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NAVARRO VERTICES AND NORMAL SUBGROUPS IN GROUPS OF ODD ORDER

JAMES P. COSSEY

ABSTRACT. Let p be a prime, and suppose G is a finite solvable group and χ is an ordinary irreducible character of G. Navarro has shown that one can associate to χ a pair (Q, δ), where Q is a p-subgroup of G and δ is an irreducible character of Q, which is unique up to conjugacy. This pair is called the Navarro vertex of χ, and in this paper we study the behavior of the Navarro vertices of χ with respect to normal subgroups of G when G is assumed to have odd order and χ is a lift of an irreducible Brauer character of G. For $N \triangleleft G$, we use these results to give a sufficient condition for the constituents of χ_N to be lifts of Brauer characters, and we develop a Clifford-type correspondence for lifts with a given Navarro vertex.

1. Introduction. Let p be a prime and G a finite solvable group. If φ is an irreducible Brauer character of G, it has long been known that one can associate to φ a p-subgroup Q of G, called the vertex of φ, and that this subgroup is unique up to conjugacy in G. The vertex subgroup is useful in proving some of the well-known results in the representation theory of finite groups, such as the solvable case of the Alperin weight conjecture [**7**].

Recently, in [**9**], Navarro generalized the notion of a vertex to ordinary irreducible characters of solvable groups. Using an approach similar to that of Isaacs in [**6**], Navarro showed how to associate to each ordinary irreducible character χ of a solvable group G a pair (Q, δ), where Q is a p-subgroup of G and δ is a character of G. We will call this pair the Navarro vertex of χ, and Navarro has shown that this pair is unique up to conjugacy.

In this paper, we will study the behavior of Navarro vertices with respect to normal subgroups in the case that G has odd order and the character χ is a lift of an irreducible Brauer character. Our first

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main result shows that, with the above conditions, if N is a normal subgroup of G, then the Navarro vertices of the constituents of χ_N behave as would be expected. Recall that if χ is a character of G, then χ^0 is the restriction of χ to the elements of G of order not divisible by p. If $\chi^0 = \varphi \in IBr_p(G)$, then we say that χ is a p-lift of φ, and if there is no ambiguity about the prime in question, we will say that $\chi \in \mathrm{Irr}\,(G)$ is a lift if $\chi^o \in IBr_p(G)$.

Theorem 1.1. *Suppose G has odd order, and let p be a prime. Let $\chi \in \mathrm{Irr}\,(G)$ be a lift. If $N \triangleleft G$ and $\psi \in \mathrm{Irr}\,(N)$ is a constituent of χ_N, then there is a Navarro vertex (Q, δ) for χ such that $(Q \cap N, \delta_{Q\cap N})$ is a Navarro vertex for ψ.*

The above result is not true without the odd order assumption (see [**2**]).

Suppose again that G has odd order and that χ is a lift of an irreducible Brauer character of G. If $N \triangleleft G$, it is not necessarily the case that the constituents of χ_N are themselves lifts of Brauer characters. However, the following corollaries of our first main theorem give conditions for the constituents of χ_N to be lifts.

Corollary 1.2. *Suppose G has odd order and that $N \triangleleft G$ with G/N a p-group. If $\chi \in \mathrm{Irr}\,(G)$ is a lift, then the constituents of χ_N are lifts. Moreover, if $\psi \in \mathrm{Irr}\,(N)$ is a constituent of χ_N and ψ is a lift of θ, then $I_G(\psi) = I_G(\theta)$.*

Corollary 1.3. *Suppose that G has odd order, and that χ is a lift with Navarro vertex (Q, δ). Suppose $N \triangleleft G$ and $\delta_{Q\cap N}$ extends to N. If $\psi \in \mathrm{Irr}\,(N)$ is a constituent of χ_N, then ψ is a lift.*

It is known (see, for instance, [**1**, Corollary 5.2]) that if G has odd order and χ is a lift with Navarro vertex (Q, δ), then δ is necessarily linear. Therefore in Corollary 1.3, the character δ necessarily restricts to $Q \cap N$.

