ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 6, 2011

POSITIVE SOLUTIONS OF EVEN ORDER PERIODIC BOUNDARY VALUE PROBLEMS

QINGKAI KONG AND MIN WANG

ABSTRACT. We study a class of periodic boundary value problems associated with even order differential equations. By applying the Krasnosel'skii fixed point theorem and the fixed point index theory, we establish a series of criteria for the problem to have one, two, an arbitrary number, and even an infinite number of positive solutions. Criteria for the nonexistence of positive solutions are also derived. These criteria are given by explicit conditions which are easy to verify. Several examples are provided to show the applications. Our results extend, improve and supplement many results in the literature, even for the second order case.

1. Introduction. In this paper, we study the existence of positive solutions of the $2m$th order periodic boundary value problem (BVP) consisting of the equation

$$(-1)^m(D^2-\rho^2)^m u = a(t)f(t,u), \quad 0<t<\omega, \tag{1.1}$$

and the boundary condition (BC)

$$u^{(i)}(0)=u^{(i)}(\omega), \quad i=0,1,\dots,2m-1, \tag{1.2}$$

where $\omega>0$, $\rho>0$, $m\in\mathbf{N}$ and $D=d/dt$ is the differential operator. This means that

$$(D^2-\rho^2)^m = \sum_{i=0}^{m}(-1)^i\begin{pmatrix} m \\ i \end{pmatrix}\rho^{2i}D^{2m-2i}, \tag{1.3}$$

and then (1.1) is the same as

$$\sum_{i=0}^{m}(-1)^{m+i}\begin{pmatrix} m \\ i \end{pmatrix}\rho^{2i}u^{(2m-2i)} = a(t)f(t,u).$$

2010 AMS *Mathematics subject classification.* Primary 34B15, Secondary 34B18.
Keywords and phrases. Even order periodic BVP, existence and nonexistence, multiple positive solution, Green's function, Krasnosel'skii fixed point theorem, fixed point index theory.
Received by the editors on October 24, 2008, and in revised form on March 16, 2009.
DOI:10.1216/RMJ-2011-41-6-1907

Throughout this paper, we assume without further mention that

(H1) $a : [0, \omega] \to [0, \infty)$ is continuous and $\int_0^\omega a(s)\,ds > 0$,

(H2) $f : [0, \omega] \times [0, \infty) \to [0, \infty)$ is continuous.

Periodic BVPs have received attention from many authors. For periodic BVPs in the category of BVP (1.1), (1.2), see Atici and Guseinov [**1**], Graef and Kong [**5**], Graef, Kong and Wang [**6**], Lan [**13**], Rachunková, Tvrdý and Vrkoč [**18**], Torres [**19**], Yao [**20**], and Zhang and Wang [**22**] for the second order case; Li [**14**] for the fourth order case; and Li [**15**] and Li, Li, and Liang [**16**] for the general even order case. Most existing work on the second order periodic BVPs was for the existence of one and two positive solutions, and [**13**] also provided conditions for the existence of a third one. Li [**14**] studied the existence of one and two positive solutions for a fourth order periodic BVP. Although [**15, 16**] obtained results on the existence and uniqueness of nontrivial solutions of higher order periodic BVPs, nothing was given on the existence of positive solutions there. Recently, [**6**] investigated a second order nonlinear periodic eigenvalue problem and obtained conditions for the existence of one and two positive solutions and for the nonexistence of positive solutions. However, to the best of the knowledge of the authors, very little is known on the existence of positive solutions of the general even order periodic BVP (1.1), (1.2), and very little is known on the existence of more than three positive solutions even for the second order case.

In this paper, by applying the Krasnosel'skii fixed point theorem and the fixed point index theory, we establish a series of criteria for BVP (1.1), (1.2) to have one, two, an arbitrary number, and even an infinite number of positive solutions. Criteria for the nonexistence of positive solutions are also derived. These criteria are given by explicit conditions which are easy to verify. When restricting to the setting of BVP (1.1), (1.2), our results extend, improve and supplement many results in the literature, even for the second order case.

