c_0, \dots, c_r$ is a sequence of such states with $c_{j_1} = b_{i_1}, \dots, c_{j_s} = b_{i_s}$ and all other c_j's being a's, then $P[X'_n = c_0, \dots, X'_{n+r} = c_r] = (\prod_{t=1}^{s} q_{i_t}) P[X_{n+j_t} = a_i,\ 1 \leq t \leq s$, $X_{n+j} = c_j$ for the other js].

With these definitions, we can now prove the following theorem.

Theorem 2.3. *Let X and Z be Bernoulli schemes with the same entropy. Then X and Z are 2-equivalent.*

Proof. The proof of the theorem follows from Lemmas 2 and 3 of [**5**] with minor modifications. First note that, if X and Z both have two states, they are the same process and clearly 0-dependent. So, we may assume that at least one of X and Z has three states. We first suppose that both have ≥ 3 states. Let $p = (p_0, p_1, \dots, p_n)$ be the probability vector of X and $q = (q_0, q_1, \dots, q_m)$ be the probability vector of Z.

Lemma 2.4. *Let X, Z, p, and q be defined as above with $n \geq 2$ and $m \geq 2$. Then there exists a Bernoulli scheme Y_2 with probability vector $r = (r_0, r_1, \dots, r_l)$ such that $h(X) = h(Y_2) = h(Z)$, and for some k and k', $r_0 = p_k$ and $r_1 = q_{k'}$.*

Proof. Suppose by reordering and without loss of generality that $p_0 \geq p_1 \geq \cdots \geq p_n$, $q_0 \geq q_1 \geq \cdots \geq q_m$, and $p_n \geq q_m$. Then set $r_0 = p_0$ and $r_1 = q_m$. Let Y_2' be a Bernoulli scheme with probability vector $(r_0, r_1, 1 - (r_0 + r_1))$. Then $h(Y_2') \leq h(X)$. If necessary, we can split $(1 - (r_0 + r_1))$ to get our desired probability vector r (so that $h(Y_2) = h(X)$). □

We now consider the case where $n = 1$.

Lemma 2.5. *Let X, Z, p, and q be defined as above with $n = 1$ and $m \geq 2$. Then there exists a Bernoulli scheme Y_1 with probability vector $s = (s_0, s_1, \dots, s_l)$ such that $h(X) = h(Y_1) = h(Z)$, and for some integer k, $p_0^k p_1 = s_0^k s_1$.*

Proof. Choose any $s_0 > \max\{p_0, p_1\}$, and then choose k large enough so that if $s_1 = (p_0/s_0)^k p_1$, then $s_0 + s_1 < 1$. Define Y_1' to be a Bernoulli scheme with probability vector $(s_0, s_1, 1 - (s_0 + s_1))$ so that $h(Y_1') < h(X)$. We are able to define such a Y_1', because $s_1 \to 0$ as $k \to \infty$. Now split $(1 - (s_0 + s_1))$ to get the desired probability vector s and Bernoulli scheme Y_1. □

We have now shown that if X and Z both have at least 3 states, then X and Z are 1-equivalent. If X has 2-states, then X is 0-equivalent to a Bernoulli scheme, Y_1, which has at least 3 states. By Lemma 2.4, Y_1 is then 1-equivalent to Z. Therefore, any two Bernoulli schemes, X and Z, which have the same entropy are 2-equivalent. $\square$

3. D-equivalence of Markov processes. We now extend our work to Markov processes. The following result follows from the combined work of Keane and Smorodinsky in [**6**] and Ackoglu, del Junco and Rahe in [**1**] with minor modifications. For completeness, we have chosen to combine their work, and state and prove this version of their joint result in our terminology. However, we will at times refer the reader to [**1, 6**] for minor details. In the next section, we provide a complete proof in the case of r-processes. Many of the omitted details in this proof parallel the details included in the proof for r-processes.

Theorem 3.1. *If X is a mixing, irreducible, finite state Markov process, then there exists a Bernoulli scheme Z such that X is 1-equivalent to Z.*

Proof. Let X be a finite state irreducible mixing Markov shift. Let X have alphabet $A = \{a_1, a_2, \dots, a_n\}$ and probability measure μ. Let W be a Bernoulli scheme with states $\{b_0, b_1, \dots, b_n\}$ and probability vector $q = (q_0, q_1, \dots, q_n)$.