Our final set of results concerns constructing a Clifford-type correspondence for lifts in the case that G has odd order. We first introduce some notation. If Q is a p-subgroup of G, and δ is a character of Q, we will let $L(G \mid Q, \delta)$ denote the set of ordinary irreducible characters of G that are lifts and have Navarro vertex (Q, δ). Also, if $H \subseteq G$ and θ is a character of H, we denote by G_θ the stabilizer of θ in $\mathbf{N}_G(H)$.

Theorem 1.4. *Suppose that G has odd order and that*

$$\chi \in L(G \mid Q, \delta).$$

Let $N \lhd G$, let $(P, \lambda) = (Q \cap N, \delta_{Q \cap N})$, and define $T_\lambda = NG_\lambda$. Then there exists a unique character $\eta \in L(T_\lambda | Q, \delta)$ such that $\eta^G = \chi$.

Notice that the above theorem does not claim that induction is a bijection from $L(T_\lambda \mid Q, \delta)$ to $L(G \mid Q, \delta)$, and we will show that if $\eta \in L(T_\lambda \mid Q, \delta)$, then η^G need not be in $L(G \mid Q, \delta)$. However, our final main result shows that induction from a slightly different subgroup is a bijection.

Theorem 1.5. *Let G be a group of odd order, let Q be a p-subgroup of G, and let δ be a linear character of Q. Suppose that N is a normal subgroup of G and that $(P, \lambda) = (Q \cap N, \delta_{Q \cap N})$. Define $T = N\mathbf{N}_G(P)$. Then induction is a bijection from $L(T \mid Q, \delta)$ to $L(G \mid Q, \delta)$.*

2. Definitions and previous results. Rather than working with Brauer characters, we will work in the more general setting of π-partial characters (see [**6**] for more details), where π is a set of primes. Here the set $I_\pi(G)$ of irreducible π-partial characters will play the role of the irreducible Brauer characters. If χ^0 denotes the restriction of the character $\chi \in \mathrm{Irr}(G)$ to the elements of G whose order is a π-number, then we say χ is a lift if $\chi^0 \in I_\pi(G)$. Of course, if π is the complement of the prime p, then the π-partial characters are exactly the Brauer characters.

We will need to use the theory of π-special and π-factorable characters. For a complete definition and important properties, see [**5, 6**]. We will also need an important fact from [**9**] regarding π-factorable characters and normal subgroups: If G is solvable and π is a set of primes, then it is shown in [**9**] that there is a unique normal subgroup M maximal with the property that the constituents of χ_M are π-factorable.

Moreover, if χ is not π-factorable, and M is as above, then χ_M is not homogeneous. This fact is used to construct the normal nucleus $(U, \alpha\beta)$ of a character $\chi \in \mathrm{Irr}\,(G)$ (where α is π-special and β is π'-special): If χ is π-factorable, then $U = G$ and $\chi = \alpha\beta$. If χ is not π-factorable, choose M as above, and let $\psi \in \mathrm{Irr}\,(M)$ be a constituent of χ_M. Then the stabilizer G_ψ is proper in G, and define the normal nucleus of χ to be the normal nucleus of the Clifford correspondent $\eta \in \mathrm{Irr}\,(G_\psi \mid \psi)$ of χ. If Q is a Hall π'-subgroup of U and $\delta = \beta_Q$, then we define (Q, δ) to be the Navarro vertex of χ. If π is the complement of the prime p, this is exactly the construction given in [**9**].

We will also need to frequently use the main result from [**4**], which we restate here for convenience.

Theorem 2.1. *Suppose that G has odd order and that $\chi \in \mathrm{Irr}\,(G)$ is a lift of $\varphi \in I_\pi(G)$. If U is a subgroup of G with Hall π'-subgroup Q, and if $\alpha\beta$ is a π-factorable character of U (with α π-special and β π'-special) that induces to χ, then the pair (Q, β_Q) is unique up to conjugacy.*

The above result generalizes the notion of the vertex subgroup Q of a character in $I_\pi(G)$ to the vertex pair (Q, β_Q) of a lift of φ. It is also known in this case that β is linear [**1**].

