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PERMANENCE IN GENERAL NON-AUTONOMOUS SINGLE-SPECIES KOLMOGOROV SYSTEMS WITH PURE DELAYS AND FEEDBACK CONTROL

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ABSTRACT. In this paper we consider whether or not feedback control has influence on a non-autonomous single-species Kolmogorov system with pure delays. In the case of a general domain, the general criterion on permanence is established, which is described by integrable form and independence of feedback control. As applications of these results, sufficient conditions on permanence are obtained for a series of special single-species systems with pure delays and feedback control.

1. Introduction. Let $\tau \geq 0$ be a constant, $R = (-\infty, \infty)$, $R_+ = (0, \infty)$ and $R_{+0} = [0, \infty)$. We define $C[-\tau, 0]$ to be the Banach space of bounded continuous functions $\phi : [-\tau, 0] \to R$ with the supremum norm defined by

$$\|\phi\|_c = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|.$$

Define $C_+[-\tau, 0] = \{\phi \in C[-\tau, 0] : \phi(\theta) \geq 0 \text{ and } \phi(0) > 0 \text{ for all } \theta \in [-\tau, 0]\}$. The aim of this paper is to investigate the following general non-autonomous single species Kolomogorov system with pure-delays and feedback control

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= x(t)f(t, x_t, u_t) \\ \frac{\mathrm{d}u(t)}{\mathrm{d}t} &= -\eta(t)u(t) + g(t, x_t). \end{aligned} \tag{1.1}$$

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System (1.1) includes many well known single species non-autonomous population growth models with pure delays and feedback control as its specific case, for example:

(1) Non-autonomous logistic system with delays and feedback control

$$\begin{aligned}\frac{\mathrm{d}x(t)}{\mathrm{d}t} &= x(t)\Big[b(t) - \sum_{i=1}^{n} a_i(t)x(t-\tau_i(t)) \\ &\quad - \sum_{j=1}^{m}\int_{-\gamma_j}^{0} K_j(t,s)x(t+s)\,\mathrm{d}s - c(t)u(t-\delta(t))\Big] \\ \frac{\mathrm{d}u(t)}{\mathrm{d}t} &= r(t) - \eta(t)u(t) + \sum_{l=1}^{q} d_l(t)x(t-\sigma_l(t)).\end{aligned} \tag{1.2}$$

(2) Non-autonomous multiplicative delayed logistic system with feedback control

$$\begin{aligned}\frac{\mathrm{d}x(t)}{\mathrm{d}t} &= x(t)\Big[b(t) - \prod_{i=1}^{n}\frac{x(t-\tau_i(t))}{K(t)} - c(t)u(t-\delta(t))\Big] \\ \frac{\mathrm{d}u(t)}{\mathrm{d}t} &= -\eta(t)u(t) + d(t)x(t-\sigma(t)).\end{aligned} \tag{1.3}$$

(3) Non-autonomous delayed Michaelis-menton system with feedback control

$$\begin{aligned}\frac{\mathrm{d}x(t)}{\mathrm{d}t} &= b(t)x(t)\Big[1 - \sum_{i=1}^{n}\frac{a_i(t)x(t-\tau_i(t))}{b_i(t)+c_i(t)x(t-\tau_i(t))} - c(t)u(t-\delta(t))\Big] \\ \frac{\mathrm{d}u(t)}{\mathrm{d}t} &= -\eta(t)u(t) + d(t)x(t-\sigma(t)).\end{aligned} \tag{1.4}$$

(4) Non-autonomous delayed Allee-effect system with feedback control

$$\begin{aligned}\frac{\mathrm{d}x(t)}{\mathrm{d}t} &= x(t)\Big[b(t) - \sum_{i=1}^{n} a_i(t)x^{\alpha_i}(t-\tau_i(t)) \\ &\quad - a(t)\int_{-\gamma}^{0} H(s)x^{\beta}(t+s)\,\mathrm{d}s - c(t)u(t-\delta(t))\Big] \\ \frac{\mathrm{d}u(t)}{\mathrm{d}t} &= r(t) - \eta(t)u(t) + \sum_{j=1}^{m} d_j(t)x^{\beta_j}(t-\sigma_j(t)).\end{aligned} \tag{1.5}$$

In past decades, some single-species growth models with finite or infinite pure delays have been extensively studied in many articles (see [**1, 4, 5, 7, 9, 15, 20, 21**] and references cited therein). In particular, in [**26**], Vance and Coddington studied the general non-autonomous single-species Kolmogorov system without time delay. They established a series of sufficient conditions on boundedness, persistence, permanence, global asymptotic stability and the existence of positive periodic solutions (see [**26**, Theorems 1–6]). In [**22**], Teng considered the general non autonomous single-species Kolmogorov system with pure delays and established a series of very general and rather weak criteria of integrable form for boundedness, persistence, permanence, global asymptotic stability and the existence of positive periodic solutions.

Ecosystems in the real world are continuously distributed by unpredictable forces which can result in changes in biological parameters such as survival rates. In ecology, a practical question is whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time. In the language of control variables, we call the disturbance functions, *control variables*. Control variables discussed in most literature are constants or time dependent [**9–11**].

Recently, we see that the dynamic behaviors for the single-species or multi-species population equation with time delays and feedback controls are studied in [**2, 3, 6, 8, 12–14, 16–19, 27**], where sufficient conditions on boundedness, permanence, global stability and the existence of positive periodic solutions and positive almost periodic solutions are obtained. However, we note that few authors consider whether or not feedback control has an influence on the permanence of system (1.1). At the same time, we also note that there are almost no studies on whether or not the feedback control has influence on the permanence of the non-autonomous logistic system with several times delays and feedback control, non-autonomous multiplicative delayed logistic system with feedback control, non-autonomous delayed Michaelis-Menton system with feedback control, non-autonomous delayed Allee-effect system with feedback control, etc.