This paper is organized as follows: After this Introduction, our main results are stated in Section 2, followed by several examples for demonstration in Section 3. All proofs of the main results are given in Section 4. In the last section, we interpret our results to an eigenvalue problem associated with BVP (1.1), (1.2).

2. Main result. Let

$$G(t) = \frac{e^{\rho t} + e^{\rho(\omega - t)}}{2\rho(e^{\rho\omega} - 1)}, \quad t \in [0, \omega], \tag{2.1}$$

$$G_1(t,s) = \begin{cases} G(t-s) & 0 \le s \le t \le \omega, \\ G(s-t) & 0 \le t \le s \le \omega, \end{cases} \tag{2.2}$$

and

$$G_i(t,s) = \int_0^\omega G_1(t,\tau) G_{i-1}(\tau, s)\, d\tau, \quad i = 2, \dots, m. \tag{2.3}$$

It is easy to see

$$G(\omega/2) \le G_1(t,s) \le G(0), \quad s, t \in [0, \omega], \tag{2.4}$$

and from (2.3), by induction, we have

$$\omega^{i-1} G^i(\omega/2) \le G_i(t,s) \le \omega^{i-1} G^i(0), \quad i = 1, 2, \dots, m. \tag{2.5}$$

Define

$$\alpha = (G(\omega/2))^m (G(0))^{-m} \quad \text{and} \quad \beta = \max_{t \in [0,\omega]} \left\{ \int_0^\omega G_m(t,s) a(s)\, ds \right\}. \tag{2.6}$$

Note from (2.4) that $0 < \alpha < 1$.

For $u \in C[0, \omega]$, the Banach space of continuous functions on $[0, \omega]$, we denote by $\|u\|$ the standard maximum norm of u. The first theorem is our basic result on the existence of positive solutions of BVP (1.1), (1.2).

Theorem 2.1. *If there exist* $0 < r_* < r^*$ (*respectively,* $0 < r^* < r_*$), *such that*

$$f(t,x) \le \beta^{-1} r_* \quad \textit{for all } (t,x) \in [0,\omega] \times [\alpha r_*, r_*] \tag{2.7}$$

and

$$f(t,x) \ge \beta^{-1} r^* \quad \textit{for all } (t,x) \in [0,\omega] \times [\alpha r^*, r^*]. \tag{2.8}$$

Then BVP (1.1), (1.2) *has at least one positive solution* u *with* $r_* \leq ||u|| \leq r^*$ (*respectively,* $r^* \leq ||u|| \leq r_*$).

In the sequel, we will use the following notation:

$$f_0 = \liminf_{x\to 0} \min_{t\in[0,\omega]} f(t,x)/x, \qquad f_\infty = \liminf_{x\to\infty} \min_{t\in[0,\omega]} f(t,x)/x;$$

$$f^0 = \limsup_{x\to 0} \max_{t\in[0,\omega]} f(t,x)/x, \qquad f^\infty = \limsup_{x\to\infty} \max_{t\in[0,\omega]} f(t,x)/x.$$

The next three theorems are derived from Theorem 2.1 using f_0, f_∞, f^0, and f^∞.

Theorem 2.2. BVP (1.1), (1.2) *has at least one positive solution if either*

(a) $f^0 < \beta^{-1}$ *and* $f_\infty > (\alpha\beta)^{-1}$; *or*

(b) $f_0 > (\alpha\beta)^{-1}$ *and* $f^\infty < \beta^{-1}$.

Theorem 2.3. *Assume there exists* $r_* > 0$ *such that* (2.7) *holds.*

(a) *If* $f_0 > (\alpha\beta)^{-1}$, *then* BVP (1.1), (1.2) *has at least one positive solution* u *with* $||u|| \leq r_*$;

(b) *if* $f_\infty > (\alpha\beta)^{-1}$, *then* BVP (1.1), (1.2) *has at least one positive solution* u *with* $||u|| \geq r_*$.

Theorem 2.4. *Assume there exists an* $r^* > 0$ *such that* (2.8) *holds.*

(a) *If* $f^0 < \beta^{-1}$, *then* BVP (1.1), (1.2) *has at least one positive solution* u *with* $||u|| \leq r^*$;

(b) *if* $f^\infty < \beta^{-1}$, *then* BVP (1.1), (1.2) *has at least one positive solution* u *with* $||u|| \geq r^*$.