3. Restriction to normal subgroups. We are now ready to prove Theorem 1.1. We point out that the proof is very similar to the proof of Theorem 1.1 in [**3**]. We will restate the theorem in the more general context of π-partial characters.

Theorem 3.1. *Suppose G has odd order. Let $\chi \in \mathrm{Irr}\,(G)$ be a lift of $\varphi \in I_\pi(G)$. If $N \triangleleft G$ and $\psi \in \mathrm{Irr}\,(N)$ lies below χ, then there is a Navarro vertex (Q, δ) for χ such that $(Q \cap N, \delta_{Q\cap N})$ is a Navarro vertex for ψ.*

Proof. Suppose first that χ is π-factorable. Then every constituent of χ_N is π-factorable, and since the π'-factor of χ necessarily lies above the π'-factor of ψ, then we are done.

Now suppose that ψ is not invariant in G. Let $\xi \in \mathrm{Irr}\,(G_\psi \mid \psi)$ be such that $\xi^G = \chi$. Note that $\xi^o \in I_\pi(G_\psi)$. By Theorem 2.1, every vertex for ξ is a vertex for χ, and by induction, we are done.

Now suppose that ψ is π-factorable. Then if K is the unique normal subgroup of G maximal with the property that the constituents of χ_K are π-factorable, then $N \subseteq K$. If $\rho \in \operatorname{Irr}(K)$ is a constituent of χ_K that lies over ψ, and $\omega \in \operatorname{Irr}(G_\rho \mid \rho)$ is the Clifford correspondent of χ lying over ρ, then since χ is not factorable, G_ρ is proper in G, and any vertex of ω is also a vertex of χ, by definition. Again, note that $\omega^o \in I_\pi(G_\rho)$. Thus, by induction, we are done in this case.

Finally we may assume that ψ is invariant in G and ψ is not factorable. If M is a normal subgroup of N maximal with the property that the constituents of ψ_M are factorable, then we see that M is in fact normal in G (by the invariance of ψ and the uniqueness of M). Let $\theta \in \operatorname{Irr}(M)$ be a constituent of ψ_M. Since ψ is not factorable, then $N_\theta < N$. Also, note by the Frattini argument that $NG_\theta = G$, and also we have $G_\theta \cap N = N_\theta$. Now let $\mu \in \operatorname{Irr}(G_\theta \mid \theta)$ be the Clifford correspondent for χ, and let $\nu \in \operatorname{Irr}(N_\theta)$ lie under μ (and thus above θ). Note that $\mu^o \in I_\pi(G_\theta)$ and that ν necessarily induces to ψ. By induction, we have a vertex (Q, δ) for μ (and thus for χ, by Theorem 2.1) such that $(Q \cap N_\theta, \delta_{Q\cap N})$ is a vertex for ν, and thus a vertex for ψ. We still need to show that $Q \cap N_\theta = Q \cap N$. It is clear that $Q \cap N_\theta \subseteq Q \cap N$. However, $Q \subseteq G_\theta$, and thus $Q \cap N \subseteq Q \cap G_\theta \cap N \subseteq Q \cap N_\theta$, and thus $Q \cap N = Q \cap N_\theta$, and we are done. □

We note that, in the above theorem, it certainly need not be the case that ψ is itself a lift, even though χ is a lift. In the final section we will give a counterexample that demonstrates this. However, our next two results can be used to prove the corollaries from the introduction, which give conditions that guarantee that the constituents of χ_N are lifts. We first restate and prove Corollary 1.2 in terms of I_π characters.

Corollary 3.2. *Suppose G has odd order, $N \triangleleft G$, and G/N is a π'-group. If $\chi \in \operatorname{Irr}(G)$ is a lift of $\varphi \in I_\pi(G)$, then the constituents of χ_N are lifts. Moreover, if $\psi \in \operatorname{Irr}(N)$ is a constituent of χ_N, and if $\psi^0 = \theta \in I_\pi(N)$, then $G_\psi = G_\theta$.*

Proof. If $\psi \in \operatorname{Irr}(N)$ is a constituent of χ_N, then the constituents of ψ^0 lie under φ. Thus, to show that ψ is a lift, it is enough to show that if $\theta \in I_\pi(N)$ is a constituent of φ_N, then $\psi(1) = \theta(1)$.