Motivated by the above questions, in this paper we study whether or not feedback control has influence on permanence for the general single-species Kolmogorov systems with pure delays, and establish the general criteria on ultimate boundedness and permanence of all positive solutions, which are described by integrable form and independence of feedback control.

This paper is organized as follows. We present two important lemmas on the single species non-autonomous system in Section 2. In Section 3, we state and prove a general theorem for the permanence of system (1.1). In the last section, as applications of the above criterions, we study the permanence of the special system (1.2)–(1.5).

2. Preliminaries. In this section, we consider the following first order differential equation with a parameter

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = g(t,\alpha) - b(t)v(t), \tag{2.1}$$

where $g(t,\alpha)$ is a continuous function defined on $(t,\alpha) \in R_{+0} \times [0,\ \alpha_0]$ and α_0 is a constant, $b(t)$ is a continuous function defined on R_{+0}. For system (2.1) we introduce the following assumptions.

(A_1) Function $g(t,\alpha)$ is a non-negative bounded on $R_{+0} \times [0,\ \alpha_0]$ and satisfies the Lipschitz condition with $\alpha \in [0,\ \alpha_0]$, i.e., there is a constant $L = L(\alpha_0) > 0$ such that $|g(t,\alpha_1) - g(t,\alpha_2)| \le L|\alpha_1 - \alpha_2|$ for all $t \in R$ and $\alpha_1,\ \alpha_2 \in [0,\ \alpha_0]$.

(A_2) Function $b(t)$ is non-negative bounded on R_{+0} and there is a constant $\omega_1 > 0$ such that $\liminf_{t\to\infty} \int_t^{t+\omega_1} b(s)\,\mathrm{d}s > 0$.

From assumptions (A_1) and (A_2), it is easy to prove that, for any $(t_0, v_0) \in R_{+0} \times R_+$ and $\alpha \in [0,\ \alpha_0]$, system (2.1) has a unique solution $v_\alpha(t)$ satisfying $v_\alpha(t_0) = v_0$. If $v_\alpha(t) > 0$ on the interval of existence, then $v_\alpha(t)$ is said to be a positive solution. It is easy to see that $v_\alpha(t)$ is positive for all $t \ge t_0$ if the initial value $v_0 > 0$.

In system (2.1), when parameter $\alpha = 0$, we obtain the following system

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = g(t,0) - b(t)v(t). \tag{2.2}$$

Let $v_\alpha^*(t)$ be a fixed positive solution of system (2.1) defined on R_{+0}. We say that $v_\alpha^*(t)$ is globally uniformly attractive on R_{+0} if, for any constants $\eta > 1$ and $\varepsilon > 0$, there is a positive constant $T = T(\eta,\varepsilon) > 0$ such that for any initial time $t_0 \in R_{+0}$ and any solution $v_\alpha(t)$ of system (2.1) with $v_\alpha(t_0) \in [\eta^{-1},\eta]$, one has $|v_\alpha(t) - v_\alpha^*(t)| < \varepsilon$ for all $t \ge t_0 + T$. By Lemma 1 given in [**24**], we have the following result.

Lemma 2.1. *Suppose that assumptions* (A_1) *and* (A_2) *hold. Then,*

(a) *there is a constant* $M > 0$ *such that* $\limsup_{t\to\infty} v_\alpha(t) \le M$ *for any positive solution* $v_\alpha(t)$ *of system* (2.1).

(b) *If there is a constant* $\omega_2 > 0$ *such that* $\liminf_{t\to\infty} \int_t^{t+\omega_2} g(s,\alpha)\,\mathrm{d}s > 0$ *for all* $\alpha \in [0,\, \alpha_0]$*, then there is a constant* $\eta > 1$ *such that*

$$\eta^{-1} \le \liminf_{t\to\infty} v_\alpha(t) \le \limsup_{t\to\infty} v_\alpha(t) \le \eta$$

for any positive solution $v_\alpha(t)$ *of system* (2.1).

(c) *Each fixed positive solution* $u_\alpha^*(t)$ *of system* (2.1) *is globally uniformly attractive on* R_{+0}.

Let $v_0 \in R_+$, $t_0 \in R_{+0}$, $\alpha \in [0,\, \alpha_0]$, and $v_\alpha(t)$, $v_0(t)$ be the solutions of systems (2.1) and (2.2) with initial value $v_\alpha(t_0) = v_0$ and $V_0(t_0) = v_0$, respectively. By [**24**, Lemma 2], we have

Lemma 2.2. *Suppose that assumptions* (A_1) *and* (A_2) *hold. Then* $v_\alpha(t)$ *converges to* $v_0(t)$ *uniformly for* $t \in [t_0,\, \infty)$ *as* $\alpha \to 0$.

Remark 2.1. In system (2.2), if function $g(t,0) \equiv 0$, then system (2.2) has a trivial equilibria $E = 0$, and E is globally asymptotically stable. For any $\Gamma > 1$ and $t_0 \in R_{+0}$, let $\alpha \in [0,\, \alpha_0]$, and $v_\alpha(t)$ be the positive solution of system (2.1) with initial value $v_\alpha(t_0) \in [\Gamma^{-1},\, \Gamma]$. By Lemmas 2.1 and 2.2, we further have the following result: solution $v_\alpha(t)$ converges to 0, as $\alpha \to 0$ and $t \to \infty$, i.e., for any $\varepsilon > 0$, there are positive constants $T = T(\varepsilon, \Gamma)$ and $\delta = \delta(\varepsilon)$ such that $v_\alpha(t) < \varepsilon$ for all $t \ge t_0 + T$ and $\alpha < \delta$.

For the convenience of statements which follow in this paper, we introduce the definition on permanence.

Definition 2.1. System (1.2) is said to be permanent, if there are positive constants m and M such that

$$m \le \liminf_{t\to\infty} x(t) \le \limsup_{t\to\infty} x(t) \le M$$

for any positive solution $(x(t), u(t))$ of system (1.2).

Remark 2.2. In system (1.2), $u(t)$ is control variable, so we do not consider the permanence of control variable.