Combining Theorems 2.3 and 2.4 we obtain results on the existence of at least two positive solutions.

Theorem 2.5. *Assume either*

(a) $f_0 > (\alpha\beta)^{-1}$ *and* $f_\infty > (\alpha\beta)^{-1}$, *and there exists* $r > 0$ *such that*

$$f(t,x) < \beta^{-1} r \quad \text{for all } (t,x) \in [0,\omega] \times [\alpha r, r]; \text{ or} \tag{2.9}$$

(b) *$f^0 < \beta^{-1}$ and $f^\infty < \beta^{-1}$, and there exists $r > 0$ such that*

$$f(t,x) > \beta^{-1} r \quad \text{for all } (t,x) \in [0,\omega] \times [\alpha r, r]. \tag{2.10}$$

Then BVP (1.1), (1.2) *has at least two positive solutions u_1 and u_2 with $||u_1|| < r < ||u_2||$.*

Note that in Theorem 2.5, the inequalities in (2.9) and (2.10) are strict and hence are different from those in (2.7) and (2.8) in Theorem 2.1. This is to guarantee that the two solutions u_1 and u_2 are different. By applying Theorem 2.1 repeatedly, we can generalize the conclusion to obtain criteria for the existence of multiple positive solutions.

Theorem 2.6. *Let $\{r_i\}_{i=1}^N \subset \mathbf{R}$ be such that $0 < r_1 < r_2 < r_3 < \cdots < r_N$. Assume either*

(a) *f satisfies* (2.9) *with $r = r_i$ when i is odd, and satisfies* (2.10) *with $r = r_i$ when i is even*; *or*

(b) *f satisfies* (2.9) *with $r = r_i$ when i is even, and satisfies* (2.10) *with $r = r_i$ when i is odd.*

Then BVP (1.1), (1.2) *has at least $N-1$ positive solutions u_i with $r_i < ||u_i|| < r_{i+1}$, $i = 1, 2, \dots, N-1$.*

Theorem 2.7. *Let $\{r_i\}_{i=1}^\infty \subset \mathbf{R}$ be such that $0 < r_1 < r_2 < r_3 < \cdots$. Assume either*

(a) *f satisfies* (2.7) *with $r_* = r_i$ when i is odd, and satisfies* (2.8) *with $r^* = r_i$ when i is even*; *or*

(b) *f satisfies* (2.7) *with $r_* = r_i$ when i is even, and satisfies* (2.8) *with $r^* = r_i$ when i is odd.*

Then BVP (1.1), (1.2) *has an infinite number of positive solutions.*

The following is an immediate consequence of Theorem 2.7.

Corollary 2.1. *Let $\{r_i\}_{i=1}^\infty \subset \mathbf{R}$ be such that $0 < r_1 < r_2 < r_3 < \cdots$. Let $E_1 = \cup_{i=1}^\infty [\alpha r_{2i-1}, r_{2i-1}]$ and $E_2 = \cup_{i=1}^\infty [\alpha r_{2i}, r_{2i}]$. Assume*

$$\limsup_{E_1 \ni x \to \infty} \max_{t \in [0,\omega]} \frac{f(t,x)}{x} < \beta^{-1} \quad \text{and} \quad \liminf_{E_2 \ni x \to \infty} \min_{t \in [0,\omega]} \frac{f(t,x)}{x} > (\alpha\beta)^{-1}.$$

Then BVP (1.1), (1.2) *has an infinite number of positive solutions.*

Now we present a result on the nonexistence of positive solutions of BVP (1.1), (1.2).

Theorem 2.8. BVP (1.1), (1.2) *has no positive solutions if*

(a) $f(t,x)/x < \beta^{-1}$ *for all* $(t,x) \in [0,\omega] \times (0,\infty)$, *or*

(b) $f(t,x)/x > (\alpha\beta)^{-1}$ *for all* $(t,x) \in [0,\omega] \times (0,\infty)$.