Since G/N is a π'-group, then $\chi(1)_\pi = \psi(1)_\pi$ and $\varphi(1)_\pi = \theta(1)_\pi$. Since $\chi(1) = \varphi(1)$, then we have $\psi(1)_\pi = \theta(1)_\pi$.

Let Q be a vertex for φ. Then, by [**8**] some conjugate P of $Q \cap N$ is a vertex for θ. Thus $\theta(1)_{\pi'} = |N : P|_{\pi'}$. Since χ is a lift of φ, then (Q, δ) is a vertex of χ for some linear character $\delta \in \mathrm{Irr}\,(Q)$. Thus by Theorem 3.1, some conjugate of $(Q \cap N, \delta_{Q\cap N})$ is a vertex for ψ. Thus $\psi(1)_{\pi'} = |N : P|_{\pi'} = \theta(1)_{\pi'}$. Therefore $\psi(1) = \theta(1)$ and ψ is a lift.

Now suppose that $\psi \in \mathrm{Irr}\,(N)$ lies under χ and thus is a lift of $\theta \in I_\pi(N)$. Clearly $G_\psi \subseteq G_\theta$. Then

$$\chi(1)_{\pi'} \geq |G : G_\psi|\psi(1)_{\pi'} \geq |G : G_\theta|\psi(1)_{\pi'} = |G : G_\theta|\theta(1)_{\pi'} = \varphi(1)_{\pi'},$$

where the last equality is because θ extends to G_θ. Since χ is a lift of φ, we have equality throughout, and therefore $G_\psi = G_\theta$. □

We mention here that in a forthcoming paper we will prove that a certain condition guarantees that the constituents of χ_N are lifts, where χ is a lift and $N \triangleleft G$ has arbitrary index.

We now restate and prove Corollary 1.3 in terms of I_π characters.

Corollary 3.3. *Suppose that G has odd order, $\chi \in \mathrm{Irr}\,(G)$ is such that $\chi^o \in I_\pi(G)$, and that (Q, δ) is a Navarro vertex of χ. Suppose $N \triangleleft G$ and $\delta_{Q\cap N}$ extends to N. If $\psi \in \mathrm{Irr}\,(N)$ is a constituent of χ_N, then ψ is a lift of some character $\theta \in I_\pi(N)$.*

Proof. Since the vertices of χ are all conjugate, and obviously the constituents of χ_N are conjugate, it is enough to show the result for one vertex of χ and one constituent of χ_N.

Suppose χ_N is not homogeneous. Choose a constituent ψ of χ_N such that the Clifford correspondent $\xi \in \mathrm{Irr}\,(G_\psi \mid \psi)$ for χ has vertex (Q, δ). (We may do this because the vertices of the Clifford correspondents are the vertices of χ, by Theorem 2.1.) Note that ξ must be a lift. By induction applied to $\xi \in \mathrm{Irr}\,(G_\psi)$, we are done in this case.

Thus we may assume that $\chi_N = e\psi$. If ψ is factorable, then since χ is a lift and G has odd order, then ψ is π-factorable with π-degree, and thus necessarily π-irreducible. Thus we may assume ψ is not factorable.

Now let M be a normal subgroup of N maximal with the property that the constituents of ψ_M are factorable. Since ψ is invariant in G, then $M \triangleleft G$, and if $\rho = \alpha\beta$ is any constituent of ψ_M (with α being π-special and β being π'-special), then N_ρ is proper in N, G_ρ is proper in G, $NG_\rho = G$, and $G_\rho \cap N = N_\rho$.

By the definition of the Navarro vertex, we can choose a constituent ρ of χ_M such that $(Q \cap M, \delta_{Q\cap M})$ is a vertex for ρ. Notice that since ψ is invariant in G, then ρ necessarily lies under ψ. If β is the π'-special factor of ρ, then since $(Q \cap M, \delta_{Q\cap M})$ is a vertex for ρ and $\beta_{Q\cap M} = \delta_{Q\cap M}$.