3. Main results. For system (1.1), we first introduce the assumptions.

(H_1) Function $f(t, x_t, u_t)$ satisfies the following conditions.

(1) The function $f(t, \phi, \psi)$ is decreasing with respect to $(\phi, \psi) \in C_+ \times C_+$, i.e., for any (ϕ_1, ψ_1), $(\phi_2, \psi_2) \in C_+ \times C_+$, if $\phi_1 \leq \phi_2$ and $\psi_1 \leq \psi_2$, then $f(t, \phi_1, \psi_1) \geq f(t, \phi_2, \psi_2)$ for all $t \in R$.

(2) For any constants $x_1 \geq 0$, $x_2 \geq 0$, function $f(t, x_1, x_2)$ is bounded on R_{+0}, and there are positive constants k and ω, such that

$$\limsup_{t \to \infty} \int_t^{t+\omega} f(s, k, 0)\, \mathrm{d}s < 0.$$

(H_2) Function $g(t, \phi)$ satisfies the following conditions.

(1) For any constant $h \geq 0$, function $g(t, h)$ is non-negative and bounded.

(2) The function $g(t, \phi)$ is increasing with respect to $\phi \in C_+$ and satisfies the Lipschitz condition with $\phi \in C_+$.

(H_3) The function $\eta(t)$ is non-negative bounded on R_{+0}, and there is a constant $\mu > 0$ such that $\liminf_{t \to \infty} \int_t^{t+\mu} \eta(s)\, \mathrm{d}s > 0$.

For any $(t_0, \phi, \psi) \in R_+ \times C_+ \times C_+$, it is well known by the fundamental theory of functional differential equations (see [**4, 25**]) that system (1.1) has a unique solution $X(t) = (x(t), u(t))$, which is through (t_0, ϕ, ψ) and continuous. It is easy to verify that solutions of system (1.1) are defined on $[0, \infty)$ and remain positive for all $t \geq 0$, if the initial value $(t_0, \phi, \psi) \in R_+ \times C_+ \times C_+$.

In this section, we proceed to discussion on the ultimate boundedness and permanence of any positive solution of system (1.1).

Firstly, on the ultimate boundedness of any positive solution for system (1.1), we have the following result.

Theorem 3.1. *Suppose that assumptions* (H_1)–(H_3) *hold. Then system* (1.1) *is ultimately bounded, in the sense that there are positive constants M and T such that, if $t \geq T$, then $x(t) \leq M$ and $u(t) \leq M$ for all positive solutions $X(t) = (x(t), u(t))$ of system* (1.1).

Proof. Let $X(t) = (x(t), u(t))$ be any positive solution of system (1.1). We first prove that component x of system (1.1) is ultimately bounded. From condition (2) of (H_1), there are positive constants T_0 and μ such that for all $t \geq T_0$

$$\int_t^{t+\omega} f(s, k, 0)\,\mathrm{d}s < -\mu. \tag{3.1}$$

By condition (1) of (H_1), we have

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \leq x(t) f(t, 0, 0) \leq \beta_1 x(t), \tag{3.2}$$

where $\beta_1 = \max_{t\geq 0}\{|f(t, 0, 0)|\}$. For any $t \geq \tau$ and $\theta \in [-\tau,\ 0]$, integrating (3.2) from $t + \theta$ to t, we obtain that

$$x(t + \theta) \geq x(t) \exp(\beta_1 \theta) \geq x(t) \exp(-\beta_1 \tau). \tag{3.3}$$

Further, from condition (1) of (H_1), we have

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \leq x(t) f(t, x(t) \exp(-\beta_1 \tau), 0). \tag{3.4}$$

We now consider the following auxiliary system

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = y(t) f(t, y(t) \exp(-\beta_1 \tau), 0). \tag{3.5}$$

Let $y(t)$ be the solution of system (3.5) with the initial condition $y(T_1) = x(T_1)$, where $T_1 = \max\{\tau, T_0\}$. By the comparison theorem of ordinary differential equations, we further obtain from (3.4)

$$x(t) \leq y(t) \quad \text{for all } t \geq T_1. \tag{3.6}$$

Next, we prove that there is a $T_2 \geq T_1$ such that, for all $t \geq T_2$,

$$y(t) \leq k^* \exp(\beta_1 \omega),$$

where $k^* = k \exp(\beta_1 \tau)$. If $y(t) \geq k^*$ for all $t \geq T_1$, then by condition (1) of (H_1), we have

$$\begin{aligned} y(t) &= y(T_1) \exp \int_{T_1}^t f(s, y(t) \exp(-\beta_1 \tau), 0)\,\mathrm{d}s \\ &\leq y(T_1) \exp \int_{T_1}^t f(s, k, 0)\,\mathrm{d}s \end{aligned}$$

for all $t \geq T_1$. From (3.1), we easily obtain $y(t) \to 0$ as $t \to \infty$, which is a contradiction. Hence, there is a positive constant $t_1 \geq T_1$ such that $y(t_1) < k^*$. Further, if there are $t_3 > t_1$ such that $y(t_3) > k^* \exp(\beta_2\omega)$, then there is a $t_2 \in (t_1,\, t_3)$ such that $y(t_2) = k^*$ and $y(t) > k^*$ for all $t \in (t_2,\, t_3]$. Therefore, we can choose an integer $p \geq 0$ such that $t_3 \in (t_2 + p\omega,\, t_2 + (p+1)\omega]$ and obtain

$$\begin{aligned}
y(t_3) &= y(t_2) \exp \int_{t_2}^{t_3} f(t, y(t) \exp(-\beta_1\tau), 0)\, \mathrm{d}t \\
&= y(t_2) \exp \left(\int_{t_2}^{t_2+\omega} + \cdots + \int_{t_2+(p-1)\omega}^{t_2+p\omega} + \int_{t_2+p\omega}^{t_3} \right) f(t, y(t) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \exp(-\beta_1\tau), 0)\, \mathrm{d}t \\
&\leq k^* \exp \int_{t_2+p\omega}^{t_3} f(t, k, 0)\, \mathrm{d}t \\
&\leq k^* \exp(\beta_1\omega),
\end{aligned}$$

which is also a contradiction. So, we have $y(t) \leq k^* \exp(\beta_1\omega)$ for all $t \in [t_1,\, \infty)$. By (3.6), we finally obtain that

$$x(t) \leq k^* \exp(\beta\omega) = k \exp(2\beta_1\tau) \quad \text{for all } t \geq t_1.$$