We observe that, in the above theorems, if one of $f_0, f_\infty, f^0, f^\infty$ is involved and it is between β^{-1} and $(\alpha\beta)^{-1}$, then the corresponding conclusions fail. Motivated by the ideas in [**4, 7, 12, 17**], we employ the first eigenvalue of a Sturm-Liouville Problem (SLP) associated with BVP (1.1), (1.2) and the fixed point index theory to improve the criteria given in Theorems 2.2–2.5.

It is clear that $\mu_0 = \rho^2$ is the first eigenvalue of

$$-u'' + \rho^2 u = \mu u, \quad u(0) = u(\omega), \quad u'(0) = u'(\omega), \tag{2.11}$$

with $u_0 \equiv 1$ an associated eigenfunction. In the following we let

$$\underline{a} = \min_{t\in[0,\omega]} a(t), \quad \overline{a} = \max_{t\in[0,\omega]} a(t), \quad \zeta = \min\{(\alpha\beta)^{-1}, \underline{a}^{-1}\rho^{2m}\}. \tag{2.12}$$

By replacing $(\alpha\beta)^{-1}$ by ζ, we obtain the theorems below which provide generalized criteria for the existence of one and two positive solutions of BVP (1.1), (1.2) given by Theorems 2.2–2.5.

Theorem 2.9. BVP (1.1), (1.2) *has at least one positive solution if one of following is satisfied*:

(a) $f^0 < \beta^{-1}$ *and* $f_\infty > \zeta$;

(b) $f_0 > \zeta$ *and* $f^\infty < \beta^{-1}$;

(c) $f_0 > \zeta$ *or* $f_\infty > \zeta$, *and there exists* r_* *such that* (2.7) *holds*;

(d) $f^0 < \beta^{-1}$ *or* $f^\infty < \beta^{-1}$, *and there exists* r^* *such that* (2.8) *holds.*

Theorem 2.10. BVP (1.1), (1.2) *has at least two positive solutions if either*

(a) $f_0 > \zeta$ *and* $f_\infty > \zeta$, *and there exists* r *such that* (2.9) *holds*; *or*

(b) $f^0 < \beta^{-1}$ *and* $f^\infty < \beta^{-1}$, *and there exists* r *such that* (2.10) *holds.*

Remark 2.1. It is easy to show that ρ^{2m} is an eigenvalue of the $2m$th order SLP consisting of the equation

$$(-1)^m(D^2 - \rho^2)^m u = \mu u \tag{2.13}$$

and BC (1.2) with $u_0 \equiv 1$ an associated eigenfunction. Thus for $t \in [0, \omega]$

$$1 \equiv u_0(t) = \rho^{2m} \int_0^\omega G_m(t,s) u_0(s)\, ds \equiv \rho^{2m} \int_0^\omega G_m(t,s)\, ds.$$

From (2.6) there exists a $t_1 \in [0, \omega]$ such that

$$\int_0^\omega G_m(t_1, s) a(s)\, ds = \beta. \tag{2.14}$$

Hence

$$\overline{a} \geq \rho^{2m} \int_0^\omega G_m(t_1, s) a(s)\, ds = \rho^{2m}\beta,$$

and for $t \in [0, \omega]$

$$\underline{a} \leq \rho^{2m} \int_0^\omega G_m(t, s) a(s)\, ds \leq \rho^{2m}\beta.$$

Therefore,

$$\beta^{-1}\underline{a} \leq \rho^{2m} \leq \beta^{-1}\overline{a}. \tag{2.15}$$

In particular, when $\underline{a} > \alpha\overline{a}$, i.e., the function $a(t)$ has a small change on $[0, \omega]$, we have $\underline{a}^{-1}\rho^{2m} < \alpha^{-1}\overline{a}^{-1}\rho^{2m} \leq (\alpha\beta)^{-1}$. In this case, Theorems 2.9 and 2.10 are real improvements of Theorems 2.2–2.5.

