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# HYPERCYCLICITY AND SUPERCYCLICITY OF $m$-ISOMETRIC OPERATORS

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ABSTRACT. An operator $T$ defined on a Hilbert space $\mathcal{H}$, satisfying the equation $\sum_{k=0}^{m}(-1)^k \binom{m}{k} T^{*m-k}T^{m-k} = 0$, is called an $m$-isometry. In this paper, we prove that the orbits of vectors under $m$-isometries are eventually norm increasing. Also, it is shown that power bounded $m$-isometries are, in fact, isometries. Moreover, we show that all $m$-isometries are neither supercyclic nor weakly hypercyclic.

**1. Introduction.** Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and $\mathcal{B}(H)$ the space of all bounded linear operators on $\mathcal{H}$. Throughout this article, by an operator we mean a bounded linear operator on $\mathcal{H}$.

For an operator $T \in \mathcal{B}(\mathcal{H})$, if there exists an element $x \in \mathcal{H}$ such that $\operatorname{orb}(T, x) = \{T^n x : n \geq 0\}$ is dense (weakly dense) in $\mathcal{H}$, then $x$ is called a *hypercyclic* (*weakly hypercyclic*) *vector* for $T$, and $T$ is said to be a *hypercyclic* (*weakly hypercyclic*) *operator*. Obviously, every hypercyclic operator is weakly hypercyclic, but not vice-versa. Chan and Sanders in [**7**] have shown that the operator $T$ on $l^p(\mathbf{Z})$, $2 \leq p < \infty$, with standard basis $\{e_n\}_n$, defined by $Te_n = e_{n-1}$ if $n \leq 0$ and $Te_n = 2e_{n-1}$ if $n \geq 1$, is a weakly hypercyclic operator that fails to be norm hypercyclic. A vector $x \in \mathcal{H}$ is called a *supercyclic vector* for $T$ if the set $\{\lambda T^n x : \lambda \in \mathbf{C}, n \geq 0\}$ is dense in $\mathcal{H}$, and $T$ is said to be a *supercyclic operator*.

An important class of operators is the class of isometries, which are not hypercyclic, because, for every isometry $T$, $\sup_n \|T^n\| < \infty$; see [**8, 11**]. They are not even supercyclic. The authors in [**10**] proved this fact for isometries on Hilbert spaces of dimension greater than 1. After that, Ansari and Bourdon in [**5**] and Miller in [**12**] generalized their result to isometries on Banach spaces.

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A larger class is the class of $m$-isometries. An operator $T \in \mathcal{B}(\mathcal{H})$ is called an *$m$-isometry* ($m \in \mathbf{N}$), if

$$(yx-1)^m(T) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} T^{*^k} T^k = 0.$$

Since $(yx-1)^m(T)$ is a self-adjoint operator, we observe that $T$ is an $m$-isometry if and only if for every $x \in \mathcal{H}$,

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} ||T^k x||^2 = 0.$$

Obviously, a 1-isometry is an isometry. Furthermore, every $m$-isometry is an $(m+1)$-isometry; but the inclusion is strict. In fact, in [**6**], Athavale has introduced examples of $(m+1)$-isometries, which are not $m$-isometries. Some of the $m$-isometries are unitarily equivalent to multiplication by $z$. Indeed, Agler and Stankus in [**1, 2, 3**] have considered the differential operator $D = (1/i)(d/d\theta)$ on $C^\infty(\partial\mathbf{D})$, the Fréchet space of infinitely differentiable functions on the unit circle, and defined a class of operators which are periodic distributions. They have shown that all finitely cyclic $m$-isometries can be considered as the operators of multiplication by $z$ on the completion of a pre-Hilbert space of analytic functions with respect to a norm defined by periodic distributions. So, taking this in mind, we see that the study of operator theoretic aspects of $m$-isometries are worthy of attention. Some nice properties of $m$-isometries are also discussed in [**9, 13**]. In [**14**], it is proved that $m$-isometric composition operators on many Hilbert spaces of analytic functions, including Hardy spaces on the ball and polydisc, and weighted Bergman spaces on the disc, are isometries. Furthermore, under some conditions on the symbols of $m$-isometric composition operators on Dirichlet space, they become isometries. Consequently, these operators are not supercyclic or hypercyclic. It is natural to ask whether hypercyclicity and supercyclicity of $m$-isometries are similar to that of isometries. Non-supercyclicity of 2-isometries is discussed in [**4**]. One of our aims is to generalize this result to all $m$-isometries. Let $T \in \mathcal{B}(\mathcal{H})$ be an $m$-isometry. Using the notation of [**1**], take $\beta_\ell(T) = (1/l!)(yx-1)^\ell(T)$, for $\ell \geq 0$, and define the covariance operator of $T$ by $\Delta_T = \beta_{m-1}(T)$. It is proved that, for an $m$-isometry, the covariance operator is, in fact, a positive operator. Also, in Lemma 1.21 of [**1**], it is shown that an $m$-isometry is one-to-one and has closed range. For 2-isometries we have more; if $T$ is a 2-isometry, then $T^*T - I \geq 0$, which implies that $||Tx|| \geq ||x||$, for all $x \in \mathcal{H}$. This means that $T$ is bounded below by 1.