Further, from the second equation of system (1.1) we have

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} \leq -\eta(t)u(t) + g(t, k\exp(2\beta_1\omega))$$

for all $t \geq t_1 + \tau$. Hence, using the comparison theorem of ordinary differential equations and conclusion (a) of Lemma 2.1, we can further obtain that there is a constant $M_1 > 0$ such that for any positive solution $(x(t), u(t))$ of system (1.1) there is a $t^* \geq t_1 + \tau$ such that $u(t) \leq M_1$ for all $t \geq t^*$. Now, we let $M = \max\{M_1, k\exp(2\beta_1\omega)\}$; then for all $t \geq t^*$,

$$0 < x(t) \leq M, \quad 0 < u(t) \leq M.$$

Therefore, the solution $X(t)$ is ultimately bounded. This completes the proof of the theorem. $\square$

In order to obtain the permanence of system (1.1), we next consider the single-species linear system

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\eta(t) + g(t,0). \tag{3.7}$$

By assumptions (H_2) and (H_3), we see that system (3.7) satisfies all the conditions of Lemma 2.1; hence, each positive solution of system (3.7) is globally asymptotically stable. Let $u_0(t)$ be some fixed positive solution of system (3.7). We further introduce the assumption

(H_4) There is a constant $\lambda > 0$ such that

$$\liminf_{t\to\infty} \int_t^{t+\lambda} f(s,0,u_{0t})\,\mathrm{d}s,$$

where $u_{0t} = u_0(t+\theta)$ for all $\theta \in [-\tau,\ 0]$.

Remark 3.1. If $g(t,0) \equiv 0$ in system (3.7), then we choose $u_0(t) \equiv 0$.

Next, on the permanence of components for system (1.1), we have the following theorem.

Theorem 3.2. *Suppose that assumptions* (H_1)–(H_4) *hold. Then system* (1.1) *is permanent.*

Proof. Let $X(t) = (x(t), u(t))$ be any positive solution of system (1.1). From Theorem 3.1, there is a constant $M > 0$ such that, for any positive solution $X(t)$ of system (1.1), there is a $T_1 \geq 0$ such that $x(t) \leq M$ and $u(t) \leq M$ for all $t \geq T_1$. Therefore, from the first equation of system (1.1) we have

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &\geq x(t)f(t,M,M) \\ &\geq -\beta_2 x(t) \end{aligned} \tag{3.8}$$

for all $t \geq T_1 + \tau$, where $\beta_2 = \sup_{t\in R_{+0}}\{|f(t,M,M)|\}$. For any $t \geq T_1 + \tau$ and $\theta \in [-\tau,\ 0]$, integrating (3.8) from $t+\theta$ to t, we obtain

$$x(t+\theta) \leq x(t)\exp(-\beta_2\theta) \leq x(t)\exp(\beta_2\tau). \tag{3.9}$$

For any t_1, t_2 and $t_2 \geq t_1 \geq 0$, integrating directly system (1.1) we have

$$x(t_2) = x(t_1) \exp \int_{t_1}^{t_2} f(t, x(t), x_t, u(t), u_t)\, \mathrm{d}t. \tag{3.10}$$

In the following, we will use two claims to complete the proof of Theorem 3.2.

Claim 3.1. *There is a constant* $\alpha > 0$ *such that* $\limsup_{t\to\infty} x(t) > \alpha$ *for any positive solution* $X(t)$ *of system* (1.1).

In fact, by assumption (H_4), we can choose small enough positive constants ε and δ, and a large enough $T_2 \geq T_1$, such that for all $t \geq T_2$

$$\int_t^{t+\lambda} f(s, \varepsilon \exp(\beta_2\tau), u_{0t} + \varepsilon)\, \mathrm{d}s > \delta. \tag{3.11}$$

Next, we consider the following system with a parameter

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\eta(t)u(t) + g(t, \alpha \exp(\beta_2\tau)), \tag{3.12}$$

where $\alpha \in [0,\, \alpha_0]$ is a parameter. Let $u_\alpha(t)$ be the solution of system (3.12) with the initial value $u_\alpha(0) = u_0(0)$, by Lemma 2.1, $u_\alpha(t)$ is globally asymptotically stable, and $u_\alpha(t) \to u_0(t)$, as $\alpha \to 0$ and $t \to \infty$. Hence, there are positive constants α, T_3 and $T_3 \geq T_2$, $\alpha < \varepsilon$ such that

$$u_\alpha(t) < u_0(t) + \frac{\varepsilon}{2} \quad \text{for all } t \geq T_3. \tag{3.13}$$

If Claim 3.1 is not true, then there is positive solution $(x(t), u(t))$ of system (1.1) such that $\limsup_{t\to\infty} x(t) < \alpha$. So, there is a constant $T_4 > T_3$ such that $x(t) < \alpha$ for all $t \geq T_4$. From (3.9) and the second equation of system (1.1) we obtain

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} \leq -\eta(t) + g(t, \alpha, \alpha \exp(\beta_2\tau)) \quad \text{for all } t \geq T_4.$$

Using the comparison theorem of ordinary differential equations and global asymptotical stability of solution $u_\alpha(t)$, we obtain that there is a $T_5 \ge T_4$ such that

$$u(t) < u_\alpha(t) + \frac{\varepsilon}{2} \quad \text{for all } t \ge T_5. \tag{3.14}$$

So, from (3.13) and (3.14) it follows that

$$u(t) < u_0(t) + \varepsilon \quad \text{for all } t \ge T_5. \tag{3.15}$$

By (3.9), (3.10) and (3.15) we obtain

$$\begin{aligned} x(t) &= x(T_5) \exp \int_{T_5}^{t} f(s, x_t, u_{0t})\, \mathrm{d}s \\ &\ge x(T_5) \exp \int_{T_5}^{t} f(s, \varepsilon \exp(\beta_2 \tau), u_{0t} + \varepsilon)\, \mathrm{d}s \end{aligned}$$

for all $t \ge T_5$. Thus from (3.11) we finally obtain $\lim_{t\to\infty} x(t) = \infty$ which leads to a contradiction. Therefore, Claim 3.1 is true.