For the second order case, an alternative approach using a different SLP provides general improvements of Theorems 2.2–2.5 without imposing any restriction on the function $a(t)$. We consider the nonlinear BVP (1.1), (1.2) with $m = 1$, i.e.,

$$-u'' + \rho^2 u = a(t) f(t, u), \quad u(0) = u(\omega), \quad u'(0) = u'(\omega). \tag{2.16}$$

Let ν_0 be the first eigenvalue of the corresponding SLP

$$-u'' + \rho^2 u = \nu a(t)u, \quad u(0) = u(\omega), \quad u'(0) = u'(\omega), \tag{2.17}$$

with an associated eigenfunction $v_0(t)$. It is known that $\nu_0 > 0$ and $v_0(t)$ has no zero in $(0, \omega)$, see [**8, 9, 10**]. Then we have the results below:

Theorem 2.11. BVP (2.16) *has at least one positive solution if one of following is satisfied*:

(a) $f^0 < \nu_0$ *and* $f_\infty > \nu_0$;

(b) $f_0 > \nu_0$ *and* $f^\infty < \nu_0$;

(c) $f_0 > \nu_0$ *and there exists* r_* *such that* (2.7) *holds*;

(d) $f_\infty > \nu_0$ *and there exists an* r_* *such that* (2.7) *holds*;

(e) $f^0 < \nu_0$ *and there exists* r^* *such that* (2.8) *holds*;

(f) $f^\infty < \nu_0$ *and there exists* r^* *such that* (2.8) *holds.*

Theorem 2.12. BVP (2.16) *has at least two positive solutions if either*

(a) $f_0 > \nu_0$, $f_\infty > \nu_0$, *and there exists an* r *such that* (2.9) *holds*;*or*

(b) $f^0 < \nu_0$, $f^\infty < \nu_0$, *and there exists an* r *such that* (2.10) *holds.*

Remark 2.2. We claim that $\beta^{-1} \le \nu_0 \le (\alpha\beta)^{-1}$, and hence Theorems 2.11 and 2.12 are real improvements of Theorems 2.2–2.5 when $m = 1$. In fact, we note that

$$v_0(t) = \nu_0 \int_0^\omega G_1(t,s)a(s)v_0(s)\,ds, \quad t \in [0, \omega].$$

Let $t_2 \in [0, \omega]$ with $v_0(t_2) = ||v_0||$. Then

$$||v_0|| = \nu_0 \int_0^\omega G_1(t_2,s)a(s)v_0(s)\,ds \le \nu_0\beta||v_0||, \tag{2.18}$$

which implies that $\beta^{-1} \le \nu_0$. On the other hand, from (2.4) we have that for any $t \in [0, \omega]$

$$\begin{aligned} v_0(t) &\ge \nu_0 \int_0^\omega G(\omega/2)a(s)v_0(s)\,ds \\ &= \nu_0\alpha \int_0^\omega G(0)a(s)v_0(s)\,ds \\ &\ge \alpha\nu_0 \int_0^\omega G_1(t_2, s)a(s)v_0(s)\,ds \\ &= \alpha v_0(t_2) = \alpha||v_0||. \end{aligned}$$

By (2.6) with $m = 1$, there exists a $t_1 \in [0, \omega]$ such that $\int_0^\omega G_1(t_1, s)a(s)\,ds = \beta$. Thus

$$||v_0|| \ge \nu_0 \int_0^\omega G_1(t_1, s)a(s)v_0(s)\,ds \ge \nu_0\alpha\beta||v_0||, \tag{2.19}$$

which implies that $\nu_0 \le (\alpha\beta)^{-1}$.

Remark 2.3. (i) Theorems 2.11 and 2.12 are practically convenient to apply. This is because the eigenvalues of SLP (2.17) are easy to compute using standard software packages for second order two-point linear self-adjoint SLPs such as those in [**2**].

(ii) The results in Theorems 2.11 and 2.12 can be extended to the general higher order BVP (1.1), (1.2) using the first eigenvalue ν_0 of the SLP consisting of the equation

$$(-1)^m(D^2 - \rho^2)^m u = \nu a(t)u \tag{2.20}$$

and BC (1.2). Theoretically, it can be shown that $\nu_0 > 0$ and the associated eigenfunction has no zero in $(0, \omega)$ using the Krein-Rutman theorem in [**21**]. However, since the eigenvalues of higher order SLPs are difficult to compute numerically, such results are not very useful practically. Therefore, we do not show the details here.