Since $\delta_{Q\cap N}$ extends to N, and $\delta_{Q\cap N}$ is linear (since G has odd order and δ is the vertex character of a lift), then $\delta_{Q\cap N}$ has a π'-special extension $\lambda \in \mathrm{Irr}(N)$. Note that λ_M is a π'-special character that necessarily lies over $\delta_{Q\cap M} = \beta_{Q\cap M}$. Thus $\lambda_M = \beta$, and therefore β is invariant in N. We now see that $N_\rho = N_\alpha = N_{\rho^o}$.

Let $\mu \in \mathrm{Irr}(G_\rho \mid \rho)$ be the Clifford correspondent for χ. Replacing (Q, δ) by a conjugate if necessary, we can assume (Q, δ) is a vertex for μ. Note that $N_\rho \triangleleft G_\rho$, and let ν be a constituent of μ_{N_ρ}. Then by Theorem 3.1, ν has vertex $(Q \cap N_\rho, \delta_{Q\cap N_\rho}) = (Q \cap N, \delta_{Q\cap N})$. Moreover ν lies over ρ, and thus induces irreducibly to N. Now $e\psi = \chi_N = (\mu^G)_N = (\mu_{N_\rho})^N$, so $\nu^N = \psi$. By induction, ν is a lift.

Now ν induces to ψ and lies over ρ, and ν and ρ are lifts and $N_\rho = N_{\rho^o}$. Thus $\psi^o = (\nu^N)^o = (\nu^o)^N$ and therefore ψ is a lift and we are done. □

4. A Clifford-type correspondence. We will now work towards proving a Clifford-type correspondence for lifts when the order of G is odd. Again, we work in the more general setting of I_π characters, and we recall that for a group G and a π'-subgroup Q of G and a character $\delta \in \mathrm{Irr}(Q)$, we denote by $L(G \mid Q, \delta)$ the set of characters $\chi \in \mathrm{Irr}(G)$ that are lifts and have Navarro vertex (Q, δ).

Theorem 4.1. *Suppose G has odd order and $\chi \in L(G \mid Q, \delta)$. Let $N \triangleleft G$, and let $(P, \lambda) = (Q \cap N, \delta_{Q\cap N})$. Let $T_\lambda = G_\lambda N$. Then there is a unique character $\eta \in L(T_\lambda \mid Q, \delta)$ such that $\eta^G = \chi$.*

Proof. We first prove the existence of η. We can choose a constituent ψ of χ_N such that if $\xi \in \mathrm{Irr}(G_\psi \mid \psi)$ is the Clifford correspondent for χ, then (Q, δ) is a vertex for ξ. Notice that, since χ is a lift, then ξ is a lift. By Theorem 3.1, (P, λ) is a vertex for ψ. If $x \in G_\psi$, then $(P, \lambda)^x = (P, \lambda)^n$ for some $n \in N$, and thus $xn^{-1} \in G_\lambda$ and $G_\psi \subseteq NG_\lambda = T_\lambda$. We let $\eta = \xi^{T_\lambda}$. Since $\chi = \eta^G$, then η is a lift, and since (Q, δ) is a vertex of ξ, then Theorem 2.1 shows that (Q, δ) is a vertex of η. Therefore $\eta \in L(T_\lambda \mid Q, \delta)$.