The proof of the following lemma relies heavily on the mixing property of X. For a complete proof, see [**5**].

Lemma 3.2. *Let X be the Markov process defined above, with $n \geq 2$. There exists a state $a_g \in A$ such that for all integers k, there is an allowable sequence*

$$\alpha^0 = \alpha_1^0 \alpha_2^0 \cdots \alpha_k^0 \in A^k$$

with $\alpha_i^0 \neq a_g$ for all i such that $1 \leq i \leq k$.

Let a_g be as in this lemma. Since X is mixing, we can choose an integer k_0 such that, for all $k \geq k_0$, and for all i such that $1 \leq i \leq n$, there is an allowable sequence

$$\alpha^i = \alpha_1^i \alpha_2^i \cdots \alpha_k^i$$

with $\alpha_1^i = a_g$ and $\alpha_k^i = a_i$.

For example, $\alpha^2 = a_g \alpha_2^2 \cdots \alpha_{k-1}^2 a_2$.

Now let $\widehat{Y} = X \times W$, and let $\widehat{Y}^{(k)}$ be the k-stringing (k-block presentation) of $\widehat{Y}$. Partition $\widehat{Y}^{(k)}$ into three disjoint subsets as follows.

$$M = \cup\{\alpha^i \times (b_0, b_0, \ldots, b_0, b_i) \quad \text{for } 1 \le i \le n\}$$
$$N = \{\alpha^0 \times (b_{i_1}, \ldots, b_{i_k}) \text{ where } (b_{i_1}, \ldots, b_{i_k}) \in \{b_0, b_1, \ldots, b_n\}^k\},$$

and

$$O = \text{ everything else.}$$

We define Y' to be the process obtained by collapsing M, N, and O. We naturally call the three states of Y', M, N and O. Since the measures of M and N tend to 0 as $k \to \infty$, we may choose k so large that $h(Y') \le h(X)$.

The following lemma follows from the results of [**1**].

Lemma 3.3. *We can choose the probability vector $q = (q_0, q_1, \ldots, q_n)$ of W so that M has the same distribution in Y' as some state in the* k*-stringing of a Bernoulli scheme Z and where $h(X) = h(Z)$.*

Proof. We only sketch the proof here. For a complete proof see [**1**].

Let

$$q_i = C\frac{\mu(a_i)}{\mu(\alpha^i)}$$

where C is a constant and $1 \le i \le n$. We then take the constant C large enough so that $\sum_{i=1}^n q_i < 1$, and set $q_0 = 1 - \sum_{i=1}^n q_i$. This ensures that M has the same distribution in Y' as some state in the k-stringing of a Bernoulli scheme Z where $h(Z) = h(X)$. □

We can now independently split the state O in Y' to create a new process Y with states $M, N, O_1, \ldots, O_p$ for some positive integer p and where $h(Y) = h(X)$. We will need the following lemma from [**6**]. The proof relies on both X being Markov and W being Bernoulli.

Lemma 3.4. *M and N are renewal states in Y.*

We have seen that M is a renewal state in Y with the same distribution as a state in the k-stringing of a Bernoulli scheme. Therefore, M has k-Bernoulli distribution, and Y is an r-process. N clearly has the same distribution in Y as α^0 in X. Therefore, X is 0-equivalent to an r-process Y. Since M is a renewal state in Y with the same distribution as a state in $Z^{(k)}$, Y and Z are 0-equivalent. Therefore, X and Z are 1-equivalent. $\square$

This theorem and the work in Section 2 imply the following.

Corollary 3.5. *Let X and Y be mixing, irreducible, finite state Markov processes with the same entropy. Then X and Y are 6-equivalent.*

4. D-equivalence of r-processes. Next, we extend our results to a new type of discrete stationary stochastic process [**15**]. We will need the following lemma, the proof of which is trivial.

Lemma 4.1. *If X is a renewal process with renewal state a, then any sequence of length k, $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ such that $\sigma_i = a$ for some $1 \leq i \leq k$ is a renewal state of $X^{(k)}$.*

We will also need the following lemma.