3. Examples. In this section, we give several examples to demonstrate the applications of the criteria obtained in Section 2. For simplicity we choose $a(t) \equiv 1$, $\rho = 1$, and $f(t, x) \equiv f(x)$ in all the examples.

Example 1. Let $f(x) = x^k$.

If $k > 1$, then $\lim_{x\to 0+} f(x)/x = 0$ and $\lim_{x\to\infty} f(x)/x = \infty$. By Theorem 2.2 (a), BVP (1.1), (1.2) has at least one positive solution.

If $0 < k < 1$, then $\lim_{x\to 0+} f(x)/x = \infty$ and $\lim_{x\to\infty} f(x)/x = 0$. By Theorem 2.2 (b), BVP (1.1), (1.2) has at least one positive solution.

Example 2. Let $f(x) = c(x^{k_1} + x^{k_2})$, where $0 < k_1 < 1 < k_2 < \infty$. Let $r = ((1 - k_1)/(k_2 - 1))^{1/(k_2 - k_1)}$. Then

(a) BVP (1.1), (1.2) has at least one positive solution when $c = r(r^{k_1} + r^{k_2})^{-1}\beta^{-1}$;

(b) BVP (1.1), (1.2) has at least two positive solutions u_1 and u_2 with $||u_1|| < r < ||u_2||$ when $0 < c < r(r^{k_1} + r^{k_2})^{-1}\beta^{-1}$;

(c) BVP (1.1), (1.2) has no positive solutions when $c > r(r^{k_1} + r^{k_2})^{-1}(\alpha\beta)^{-1}$.

In fact, it is clear that $\lim_{x\to 0+} f(x)/x = \lim_{x\to\infty} f(x)/x = \infty$, $f(x)$ is strictly increasing and r is the minimum point of $f(x)/x$ on $(0, \infty)$.

When $c = r(r^{k_1} + r^{k_2})^{-1}\beta^{-1}$, we have $f(x) \le f(r) = \beta^{-1}r$ for all $x \in [\alpha r, r]$. Then from Theorem 2.3 (a), BVP (1.1), (1.2) has a positive solution u_1 with $||u_1|| \le r$. Similarly from Theorem 2.3 (b), BVP (1.1), (1.2) has a positive solution u_2 with $||u_2|| \ge r$. However, u_1 and u_2 may be the same for the case when $||u_1|| = ||u_2|| = r$.

When $0 < c < r(r^{k_1} + r^{k_2})^{-1}\beta^{-1}$, by a similar argument and from Theorem 2.5 (a), we obtain the conclusion.

When $c > r(r^{k_1} + r^{k_2})^{-1}(\alpha\beta)^{-1}$, $f(x)/x \ge f(r)/r > (\alpha\beta)^{-1}$ on $(0, \infty)$. Then the conclusion follows from Theorem 2.8 (b).

Example 3. Let

$$f(x) = \begin{cases} c/(x^{-k} + 1) & x > 0, \\ 0 & x = 0, \end{cases}$$

where $k > 1$. Let $r = (k - 1)^{1/k}$. Then

(a) BVP (1.1), (1.2) has at least one positive solution when $c = (r^{1-k} + r)(\alpha\beta)^{-1}$;

(b) BVP (1.1), (1.2) has at least two positive solutions u_1 and u_2 with $||u_1|| < r < ||u_2||$ when $c > (r^{1-k} + r)(\alpha\beta)^{-1}$;

(c) BVP (1.1), (1.2) has no positive solutions when $0 < c < (r^{1-k} + r)\beta^{-1}$.

In fact, it is clear that $\lim_{x\to 0+} f(x)/x = \lim_{x\to\infty} f(x)/x = 0$, $f(x)$ is strictly increasing and r is the maximum point of $f(x)/x$ on $(0,\infty)$.