Now suppose that η_1 and η_2 are in $L(T_\lambda \mid Q, \delta)$ and that $\eta_1^G = \eta_2^G = \chi \in L(G|Q, \lambda)$. Since η_1 and η_2 have vertex (Q, δ), then we know there is a constituent ψ_1 of η_1 and a constituent ψ_2 of η_2 that each have vertex (P, λ). Since ψ_1 and ψ_2 both lie under χ, then there is an element $x \in G$ such that $\psi_1 = \psi_2^x$. Thus $(P, \lambda)^x$ is also a vertex for ψ_2, and therefore $(P, \lambda)^x = (P, \lambda)^n$ for some element $n \in N$, and $xn^{-1} \in G_\lambda$. Now $\psi_1 = \psi_2^x = \psi_2^{xn^{-1}}$, so ψ_1 and ψ_2 are conjugate via an element of $G_\lambda \subseteq T_\lambda$. Therefore, $(\eta_1)_N$ and $(\eta_2)_N$ have a constituent ψ in common. We know that ψ has vertex (P, λ), and as before, we see that $G_\psi \subseteq T_\lambda$. If ξ_1 and ξ_2 are the Clifford correspondents of η_1 and η_2, respectively, in G_ψ, then $\xi_1^G = \xi_2^G = \chi$, and thus $\xi_1 = \xi_2$. Therefore, $\eta_1 = \eta_2$. □

Notice that we are not claiming that if $\eta \in L(T_\lambda \mid Q, \delta)$ then $\eta^G \in L(G \mid Q, \delta)$. In fact, the following counterexample shows that this is not true. Let C be a cyclic group of order 5, and let E be an extra-special group of order 27. Let $F \triangleleft E$ have index 3, and let $\lambda \in \mathrm{Irr}(F)$ be such that $\lambda^E \in \mathrm{Irr}(E)$. Let $\alpha \in \mathrm{Irr}(C)$ be nontrivial. Let $G = C \times E$, and let $\pi = \{5\}$. Note $CF \triangleleft G$, and $T_\lambda = CF$. We see that $\alpha \times \lambda \in L(CF \mid F, \lambda)$, but $(\alpha \times \lambda)^G$ is not a lift. However, if we alter slightly the subgroup we look at, or require a certain condition on λ, we do get a bijection.

Theorem 4.2. *Let G be a group of odd order, let Q be a π'-subgroup of G, let δ be a linear character of Q, and let $N \triangleleft G$ with $(P, \lambda) = (Q \cap N, \delta_{Q\cap N})$. If $T = N\mathbf{N}_G(P)$, then induction is a bijection from $L(T \mid Q, \delta)$ to $L(G \mid Q, \delta)$.*

Proof. We first show that induction is a well-defined map from $L(T \mid Q, \delta)$ to $L(G \mid Q, \delta)$. Suppose that $\eta \in L(T \mid Q, \delta)$. Then Theorem 3.4 of [**3**] shows that η^o induces irreducibly to G, and thus if

$\chi = \eta^G$, then $\chi \in \mathrm{Irr}(G)$ and $\chi^o \in I_\pi(G)$. Moreover, since (Q, δ) is a vertex for η, then Theorem 2.1 implies that (Q, δ) is a vertex for χ.

To show surjectivity, suppose $\chi \in L(G \mid Q, \delta)$. Then Theorem 4.1 implies there is a character $\mu \in L(T_\lambda \mid Q, \delta)$ such that $\mu^G = \chi$. Therefore, $\eta = \mu^T$ is a lift and thus (by Theorem 2.1) η has (Q, δ) as a vertex. Thus, $\eta \in L(T \mid Q, \delta)$ and $\eta^G = \chi$.

To show injectivity, suppose $\chi \in L(G \mid Q, \delta)$, and suppose η_1 and η_2 are in $L(T \mid G, \delta)$ such that $\eta_1^G = \eta_2^G = \chi$. Then Theorem 4.1 applies (in T) to show that there are characters μ_1 and μ_2 in $L(T_\lambda \mid Q, \delta)$ such that $\mu_1^T = \eta_1$ and $\mu_2^T = \eta_2$. Thus, $\mu_1^G = \chi = \mu_2^G$, and the injectivity in Theorem 4.1 shows that $\mu_1 = \mu_2$. Therefore, $\eta_1 = \eta_2$, and we are done. □

As a consequence of Corollary 4.2, we have the following, which is useful in attempting to count lifts. For $\varphi \in I_\pi(G)$, we define $L_\varphi(G \mid Q, \delta)$ to be the set of lifts of φ with vertex (Q, δ).