In Section 2 of this paper, we present some new properties of $m$-isometries. It is proved that the orbits of elements in $\mathcal{H}$ under $m$-isometries are norm increasing, after deleting possibly a finite number of terms. Then it is shown that power bounded $m$-isometries are isometries. Furthermore, we will prove that all $m$-isometries are not supercyclic.

In Section 3, we turn our attention to weak hypercyclicity and prove that $m$-isometries are never weakly hypercyclic.

**2. Supercyclicity of $m$-isometries.** The authors have shown in [**4**] that 2-isometries are not supercyclic. The question that is posed runs as follows:

**Question.** Are there supercyclic $m$-isometries with $m > 2$ ?

In this section, we give a negative answer to this question. Indeed, the supercyclic behavior of $m$-isometries is similar to that of isometries. First, we prove some properties of $m$-isometries.

Let $T$ be an $m$-isometry. The symbol of $T$ which is defined by $S_T(n) = T^{*n}T^n$, $n \geq 0$, can be considered as the formal difference series

$$S_T(n) = \sum_{k=0}^{m-1} n^{(k)}\beta_k(T) \tag{1}$$

where $n^{(k)} = 1$, for $n = 0$ or $k = 0$, and otherwise $n^{(k)} = n(n-1)\cdots(n-k+1)$.

An important property of the orbits of $m$-isometries is discussed in the following theorem.

**Theorem 1.** *Let $m$ be an arbitrary positive integer. Suppose that $T \in \mathcal{B}(\mathcal{H})$ is an $m$-isometry and $x \in \mathcal{H}$. Then, except possibly for a finite number of terms,* $\operatorname{orb}(T,x)$ *is norm increasing.*

*Proof.* If $m=1$, the result is obvious. So let $m>1$. Take $x \in \mathcal{H}$ and suppose that $\langle \beta_j(T)x, x\rangle = 0$ for $j=1,2,\ldots,m-1$. Then (1) implies that

$$\|T^{n+1}x\|^2 - \|T^n x\|^2 = \langle (S_T(n+1) - S_T(n))x, x\rangle = 0,$$

for every positive integer $n$, and the result holds. Otherwise, let $k$ be the largest integer with $1 \le k \le m-1$ such that $\langle \beta_k(T)x, x\rangle \neq 0$. We claim that $\langle \beta_k(T)x, x\rangle > 0$. It clearly holds for $k=m-1$. So suppose that $1 \le k < m-1$. Define the operators $T_j$ $(0 \le j \le m-1)$ inductively by

$$\begin{cases} T_0 = T, \\ T_j = T_{j-1}|_{\ker \beta_{m-j}(T_{j-1})}. \end{cases}$$

For $j = 1,\ldots, m-(k+1)$, Proposition 1.6 of [**1**] implies that $\ker \beta_{m-j}(T_{j-1})$ is invariant for $T_{j-1}$; furthermore, $T_j$ is an $m-j$-isometry which, in turn, implies that $\beta_{m-j-1}(T_j)$ is a positive operator. This coupled with the fact that $\langle \beta_{m-j}(T_{j-1})x, x\rangle = \langle \beta_{m-j}(T)x, x\rangle = 0$, shows that $x \in \ker \beta_{m-j}(T_{j-1})$. Especially, $T_{m-k-1}$ is a $k+1$-isometry, and so $\beta_k(T_{m-k-1})$ is a positive operator. Moreover, since the vector $x \in \ker \beta_{k+1}(T_{m-k-2})$ and we assumed that $\langle \beta_k(T)x, x\rangle \neq 0$, we get

$$\langle \beta_k(T_{m-k-1})x, x\rangle = \langle \beta_k(T)x, x\rangle > 0.$$

Now, for every positive integer $n$, using (1), we observe that

$$\begin{aligned}\|T^{n+1}x\|^2 - \|T^n x\|^2 &= \langle (S_T(n+1) - S_T(n))x, x\rangle \\ &= \sum_{i=0}^{k} [(n+1)^{(i)} - n^{(i)}]\langle \beta_i(T)x, x\rangle.\end{aligned}$$