Claim 3.2. *There is a constant* $\gamma > 0$ *such that* $\liminf_{t\to\infty} x(t) > \gamma$ *for any positive solution* $X(t)$ *of system* (1.1).

In fact, if Claim 3.2 is not true, then there is a sequence of initial value $\{X_n = (\phi_n, \psi_n)\} \subset C_+ \times C_+$ such that, for the solution $(x(t, X_n), u(t, X_n))$ of system (1.1),

$$\liminf_{t\to\infty} x(t, X_n) < \frac{\alpha}{n^2}, \quad n = 1, 2, \dots ,$$

where constant α is given in Claim 3.1. By Claim 3.1, for every n there are two time sequences $\{s_q^{(n)}\}$ and $\{t_q^{(n)}\}$, satisfying $0 < s_1^{(n)} < t_1^{(n)} < s_2^{(n)} < t_2^{(n)} < \dots < s_q^{(n)} < t_q^{(n)} < \cdots$ and $\lim_{q\to\infty} s_q^{(n)} = \infty$, such that

$$x\big(s_q^{(n)}, X_n\big) = \frac{\alpha}{n}, \qquad x\big(t_q^{(n)}, X_n\big) = \frac{\alpha}{n^2}, \tag{3.16}$$

and

$$\frac{\alpha}{n^2} < x(t, X_n) < \frac{\alpha}{n} \quad \text{for all } t \in \big(s_q^{(n)},\ t_q^{(n)}\big). \tag{3.17}$$

From the ultimate boundedness of system (1.1), we can choose a positive constant $T^{(n)}$ for every n such that $x(t, X_n) < M$ and $u(t, X_n) < M$ for all $t > T^{(n)}$. Further, there is an integer $K_1^{(n)} > 0$ such that $s_q^{(n)} > T^{(n)} + \tau$ for all $q > K_1^{(n)}$. Let $q > K_1^{(n)}$; then, for any $t \in [s_q^{(n)}, t_q^{(n)}]$, we have

$$\begin{aligned}\frac{\mathrm{d}x(t, X_n)}{\mathrm{d}t} &\geq x(t, X_n) f(t, M, M) \\ &\geq -\beta_2 x(t, X_n).\end{aligned}$$

Integrating the above inequality from $s_q^{(n)}$ to $t_q^{(n)}$, we further have

$$x(t_q^{(n)}, X_n) \geq x(s_q^{(n)}, X_n) \exp[-\beta_2 (t_q^{(n)} - s_q^{(n)})].$$

Consequently, by (3.16),

$$\frac{\alpha}{n^2} \geq \frac{\alpha}{n} \exp\left[-\beta_2 (t_q^{(n)} - s_q^{(n)})\right].$$

Hence,

$$t_q^{(n)} - s_q^{(n)} \geq \frac{\ln n}{\beta_2} \quad \text{for all } q > K_1^{(n)}. \tag{3.18}$$

By (3.11), there are positive constants P and ϱ such that

$$\int_t^{t+\kappa} f(s, \varepsilon \exp(\beta_2 \tau), u_{0t} + \varepsilon)\, \mathrm{d}s > \varrho \tag{3.19}$$

for all $t \geq 0$ and $\kappa \geq P$.

Let $\widetilde{u}_\alpha(t)$ be the solution of system (3.12) with the initial value $\widetilde{u}_\alpha(t) = u(s_q^{(n)}, X_n)$. By (3.9), (3.17) and condition (1) of (H_2), we have

$$\frac{\mathrm{d}u(t, X_n)}{\mathrm{d}t} \leq -\eta(t) u(t, X_n) + g(t, \alpha \exp(\beta_2 \tau))$$

for any n, q and $t \in [s_q^{(m)}, t_q^{(m)}]$. Using the comparison theorem of ordinary differential equations, we have

$$u(t, X_n) \leq \widetilde{u}_\alpha(t) \quad \text{for all } t \in \left[s_q^{(n)}, t_q^{(n)}\right]. \tag{3.20}$$

By Lemma 2.1, solution $u_\alpha(t)$ is globally uniformly attractive on R_{+0}. We obtain that there is a constant $T_0 \geq P$, and T_0 is independent of any n and $q \geq K^{(n)}$, such that

$$\widetilde{u}_\alpha(t) < u_\alpha(t) + \frac{\varepsilon}{2} \quad \text{for all } t \geq s_q^{(n)} + T_0. \tag{3.21}$$

Choose an integer $N_0 > 0$ such that, when $n \geq N_0$ and $q \geq K^{(n)}$,

$$t_q^{(n)} - s_q^{(n)} > T_0 + P.$$

Further, from (3.13) and (3.21), we obtain

$$u(t) < u_0(t) + \varepsilon \quad \text{for all } t \in \left[s_q^{(n)} + T_0,\, t_q^{(n)}\right]. \tag{3.22}$$

So, when $n \geq N_0$ and $q \geq K^{(n)}$, by (3.9), (3.10), (3.16), (3.17), (3.19) and condition (1) of (H_1) it follows

$$\begin{aligned}
\frac{\alpha}{n^2} &= x\big(t_q^{(n)}, X_n\big) \\
&= x\big(s_q^{(n)} + T_0, X_n\big) \exp \int_{s_q^{(n)}+T_0}^{t} f(s, x_t, u_t)\,\mathrm{d}s \\
&\geq \frac{\alpha}{n^2} \exp \int_{s_q^{(n)}+T_0}^{t} f(s, \varepsilon \exp(\beta_2 \tau), u_{0t} + \varepsilon)\,\mathrm{d}s \\
&> \frac{\alpha}{n^2},
\end{aligned}$$

which leads to a contradiction. Therefore, Claim 3.2 is true.