Corollary 4.3. *Let G be a group of odd order. Let $\varphi \in I_\pi(G)$ have vertex subgroup Q, let $N \triangleleft G$ with $P = Q \cap N$, let $T\mathbf{N}_G(P)$, and let $\xi \in I_\pi(T)$ be the unique character of T with vertex Q that induces to φ. If δ is a linear character of Q, then induction is a bijection from $L_\xi(T \mid Q, \delta)$ to $L_\varphi(G \mid Q, \delta)$.*

The existence and uniqueness of ξ in the statement of the corollary is guaranteed by [**3**, Theorem 3.4].

Proof. Theorem 4.2 shows that induction is a bijection from $L(T \mid Q, \delta)$ to $L(G \mid Q, \delta)$. Theorem 3.4 of [**3**] shows that if $\eta \in L(T \mid Q, \delta)$ is such that $\eta^G = \chi \in L(G \mid Q, \delta)$, then η is a lift of ξ if and only if χ is a lift of φ.

Corollary 4.4. *With the assumptions in Theorem 4.2, if $\lambda \in \mathrm{Irr}(P)$ is invariant in $\mathbf{N}_G(P)$, then induction is a bijection from $L(T_\lambda \mid Q, \delta)$ to $L(G \mid Q, \delta)$.*

Proof. Since λ is invariant in $\mathbf{N}_G(P)$, then $T = T_\lambda$, and we are immediately done by the above corollary. □

5. A counterexample and some open questions. We now provide a counterexample that shows that if G is a group of odd order and χ is a lift in G, then the constituents of χ_N for some $N \triangleleft G$ need not be lifts.

Let V be an elementary Abelian group of order 5^3, and let E be an extra special group of order 3^3. Choose a normal subgroup A of E of order 9, define an action of E on V by letting A act trivially on V, and let E/A permute the three cyclic factors of V.

Define G to be the semi-direct product of E acting on V. Let $\lambda \in \mathrm{Irr}\,(V)$ be a character not fixed by E, and let $\mu \in \mathrm{Irr}\,(A)$ be a linear character of A such that $\mu^E \in \mathrm{Irr}\,(E)$. Note that $VA = V \times A$, and define $\chi \in \mathrm{Irr}\,(G)$ by $\chi = (\lambda \times \mu)^G$. Then χ is irreducible and χ^o is a 5-Brauer character of G, since χ restricts irreducibly to E.

Let B be a normal subgroup of E of index 3 such that $A \neq B$. Then B does not fix λ, and thus if $N = VB \triangleleft G$, then χ restricts irreducibly to N. But the 5-Brauer characters of N are linear, and therefore χ_N is not a lift.

We end with a mention of possible future research. Corollary 3.2 shows that if G has odd order and $\chi \in \mathrm{Irr}\,(G)$ is a lift, then the constituents of χ_N are lifts if G/N is assumed to be a π'-group. The above counterexample shows that this is not true if G/N is a π-group. However, in a forthcoming paper, we will show that if (Q, δ) is a vertex of χ, and if $(P, \lambda) = (Q \cap N, \delta_{Q \cap N})$, then the constituents of χ_N are lifts if λ is invariant in $\mathbf{N}_G(P)$. Corollary 4.4 would seem to be a useful result to study the more general case where we make no assumption on the invariance of λ in $\mathbf{N}_G(P)$, as it allows one to reduce to the case that $T_\lambda = G$.

Also, it can be shown that if $N \triangleleft G$ (where G has odd order) and G/N is a π-group and $\psi \in \mathrm{Irr}\,(N)$ is a lift with vertex (P, λ), then a sufficient condition for the existence of lifts in $\mathrm{Irr}\,(G)$ lying above ψ is that λ is invariant in $\mathbf{N}_G(P)$. If G/N is a π'-group, a similar result can be shown. Again, it is unknown what is true in the general case that N is an arbitrary normal subgroup of G.

Of course, we have assumed throughout that G has odd order. While many of our main results are not true without this assumption, it is certainly possible that there are similar statements that may be true for arbitrary solvable or π-separable groups.

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Department of Theoretical and Applied Mathematics, University of Akron, Akron, OH 44325
Email address: cossey@uakron.edu