Consequently,

$$\lim_{n\longrightarrow\infty} \frac{\|T^{n+1}x\|^2 - \|T^n x\|^2}{(n+1)^{(k)} - n^{(k)}} = \langle \beta_k(T)x, x\rangle = \langle \beta_k(T_{m-k-1})x, x\rangle > 0;$$

hence, there exists a positive integer $n_0$ so that the sequence $\{\|T^n x\|\}_{n\ge n_0}$ is strictly increasing. □

It is worthy of attention to note that the proof of Theorem 1 shows that for an $m$-isometry $T$ with $m>1$ two cases may be considered:

(i) If $\langle \beta_k(T)x, x\rangle = 0$ for all $k$ with $1 \le k \le m-1$, then $\|T^n x\| = \|x\|$ for all $n \ge 1$.

(ii) If the largest integer $k$ exists with $1 \le k \le m-1$ such that $\langle \beta_k(T)x, x\rangle \ne 0$, then $\{\|T^n x\|\}_{n=1}^{\infty}$ is eventually strictly increasing.

**Theorem 2.** *Suppose that $T \in \mathcal{B}(H)$ is an $m$-isometry, and $x \in \mathcal{H}$. If for a strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers there exists a constant $M$ such that $\|T^{n_i} x\| \le M$ for $i = 1, 2, \dots$, then $\|T^n x\| = \|x\|$ for all nonnegative integers $n$.*

*Proof.* The result is clear, when $T$ is an isometry. So let $m > 1$. Using (1), we see that

$$\|T^{n_i} x\|^2 = \sum_{k=0}^{m-1} n_i^{(k)} \langle \beta_k(T)x, x\rangle, \quad i = 1, 2, \dots$$

which is not greater than $M$, by the hypothesis. On the other hand, if $\langle \beta_k(T)x, x\rangle \ne 0$ for some $k$ with $1 \le k \le m-1$, then

$$\lim_{i \longrightarrow \infty} \sum_{k=0}^{m-1} n_i^{(k)} \langle \beta_k(T)x, x\rangle = +\infty,$$

which is a contradiction. Hence, $\langle \beta_k(T)x, x\rangle = 0$ for every $k$ with $1 \le k \le m-1$, and so $\|T^n x\| = \|x\|$ for $n = 0, 1, 2, \dots$, by Remark (i) stated above. □

Recall that an operator $T \in B(\mathcal{H})$ is power bounded, if the sequence $\{\|T^n\|\}_{n=1}^{\infty}$ is bounded. The following result is an important useful consequence of the above theorem.

**Corollary 1.** *Suppose that $T$ is an $m$-isometry and a subsequence of $\{\|T^n\|\}_{n=1}^{\infty}$ is bounded. Then $T$ is an isometry. In particular, every power bounded $m$-isometry is an isometry.*

Thus, power bounded $m$-isometries cannot be supercyclic.

**Lemma 1.** *If $T$ is an invertible $m$-isometry, then so is $T^{-1}$.*

*Proof.* Since $T$ is an $m$-isometry,

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}||T^k x||^2 = 0.$$

Substitute $x$ by $T^{-m}y$ to get the result. □

Corollary 1 states that if $T$ is an $m$-isometry but not an isometry, then $\|T^n\| \to \infty$ as $n \to \infty$. Furthermore, we have the following result on the norms of integer powers of $T$.

**Corollary 2.** *Suppose that $T$ is an $m$-isometry, which is not an isometry. Then $\|T^n\| > 1$, for all $n \geq 1$, and if $T$ is invertible, $\|T^{-n}\| > 1$ for all $n \geq 1$.*

*Proof.* Assume, on the contrary, that there exists a positive integer $k$ such that $\|T^k\| \leq 1$. Then for all positive integers $n$, $\|T^{nk}\| \leq 1$, and so considering the previous corollary, $T$ must be an isometry, which is a contradiction.