Finally, from Claims 3.1 and 3.2 we see that Theorem 3.2 is proved. □

Remark 3.2. From the proof of Theorem 3.2, we note that $u_0(t)$ is some fixed positive solution of system (3.7), which is independent of the feedback control. So, the feedback control has no influence on the permanence of system (1.1).

Further, using Theorem 1 given by Teng and Chen in [**23**] on the existence of positive periodic solutions for general n-species periodic Kolmogorov type systems with delays, we have the following theorem on the existence of positive periodic solutions for the periodic system (1.1).

Theorem 3.3. *Suppose system* (1.1) *is ω-periodic and assumptions* (H_1)–(H_4) *hold. We further assume that there is a constant $\mu > 0$ such that* $\liminf_{t\to\infty} \int_t^{t+\mu} g(s, k_1)\,\mathrm{d}s > 0$ *for all $k_1 > 0$. Then system* (1.1) *has at least one positive ω-periodic solution.*

Proof. By assumptions (H_1)–(H_4) and Theorems 3.1 and 3.2, we obtain that the component x of system (1.1) is permanent, that is, there are positive constants m_1, M_1 and T_1 such that, for any positive solution $(x(t), u(t))$ of system (1.1) we have $m_1 \le x(t) \le M_1$ for all $t \ge T_1$. Further, from this and the second equation of system (1.1) we have

$$-\eta(t) + g(t, m_1) \le \frac{\mathrm{d}u(t)}{\mathrm{d}t} \le -\eta(t) + g(t, M_1) \quad \text{for all } t \ge T_1 + \tau.$$

Hence, using the comparison theorem of ordinary differential equations and conclusion (b) of Lemma 2.1, we can further obtain that there are positive constants m_2, M_2 and $T_2 \ge T_1 + \tau$ such that, for any positive solution $(x(t), u(t))$ of system (1.1), we have $m_2 \le u(t) \le M_2$ for all $t \ge T_2$. Now, we let $m = \min\{m_1, m_2\}$ and $M = \max\{M_1, M_2\}$; then for all $t \ge T_2$

$$m \le x(t) \le M, \quad m \le u(t) \le M.$$

Finally, by Theorem 1 in [**23**], it follows that the system has at least a positive ω-periodic solution, and this completes the proof of this theorem. □

4. Applications. In this paper we consider the permanence of the general single-species Kolmogorov system with pure delays and feedback control. In the following, we shall apply the results given in Section 3 to special system (1.2)–(1.5), which has been studied extensively in the literature.

For the convenience of statement in this section, we use the following notations: $g^u = \sup_{t\in R_{+0}} g(t)$ and $g^l = \inf_{t\in R_{+0}} g(t)$.

Consider the single species non-autonomous linear system

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = r(t) - \eta(t)u(t). \tag{4.1}$$

Let $u_{10}(t)$ be some fixed positive solution of system (4.1).

We first consider system (1.2). For system (1.2), we further introduce the following assumptions

(A_1) Functions $b(t)$, $c(t)$, $r(t)$, $\eta(t)$, $\delta(t)$, $a_i(t)$, $\tau_i(t)$, $d_l(t)$ and $\sigma_l(t)$ are bounded and continuous functions defined on R_{+0}, and $c(t)$, $r(t)$, $\eta(t)$, $\delta(t)$, $a_i(t)$, $\tau_i(t)$, $d_k(t)$ and $\sigma_k(t)$ are non-negative for all $t \in R_{+0}$; $K_j(t,s)$ is non-negative, bounded and continuous with respect to $t \in R_{+0}$, and integrable with respect to $s \in [-\gamma_j, 0]$, $\gamma_j \geq 0$ is constant, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $l = 1, 2, \ldots, q$.

(A_2) There is a constant $\lambda > 0$ such that

$$\liminf_{t\to\infty} \int_t^{t+\lambda} \eta(s)\,\mathrm{d}s > 0, \qquad \liminf_{t\to\infty} \int_t^{t+\lambda} a(s)\,\mathrm{d}s > 0,$$

and

$$\liminf_{t\to\infty} \int_t^{t+\lambda} \big[b(s) - u_{10}(s)\big]\,\mathrm{d}s > 0,$$

where $a(t) = \sum_{i=1}^n a_i(t) + \sum_{j=1}^m \int_{-\gamma_j}^0 K(t,s)\,\mathrm{d}s$.

On the permanence and ultimately boundedness for system (1.2), directly applying Theorems 3.1–3.3, we have the following result.

Theorem 4.1. *Suppose that assumptions* (A_1) *and* (A_2) *hold. Then,*

(a) *system* (1.2) *is permanent.*

(b) *If system* (1.2) *is ω-periodic and there exists a positive constant μ such that* $\liminf_{t\to\infty} \int_t^{t+\mu} r(s)\,\mathrm{d}s > 0$ *or* $\liminf_{t\to\infty} \int_t^{t+\mu} \sum_{l=1}^q d_l(s)\,\mathrm{d}s > 0$*, then system* (1.2) *has at least a positive ω-periodic solution.*

Proof. Let $\tau = \max\{\delta^u, \tau_i^u, \sigma_l^u, \gamma_j : i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m,\ l = 1, 2, \ldots, q\}$. Directly from assumptions (A_1) and (A_2), we can choose a constant $k > 0$ such that

$$\limsup_{t\to\infty} \int_t^{t+\lambda} \big[b(s) - ka(s)\big]\,\mathrm{d}s < 0.$$

Then, from assumptions (A_1) and (A_2), we easily prove that the conditions of Theorems 3.1–3.3 hold. Therefore, by Theorems 3.1–3.3 we obtain that conclusions (a) and (b) of Theorem 4.1 are true. This completes the proof of the theorem. $\square$

Remark 4.1. From conclusion (a) of Theorem 4.1, we note that the feedback control has no influence on the permanence of system (1.2).

