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MAZURKIEWICZ MANIFOLDS AND HOMOGENEITY

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ABSTRACT. It is proved that no region of a homogeneous locally compact, locally connected metric space can be cut by an F_σ-subset of a "smaller" dimension. The result applies to different finite or infinite topological dimensions of metrizable spaces.

The classical Hurewicz-Menger-Tumarkin theorem in dimension theory says that connected topological n-manifolds (with or without boundary) are Cantor manifolds (i.e., no subset of covering dimension $\leq n-2$ separates the space). The theorem was almost immediately strengthened by Mazurkiewicz who proved that regions (i.e., open connected subsets) in Euclidean spaces (and, in fact, in topological manifolds) cannot be cut by subsets of codimension at least two (a subset *cuts* if its complement is not continuum-wise connected [**2**]). The Hurewicz-Menger-Tumarkin theorem has many generalizations. In particular, it is known that regions of homogeneous locally compact metric spaces are Cantor manifolds (including their infinite-dimensional versions) [**5, 6**]. It was proved in [**3**] that no weakly infinite-dimensional subset cuts the product of a countable number of nondegenerate metric continua.

In this paper, we obtain a generalization in the spirit of the Mazurkiewicz theorem: regions in homogeneous locally compact, locally connected metric spaces cannot be cut by F_σ-subsets of codimension at least two. Moreover, our result holds true for a very general dimension function $D_{\mathcal{K}}$ considered in [**4**] which captures covering dimension, cohomological dimension $\dim_G$ with respect to any Abelian group G and extraordinary dimension $\dim_L$ with respect to a given CW-complex L

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and has its counterparts in infinite dimensions including C-spaces and weakly infinite-dimensional spaces.

Basic facts on Cantor manifolds and their stronger variations with respect to dimension $D_{\mathcal{K}}$ or to the above-mentioned infinite dimensions have been presented in [**4**]. We recall some necessary terminology and results from that paper. We restrict our considerations to metrizable spaces.

A sequence $\mathcal{K} = \{K_0, K_1, \dots\}$ of CW-complexes is called a *stratum* for a dimension theory [**1**] if

• for each space X admitting a perfect map onto a metrizable space, $K_n \in AE(X)$ implies both $K_{n+1} \in AE(X \times \mathbf{I})$ and $K_{n+j} \in AE(X)$ for all $j \geq 0$.

Here, $K_n \in AE(X)$ means that K_n is an absolute extensor for X. Given a stratum $\mathcal{K}$, the dimension function $D_{\mathcal{K}}$ for a metrizable space X is defined as follows:

(1) $D_{\mathcal{K}}(X) = -1$ if and only if $X = \varnothing$;

(2) $D_{\mathcal{K}}(X) \leq n$ if $K_n \in AE(X)$ for $n \geq 0$; if $D_{\mathcal{K}}(X) \leq n$ and $K_m \notin AE(X)$ for all $m < n$, then $D_{\mathcal{K}}(X) = n$;

(3) $D_{\mathcal{K}}(X) = \infty$ if $D_{\mathcal{K}}(X) \leq n$ is not satisfied for any n.

According to the countable sum theorem in extension theory, it follows directly from the above definition that $D_{\mathcal{K}}(X) \leq n$ implies $D_{\mathcal{K}}(A) \leq n$ for any F_σ-subset $A \subset X$.

Henceforth, $\mathcal{C}$ will denote one of the four classes of metrizable spaces:

(4) the class $\mathcal{D}^k_{\mathcal{K}}$ of at most k-dimensional spaces with respect to dimension $D_{\mathcal{K}}$,

(5) the class $\mathcal{D}^{<\infty}_{\mathcal{K}}$ of strongly countable $D_{\mathcal{K}}$-dimensional spaces, i.e., all spaces represented as a countable union of closed finite-dimensional subsets with respect to $D_{\mathcal{K}}$,

(6) the class $\mathbf{C}$ of C-spaces,

and

(7) the class $\mathcal{WID}$ of weakly infinite-dimensional spaces.

(for definitions of a weakly (strongly) infinite-dimensional or of a C-space, see [**2**]).

A metrizable space X is a *Cantor manifold with respect to a class* $\mathcal{C}$ if X cannot be separated by a closed subset which belongs to $\mathcal{C}$.