Lemma 4.2. *Let X be an* r*-process with alphabet $A = \{a_1, a_2, \ldots, a_m\}$ where $a = a_i$ for some $1 \leq i \leq m$ is the renewal state with n-Bernoulli distribution, and suppose there does not exist a positive integer k such that $X^{(k-1)}$ is Markov. For all positive integers $k \geq 2$, there exists an allowable sequence (in X) $\alpha = \alpha_1\alpha_2\cdots\alpha_k$ such that $\alpha_k = a$ and $\alpha_i \neq a$ for $1 \leq i \leq k-1$.*

Proof. If such a sequence did not exist, then by Lemma 4.1 any state in the $(k-1)$-stringing of X would be a renewal state, and $X^{(k-1)}$ would be Markov. □

We are now ready to present our main result.

Theorem 4.3. *If X is an* r*-process, there exists a Bernoulli scheme Z, such that X and Z are* 1*-equivalent.*

Proof. Let X be an r-process with alphabet $A = \{a_1, a_2, \dots, a_m\}$ where $a = a_i$ for some $1 \le i \le m$ is the renewal state with n-Bernoulli distribution. Since our theorem is true for Markov processes by the work of the previous section, we may assume that there does not exist a positive integer k such that $X^{(k-1)}$ is Markov. By Lemma 4.2, for all positive integers $k \ge 2$, there exists an allowable sequence (in X) $\alpha = \alpha_1\alpha_2\cdots\alpha_k$ such that $\alpha_k = a$ and $\alpha_i \ne a$ for $1 \le i \le k-1$. For the rest of this proof, let α denote this allowable sequence.

Remark 4.4. By Lemma 4.1, α is a renewal state of $X^{(k)}$.

Let $W = (W_n)_{n\in\mathbf{Z}}$ be a Bernoulli scheme with states $b_0, b_1, \dots, b_m \in B$ and probability vector $q_0, q_1, \dots, q_m$. We will choose k large enough to meet certain specifications later, but for now we may assume k is a fixed integer ≥ 2. Let $Y' = X \times W$ and $Y'^{(k)}$ be the k-stringing of Y'. We partition the states of $Y'^{(k)}$ into three disjoint subsets M, N, and O defined as follows.

$$
\begin{aligned}
M &= \{(a, x_2, \dots, x_k) \times (b_0, \dots, b_0, b_1) : (x_2, \dots, x_k) \in A^{k-1}\}\\
N &= \{\alpha \times (\beta_1, \dots, \beta_k) : (\beta_1, \dots, \beta_k) \in B^k\}\\
O &= \text{all other states of } Y'^{(k)}.
\end{aligned}
$$

Let Y'' denote the process obtained from $Y'^{(k)}$ by separately collapsing M, N and O. Thus Y'' is a process on three states, which we shall of course denote by M, N and O. The probabilities of M and N tend to zero as $k \to \infty$, so we may choose k so large that $h(Y'') \le h(X)$. We can then split O independently to obtain a new process Y with states M, N, $O_1, \dots, O_p$ such that $h(Y) = h(X)$.

Next, we verify that M and N are renewal states in Y. Since any independent splitting of O will not destroy the renewal property of M or N, we need only check that M and N are renewal states in Y''.

Lemma 4.5. *M and N are renewal states of Y'' (and therefore Y).*

Proof. $[Y_0'' = M]$ forces $Y_{-1}'' = O, \dots, Y_{-(k-2)}'' = O$. Thus the renewal property of M in Y'' follows from the renewal property of a in X. More explicitly, $\mathcal{U}(\dots Y_{-2}'', Y_{-1}'')$ and $\mathcal{U}(Y_1'', Y_2'' \dots)$ are conditionally independent given $[Y_0'' = M]$ if and only if $\mathcal{U}(\dots Y_{-k}'', Y_{-(k-1)}'')$ and $\mathcal{U}(Y_1'', Y_2'' \dots)$ are conditionally independent given $[Y_0'' = M]$. Since W is a Bernoulli scheme, $\mathcal{U}(\dots Y_{-k}'', Y_{-(k-1)}'')$ and $\mathcal{U}(Y_1'', Y_2'' \dots)$ are conditionally independent given $[Y_0'' = M]$ if $\mathcal{U}(\dots X_{-k}^{(k)}, X_{-(k-1)}^{(k)})$ and $\mathcal{U}(X_1^{(k)}, X_2^{(k)} \dots)$ are conditionally independent given $[X_0 = a]$. $\mathcal{U}(\dots X_{-k}^{(k)}, X_{-(k-1)}^{(k)})$ and $\mathcal{U}(X_1^{(k)}, X_2^{(k)} \dots)$ are conditionally independent given $[X_0 = a]$ if a is a renewal state in X.