Moreover, if $T^{-1}$ exists, then Lemma 1 along with the above argument for $T^{-1}$ instead of $T$, proves the last part of the corollary. □

Now, we are ready to prove the main result of this section.

**Theorem 3.** *Every $m$-isometry is not supercyclic.*

*Proof.* Assume that $T \in \mathcal{B}(H)$ is a supercyclic $m$-isometry, with a supercyclic vector $y \in \mathcal{H}$. For each $x \in \mathcal{H}$, there is a sequence of positive integers $\{k_i\}$ and a sequence of scalars $\{\lambda_i\}_i$ so that

$$\lim_{i\to\infty} \lambda_i T^{k_i} y = x.$$

Now, considering Theorem 1, orb$\,(T, y)$ (after deleting, possibly, a finite number of terms) is norm increasing; then

$$\|x\| = \lim_{i\to\infty} |\lambda_i| \cdot \|T^{k_i} y\| \leq \lim_{i\to\infty} |\lambda_i| \cdot \|T^{k_i+1} y\| = \|Tx\|.$$

The above inequality and the fact that the operator $T$ has a dense range proves the existence of $T^{-1}$ and that $T^{-1}$ is power bounded. By Lemma 1 and the fact that the inverse of a supercyclic operator is supercyclic, we get that $T^{-1}$ is a power bounded, supercyclic $m$-isometry. However, Corollary 1 gives us that $T^{-1}$ is, in fact, a supercyclic isometry which gives us our contradiction; see [**5, 12**]. $\square$

**3. Weak hypercyclicity of $m$-isometries.** A direct consequence of Theorem 3 states that all $m$-isometries are not hypercyclic. In the present section, we show that they are not even weakly hypercyclic. The proof of our main result is based upon the following proposition due to Shkarin.

**Proposition 1** [**15**]. *Let $\mathcal{H}$ be a Hilbert space and $\{x_n\}_{n=0}^{\infty}$ a sequence of elements of $\mathcal{H}$, such that $\sum_{n=0}^{\infty} \|x_n\|^{-2} < \infty$. Then the set $\{x_n : n = 0, 1, 2, \cdots\}$ is weakly closed in $\mathcal{H}$.*

If $T$ is an isometry and $x \in \mathcal{H}$, then $\operatorname{orb}(T, x)$ lies in the ball $B(0, \|x\|)$, and so $T$ cannot be weakly hypercyclic. Suppose that $T$ is a 2-isometry. Then $T$ is bounded below. If we, further, assume that $T$ is weakly hypercyclic, then $T$ will be a dense range operator. Hence, $T$ is invertible and Lemma 1 implies that $T^{-1}$ is also a 2-isometry. Consequently, $T$ is an isometry, which is impossible. Thus, neither isometries nor 2-isometries are weakly hypercyclic.

**Theorem 4.** *Every $m$-isometry is not weakly hypercyclic.*

*Proof.* We have already seen that the result holds for $m = 1, 2$. Let $T \in B(\mathcal{H})$ and $m > 2$. Assume, on the contrary, that $T$ is a weakly hypercyclic $m$-isometry with a weakly hypercyclic vector $x$. Then

$$\|T^n x\|^2 = \sum_{k=0}^{m-1} n^{(k)} \langle \beta_k(T)x, x\rangle, \quad n = 0, 1, 2, \ldots .$$

If $\beta_{m-1}(T)x \neq 0$, then the positivity of $\beta_{m-1}(T)$ shows that $\langle \beta_{m-1}(T)x, x\rangle > 0$. This, in turn, implies the convergence of the series $\sum_{n=1}^{\infty} \|T^n x\|^{-2}$. Thus, in view of the preceding proposition, we get a contradiction.

Hence, $\beta_{m-1}(T)x = 0$. Since, for every $n$, $T^n x$ is also a weakly hypercyclic vector for $T$, we see that

$$\beta_{m-1}(T)(T^n x) = 0, \quad n = 0, 1, 2, \dots .$$

This along with the fact that $\ker \beta_{m-1}(T)$ is weakly closed, implies that $\beta_{m-1}(T) = 0$. Hence, $T$ is an $m-1$-isometry. Continuing the above process, we finally conclude that $T$ is a 2-isometry, which is impossible. $\square$

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