X is a *Mazurkiewicz manifold with respect to* $\mathcal{C}$ if for every two closed, disjoint subsets $X_0, X_1 \subset X$, both having non-empty interiors in X, and every F_σ-subset $F \subset X$ with $F \in \mathcal{C}$, there exists a continuum in $X \setminus F$ joining X_0 and X_1.

Obviously, Mazurkiewicz manifolds with respect to $\mathcal{C}$ are Cantor manifolds with respect to $\mathcal{C}$. It was observed in [**4**] that if no F_σ-subset from a class $\mathcal{C}$ cuts a compact space X, then X is a Mazurkiewicz manifold with respect to $\mathcal{C}$; the converse implication holds for locally connected compact spaces X.

Our main result is Theorem 4. We are going to use the following theorem from [**4**] (see Theorem 2.6, Theorem 3.1, Theorem 3.4 and Theorem 3.6).

Theorem 1. *Let Z be a metric compact space and $Z \notin \mathcal{C}$, where $\mathcal{C} \in \{\mathcal{WID}, \mathbf{C}, \mathcal{D}^{n-2}_{\mathcal{K}}, \mathcal{D}^{<\infty}_{\mathcal{K}}\}$ and $n \geq 1$. In the case $\mathcal{C} = \mathcal{D}^{n-2}_{\mathcal{K}}$ we additionally assume $D_{\mathcal{K}}(Z) = n$, and in the case $\mathcal{C} = \mathcal{D}^{<\infty}_{\mathcal{K}}$ assume that Z does not contain closed subsets of arbitrarily large finite dimension $D_{\mathcal{K}}$. Then Z contains a compact Mazurkiewicz manifold with respect to $\mathcal{C}$.*

We also need the following version of the Effros theorem (see [**6**, Proposition 1.4]).

Theorem 2. *If X is a homogeneous locally compact metric space (with metric ρ), then for every $a \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $\rho(x,a) < \delta$, then there is an ε-homeomorphism $h : X \to X$ (i.e., $\rho(h(y), y) < \varepsilon$ for each y) such that $h(a) = x$.*

The following lemma is a slight generalization of [**7**, Theorem 8, page 243].

Lemma 3. *If X is a locally compact, locally connected metric space and the union $\bigcup_{i=1}^{\infty} F_i$ cuts a region U of X, where F_i is a closed subset of U for each i, then there is i such that F_i cuts a region $V \subset U$.*

Proof. Choose two distinct points $a, b \in U$ such that $\bigcup_{i=1}^{\infty} F_i$ cuts U between them. Suppose no set F_i cuts any subregion of U. So, there is a subregion $U_1 \subset U \setminus F_1$ containing a and b. Since U_1 is completely metrizable, it is arcwise connected (by the Mazurkiewicz-Moore-Menger theorem [**7**]); hence, there is an arc $\alpha_1 \subset U_1$ from a to b. The local compactness and local connectedness allows us to get a region U_1' such that $\alpha_1 \subset U_1' \subset \mathrm{cl}\,(U_1') \subset U_1$ and $\mathrm{cl}\,(U_1')$ is compact. Similarly, we find an arc α_2 from a to b and regions $U_2 \subset U_1' \setminus F_2$ and U_2' such that $\alpha_2 \subset U_2' \subset \mathrm{cl}\,(U_2') \subset U_2$ and $\mathrm{cl}\,(U_2')$ is compact. Continuing this way, we get a decreasing sequence of continua $\mathrm{cl}\,(U_n')$ whose intersection is a continuum in $U \setminus \bigcup_{i=1}^{\infty} F_i$ containing a and b, a contradiction. □

Theorem 4. *Let X be a homogeneous locally compact, locally connected metric space. Suppose U is a region in X and $U \notin \mathcal{C}$, where $\mathcal{C} \in \{\mathcal{WID}, \mathbf{C}, \mathcal{D}_{\mathcal{K}}^{n-2}, \mathcal{D}_{\mathcal{K}}^{<\infty}\}$ and $n \geq 1$. In case $\mathcal{C} = \mathcal{D}_{\mathcal{K}}^{n-2}$ assume $D_{\mathcal{K}}(U) = n$. Then U is a Mazurkiewicz manifold with respect to $\mathcal{C}$.*