$[Y_0'' = N]$ forces $Y_1'' = O, \dots, Y_{k-2}'' = O$. Thus the renewal property of N in Y'' also follows from the renewal property of a in X. More explicitly, $\mathcal{U}(\dots Y_{-2}'', Y_{-1}'')$ and $\mathcal{U}(Y_1'', Y_2'' \dots)$ are conditionally independent given $[Y_0'' = N]$ if and only if $\mathcal{U}(\dots Y_{-2}'', Y_{-1}'')$ and $\mathcal{U}(Y_{(k-1)}'', Y_k'' \dots)$ are conditionally independent given $[Y_0'' = N]$. Since W is a Bernoulli scheme, $\mathcal{U}(\dots Y_{-2}'', Y_{-1}'')$ and $\mathcal{U}(Y_{(k-1)}'', Y_k'' \dots)$ are conditionally independent given $[Y_0'' = N]$ if $\mathcal{U}(\dots X_{-2}^{(k)}, X_{-1}^{(k)})$ and $\mathcal{U}(X_{(k-1)}^{(k)}, X_k^{(k)} \dots)$ are conditionally independent given $[X_0 = a]$. $\mathcal{U}(\dots X_{-2}^{(k)}, X_{-1}^{(k)})$ and $\mathcal{U}(X_{(k-1)}^{(k)}, X_k^{(k)} \dots)$ are conditionally independent given $[X_0 = a]$ if a is a renewal state in X. $\square$

It is clear that N has the same distribution in Y as α in $X^{(k)}$. So all that is left to check to prove our proposition is that there exists a Bernoulli scheme Z such that some state in $Z^{(k)}$ has the same distribution as M in Y, and so that $h(Y) = h(Z)$.

Lemma 4.6. *There exists a Bernoulli scheme Z such that Y and Z are 0-equivalent.*

Proof. We know that $P[Y_0 = M] = P[X_0 = a] \cdot P[W_0 = W_1 \cdots = W_{k-2} = b_0] \cdot P[W_{k-1} = b_1]$. So $P[Y_0 = M] = P[X_0 = a] \cdot q_0^{k-1} q_1$. Since we may choose our probability vector $q_0, \ldots, q_m$ freely, and M is a renewal state in Y, we may choose our Bernoulli scheme Z with alphabet $\{c_0, \ldots, c_l\}$ such that $P[Z_0 = c_0] = q_0$ and $P[Z_k = c_1] = P[X_0 = a] \cdot q_1$ and $h[Z] = h[Y]$. Now consider the state $C^0 = (c_0, \ldots, c_0, c_1)$ in $Z^{(k)}$. $P[Z_0^{(k)} = C^0] = P[Y_0 = M]$. Since a has n-Bernoulli distribution, we may now choose $k > n$ and C^0 will have the same distribution as M. □

We have shown that, given a finite state r-process X, there exists a Bernoulli scheme Z, such that X and Z are 1-equivalent. □

From this theorem and the theorems of the previous sections, we have the following corollaries.

Corollary 4.7. *Let X and Y be finite state* r*-processes with the same entropy. Then X and Y are* 6*-equivalent.*

Corollary 4.8. *Let X be a finite state* r*-process, and let Y be a finite state irreducible mixing Markov chain such that X and Y have the same entropy. Then X and Y are* 6*-equivalent.*

5. Open questions. We do not believe that all of the bounds in the above theorems and corollaries are strict. For instance, we believe that if X and Y are two finite state mixing irreducible Markov processes with the same entropy, then X and Y are d-equivalent for some positive integer d where $d < 6$. If stricter bounds exist, we would like to find them.

We would also very much like to extend the above results to countable state processes. Recall that the definition of d-equivalence arose while trying to prove processes are finitarily isomorphic. If we were able to extend the above result to show that two countable state Markov chains, X and Y, with the same entropy are d-equivalent for some positive integer d, we would be able to extend the original methods of Keane and Smorodinsky to show that entropy is a complete finitary isomorphism invariant for countable state mixing Markov chains with exponentially decaying return times. This would be a significantly simpler proof than that in [**13, 14**].

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Department of Mathematics, St. Anselm College, 100 St. Anselm Drive 1792, Manchester, NH 03102
Email address: sshea@anselm.edu