Remark 4.2. In [**8**], Hu and Li discussed the following system

$$
\begin{aligned}
\frac{\mathrm{d}x(t)}{\mathrm{d}t} &= x(t)\left[b(t) - \sum_{i=1}^{n} a_i(t)x(t-\tau_i(t)) - c(t)u(t-\delta(t))\right] \\
\frac{\mathrm{d}u(t)}{\mathrm{d}t} &= -\eta(t)u(t) + d(t)x(t-\sigma(t)).
\end{aligned}
\tag{4.2}
$$

They obtained sufficient conditions on the existence of positive periodic solutions (see [**8**, Theorem 3.1]). Obviously, system (4.2) is a special case of system (1.2), and their method is totally different from ours in this paper. The conditions of Theorem 3.1 in [**8**] clearly imply the conditions of conclusion (b) of Theorem 4.1. So our results improve the results given in [**8**].

Next, we consider system (1.3). For system (1.3) we introduce the assumptions

(B_1) $b(t)$, $c(t)$, $\eta(t)$, $d(t)$, $K(t)$, $\delta(t)$, $\sigma(t)$ and $\tau_i(t)$ are bounded and continuous functions defined on R_{+0}, and $c(t)$, $d(t)$, $\eta(t)$, $\delta(t)$, $\sigma(t)$ and $\tau_i(t)$ $(i = 1, 2, \dots, n)$ are non-negative for all $t \in R_{+0}$, and $\inf_{t\in R_{+0}} K(t) > 0$.

(B_2) There is a constant $\lambda > 0$ such that

$$
\liminf_{t\to\infty} \int_t^{t+\lambda} b(s)\,\mathrm{d}s > 0 \quad \text{and} \quad \liminf_{t\to\infty} \int_t^{t+\lambda} \eta(s)\,\mathrm{d}s > 0.
$$

On system (1.3), applying Theorems 3.1–3.3, we have the following theorem.

Theorem 4.2. *Suppose that assumptions* (B_1) *and* (B_2) *hold. Then,*

(a) *system* (1.3) *is permanent.*

(b) *If system* (1.3) *is* ω*-periodic and there exists a positive constant* α *such that* $\liminf_{t\to\infty} \int_t^{t+\alpha} d(s)\,\mathrm{d}s > 0$, *then system* (1.3) *has at least a positive* ω*-periodic solution.*

Remark 4.3. From Theorem 4.2, we note that the feedback control has no influence on the permanence of system (1.3).

Remark 4.4. In [**8**], Huo and Li consider system (1.3) and established the sufficient conditions on the existence of positive periodic solutions (see [**8**, Theorem 3.2]) by Gaines and Mawhin's coincidence degree. Obviously, our results improve the results given in [**8**].

Further, we consider system (1.4). For system (1.4) we introduce the following assumptions.

(C_1) $b(t)$, $c(t)$, $d(t)$, $\eta(t)$, $\delta(t)$, $\sigma(t)$, $a_i(t)$, $b_i(t)$, $c_i(t)$ and $\tau_i(t)$ are non-negative, bounded and continuous functions defined on R_{+0}.

(C_2) $a^l > 0$, and there is a constant $\lambda > 0$ such that

$$\liminf_{t\to\infty}\int_t^{t+\lambda} b(s)\,\mathrm{d}s > 0 \quad\text{and}\quad \liminf_{t\to\infty}\int_t^{t+\lambda}\sum_{i=1}^n \frac{a_i(s)}{c_i(s)}\,\mathrm{d}s > 1,$$

where $a(t) = \sum_{i=1}^n a_i(t)b_i(t)$.

On system (1.4), we have the following result.

Theorem 4.3. *Suppose that assumptions* (C_1) *and* (C_2) *hold. Then,*

(a) *system* (1.4) *is permanent.*

(b) *If system* (1.4) *is ω-periodic and there exists a positive constant α such that* $\liminf_{t\to\infty}\int_t^{t+\alpha} d(s)\,\mathrm{d}s > 0$*, then system* (1.4) *has at least a positive ω-periodic solution.*

Proof. Let $\tau = \max\{\delta^u, \sigma^u, \tau_i^u, : i = 1, 2, \ldots, n\}$. Directly from assumptions (C_1) and (C_2) we can choose constant $k > 0$ such that

$$\limsup_{t\to\infty}\int_t^{t+\lambda} b(s)\left[1 - \sum_{i=1}^n \frac{a_i(s)k}{b_i(s) + c_i(s)k}\right]\mathrm{d}s < 0.$$

Hence, from assumptions (C_1) and (C_2) we easily prove that the conditions of Theorems 3.1–3.3 hold. Therefore, by Theorems 3.1–3.3 we obtain that conclusions (a) and (b) of Theorem 4.3 are true. This completes the proof of this theorem. □

Remark 4.5. From Theorem 4.3, we note that the feedback control has no influence on the permanence of system (1.4).

Finally, we consider system (1.5). For system (1.5) we introduce the assumptions

(D_1) $a(t)$, $b(t)$, $c(t)$, $r(t)$, $\eta(t)$, $\delta(t)$, $a_i(t)$, $d_j(t)$, $\tau_i(t)$ and $\sigma_j(t)$ are bounded and continuous functions defined on R_{+0}, and $a(t)$, $c(t)$, $r(t)$, $\eta(t)$, $\delta(t)$, $a_i(t)$, $\tau_i(t)$, $d_j(t)$ and $\sigma_j(t)$ are non-negative for all $t \in R_{+0}$, and $H(s)$ is an integrable function on $[-\gamma, 0]$ and $\int_{-\gamma}^{0} H(s)\,\mathrm{d}s = 1$, $\gamma \geq 0$ is constant, $\beta \geq 1$, $\alpha_i \geq 1$ and $\beta_j \geq 1$ are constants, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.