Proof. One may assume that $U \neq \varnothing$. Notice that U is second countable. It follows (by the countable sum theorem for spaces in the class $\mathcal{C}$) that U contains compact sets of arbitrarily small diameters which do not belong to $\mathcal{C}$. Suppose U is not a Mazurkiewicz manifold with respect to $\mathcal{C}$, and let an F_σ-subset $\bigcup_{i=1}^{\infty} F_i$ of U cut U with each $F_i \in \mathcal{C}$ being closed in U. It follows by Lemma 3 that there is a j such that F_j cuts a region $V \subset U$. Thus, F_j also separates V since V is locally connected [**7**, Theorem 1, page 238]. Without loss of generality one can assume that F_j is nowhere dense. Let $V \setminus F_j = V_1 \cup V_2$, where V_1 and V_2 are nonempty, open and disjoint. Fix a point $a \in F_j \cap V$.

Assume first that in the case $\mathcal{C} = \mathcal{D}_{\mathcal{K}}^{<\infty}$, the region U does not contain closed subsets of arbitrarily large finite dimension $D_{\mathcal{K}}$. Recall that U contains arbitrarily small compact sets which do not belong to $\mathcal{C}$ and, according to Theorem 1, every such set contains a compact Mazurkiewicz manifold with respect to $\mathcal{C}$. Consequently, there are arbitrarily small compact Mazurkiewicz manifolds with respect to $\mathcal{C}$ in U. By homogeneity and Theorem 2, there is a compact Mazurkiewicz manifold (with respect to $\mathcal{C}$) $M \subset V$ containing a. M being a Cantor manifold with respect to $\mathcal{C}$, it is not in $\mathcal{C}$. Indeed, otherwise every closed subset of M would be in $\mathcal{C}$, so we can find a closed subset of M in $\mathcal{C}$ separating M, a contradiction. Consequently, M is not contained in F_j. Suppose M intersects V_1. Then the Effros theorem allows us to push M toward V_2 by a small homeomorphism so that it meets both sets V_1 and V_2. This means that the displaced M is separated by F_j, a contradiction.

The case when $\mathcal{C} = \mathcal{D}_{\mathcal{K}}^{<\infty}$ and U contains a family $\mathcal{A}$ of closed subsets of arbitrarily large finite dimension $D_{\mathcal{K}}$ can be handled in a similar way. Indeed, by the σ-compactness of U and by the countable sum theorem for dimension $D_{\mathcal{K}}$, we can assume that these closed subsets are compact and of arbitrarily small diameters. We can also assume that each F_i is of finite dimension $D_{\mathcal{K}}$ and let $D_{\mathcal{K}}(F_j) = m$. By Theorem 1 every element of $\mathcal{A}$ contains a compact Mazurkiewicz manifolds with respect to the corresponding finite dimension $D_{\mathcal{K}}$. Thus there are arbitrarily small compact Cantor manifolds with respect to $\mathcal{D}_{\mathcal{K}}^{k-2}$ in U for some $k > m+2$. By homogeneity and Theorem 2, there is such a Cantor manifold $M \subset V$ containing a. Now, using the Effros theorem, we get a contradiction as in the previous paragraph. $\square$

Remark 5. If $\mathcal{C} = \mathcal{D}_{\mathcal{K}}^{<\infty}$, then the hypothesis $U \notin \mathcal{C}$ in Theorem 4 can be equivalently replaced by $D_{\mathcal{K}}(U) = \infty$. More precisely, we have the following proposition (cf. [**4**, Proposition 4.3]).

If X is a homogeneous locally compact metric space and U is a second countable open subset of X, then $D_{\mathcal{K}}(U) = \infty$ if and only if $U \notin \mathcal{D}_{\mathcal{K}}^{<\infty}$.

Indeed, suppose $D_{\mathcal{K}}(U) = \infty$ but $U = \bigcup_{i=1}^{\infty} F_i$, where each F_i is a closed subset of U of finite dimension $D_{\mathcal{K}}$. Since each closed subset of U can be represented as a countable union of compact subsets, we can assume that each F_i is compact. Then the Baire theorem and homogeneity easily imply that U is contained in a union of countably many homeomorphic copies of some F_{i_0}. So, as an F_σ subset of the union, its dimension $D_{\mathcal{K}}(U)$ is finite, a contradiction. The converse implication is obvious.

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