(D_2) There is a constant $\lambda > 0$ such that

$$\liminf_{t\to\infty} \int_t^{t+\lambda} \eta(s)\,\mathrm{d}s > 0, \qquad \liminf_{t\to\infty} \int_t^{t+\lambda} e(s)\,\mathrm{d}s > 0,$$

and

$$\liminf_{t\to\infty} \int_t^{t+\lambda} \big[b(s) - u_{10}(s)\big]\,\mathrm{d}s > 0,$$

where $e(t) = \sum_{i=1}^{n} a_i(t) + a(t) \int_{-\gamma}^{0} H(s)\,\mathrm{d}s$.

On system (1.5), applying Theorems 3.1–3.3, we have the following result.

Theorem 4.4. *Suppose that assumptions* (D_1) *and* (D_2) *hold. Then,*

(a) *system* (1.5) *is permanent.*

(b) *If system* (1.5) *is ω-periodic and there exists a positive constant α such that* $\liminf_{t\to\infty} \int_t^{t+\alpha} r(s)\,\mathrm{d}s > 0$ *or* $\liminf_{t\to\infty} \int_t^{t+\alpha} \sum_{j=1}^{m} d_j(s)\mathrm{d}s > 0$, *then system* (1.5) *has at least a positive ω-periodic solution.*

Remark 4.6. From conclusion (a) of Theorem 4.4, we note that the feedback control has no influence on the permanence of system (1.5).

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REFERENCES

1. Y. Cao and T.C. Gard, *Ultimate bounds and global asymptotic stability for differential delay equations*, Rocky Mountain J. Math. **25** (1995), 119–131.

2. X. Chen, *Almost periodic solutions of nonlinear delay population equation with feedback control*, Nonlinear Anal. RWA **8** (2007), 62–72.

3. X. Chen and F. Chen, *Almost-periodic solutions of a delay population equation with feedback control*, Nonlinear Anal. RWA **7** (2006), 559–571.

4. H.I. Freedman and J. Wu, *Periodic solutions of single-species models with periodic delay*, SIAM J. Math. Anal. **23** (1992), 689–701.

5. K. Gopalsamy, *Stability and oscillations in delay different equations of population dynamics*, Kluwer Academic, Dordrecht, 1992.

6. K. Gopalsamy and P. Wen, *Feedback regulation of logistic growth*, Internat. J. Math Math. Sci. **16** (1993), 177–192.

7. J. Hale, *Theory of functional differential equations*, Springer-Verlag, Heidelberg, 1977.

8. H. Huo and W. Li, *Positive periodic solutions of a class of delay differential system with feedback control*, Appl. Math. Comput. **148** (2004), 35–46.

9. Y. Kuang, *Delay differential equations with applications in population dynamics*, Academic Press, New York, 1993.

10. J. Lasalle and S. Lefschetz, *Stability by Lyapunov's direct method*, Academic Press, New York, 1961.

11. S. Lefschetz, *Stability of nonlinear control systems*, Academic Press, New York, 1965.

12. P. Li and Y. Li, *Multiple positive periodic solutions of nonlinear functional differential systems with feedback control*, J. Math. Anal. Appl. **288** (2003), 819–832.

13. Y. Li and P. Liu, *Positive periodic solutions of a class of functional differential systems with feedback controls*, Nonlinear Anal. **57** (2004), 655–666.

14. W. Li and L. Wang, *Existence and global attractivity of positive periodic solutions of functional differential equations with feedback control*, J. Comput. Appl. Math. **180** (2005), 293–309.

15. B. Lisena, *Global attractivity in nonautonomous logistic equations with delay*, Nonlinear Anal. RWA **9** (2008), 53–63.

16. Q. Liu and R. Xu, *Persistence and global stability for a delayed nonautonomous single-species model with dispersal and feedback control*, Diff. Equat. Dynam. Syst. **11** (2003), 353–367.

17. L. Nie, J. Peng and Z. Teng, *Harmless feedback control for permanence and global asymptotic stability in nonlinear delay population equation*, Stud. Appl. Math. **120** (2008), 247–263.

18. ———, *Permanence and stability in multi-species non-autonomous Lotka-Volterra competitive systems with delays and feedback controls*, Math. Comput. Model. **49** (2009), 295–306.

19. ———, *Permanence in multi-species competitive systems with delays and feedback controls*, Rocky Mountain J. Math. **38** (2008), 1609–1631.

20. G. Seifert, *On a delay-differential equation for single specie population variation*, Nonlinear Anal. **11** (1987), 1051–1059.

21. ———, *Almost periodic solutions for delay logistic equations with almost periodic time dependence*, Diff. Integral Equat. **9** (1996), 335–342.

22. Z. Teng, *Permanence and stability in general nonautonomous single-species Kolmogorov systems with delays*, Nonlinear Anal. **8** (2007), 230–248.

23. Z. Teng and L. Chen, *The positive periodic solutions of periodic Kolmogorov type systems with delays*, Acta Math. Appl. Sci. **22** (1999), 456–464.

24. Z. Teng, Z. Li and H. Jiang, *Permanence criteria in non-autonomous predator-prey Kolmogorov systems and its applications*, Dynam. Syst. **19** (2004), 171–194.

25. Z. Teng and Z. Lu, *The effect of dispersal on single-species nonautonomous dispersal models with delays*, J. Math. Biol. **42** (2001), 439–454.

26. R.R. Vance and E.A. Coddington, *A nonautonomous model of population growth*, J. Math. Biol. **27** (1989), 491–506.

27. Y. Xia and J. Cao, *Almost periodic solutions of n-species competitive system with feedback controls*, J. Math. Anal. Appl. **294** (2004), 503–522.

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