When $c = (r^{1-k} + r)(\alpha\beta)^{-1}$, $f(r) = (\alpha\beta)^{-1}r$. Then $f(x) \geq f(r) = (\alpha\beta)^{-1}r$ on $[r, \alpha^{-1}r]$, i.e., $f(x) \geq \beta^{-1}r^*$ on $[\alpha r^*, r^*]$, where $r^* = \alpha^{-1}r$. By Theorem 2.4 (a) or (b), There exists at least one positive solution.

When $c > (r^{1-k}+r)(\alpha\beta)^{-1}$, by a similar argument and from Theorem 2.5 (b), we obtain the conclusion.

When $0 < c < (r^{1-k} + r)\beta^{-1}$, $f(x)/x \leq f(r)/r < \beta^{-1}$ on $(0,\infty)$. Then the conclusion follows from Theorem 2.8 (a).

Example 4. Let

$$f(x) = \begin{cases} (\alpha^{-1} + 1)\beta^{-1}x(\sin(b\ln x) + 1)/2 & x > 0, \\ 0 & x = 0, \end{cases}$$

where $0 < b < (\pi - 2\sin^{-1}\delta)/\ln(\alpha^{-1})$ with $\delta = (\alpha^{-1}-1)/(\alpha^{-1}+1)$. We claim that BVP (1.1), (1.2) has an infinite number of positive solutions.

To show this, for $k = 2i + 1$, $i \in \mathbf{N}$, let

$$\xi_k = \exp(b^{-1}(\sin^{-1}\delta + (k-1)\pi)), \qquad \eta_k = \exp(b^{-1}(k\pi - \sin^{-1}\delta)).$$

Then

$$\eta_k/\xi_k = \exp(b^{-1}(\pi - 2\sin^{-1}\delta)) > \exp(\ln(\alpha^{-1})) = \alpha^{-1},$$

hence $\xi_k < \alpha\eta_k$. Note that for $x \in [\alpha\eta_k, \eta_k] \subset [\xi_k, \eta_k]$, $\sin(b\ln x) \geq \sin(\sin^{-1}\delta) = \delta$. Therefore, for $x \in [\alpha\eta_k, \eta_k]$,

$$f(x) \geq (\alpha^{-1} + 1)\beta^{-1}\alpha\eta_k(\delta + 1)/2 = \beta^{-1}\eta_k,$$

i.e., (2.7) holds with $r_* = \eta_k$.

For $k = 2i$, $i \in \mathbf{N}$, let

$$\xi_k = \exp(b^{-1}((k-1)\pi - \sin^{-1}\delta)), \qquad \eta_k = \exp(b^{-1}(k\pi + \sin^{-1}\delta)).$$

Then

$$\eta_k/\xi_k = \exp(b^{-1}(\pi + 2\sin^{-1}\delta)) > \exp(\ln(\alpha^{-1})) = \alpha^{-1},$$

hence $\xi_k < \alpha\eta_k$. Note that for $x \in [\alpha\eta_k, \eta_k] \subset [\xi_k, \eta_k]$, $\sin(b \ln x) \le -\delta$. Therefore, for $x \in [\alpha\eta_k, \eta_k]$,

$$f(x) \le (\alpha^{-1} + 1)\beta^{-1}\eta_k(-\delta + 1)/2 = \beta^{-1}\eta_k,$$

i.e., (2.8) holds with $r^* = \eta_k$.

Therefore by Theorem 2.7, BVP (1.1), (1.2) has an infinite number of positive solutions.

Example 5. Let

$$f(x) = \left(1 + k(\tan^{-1} x - \frac{\pi}{4})\right)x, \quad x \ge 0,$$

where $k \in (0, 4/\pi)$. Then BVP (1.1), (1.2) has at least one positive solution.

In fact, note that $a(t) \equiv 1$ and $\rho = 1$ in the equation. From (2.15), $\beta = \rho^{2m} = 1$, and hence $\zeta = 1$. Since $\lim_{x\to 0} f(x)/x = 1 - k\pi/4$ and $\lim_{x\to\infty} f(x)/x = 1 + k\pi/4$, the conclusion follows from Theorem 2.9 (a). However, when k is small enough, the condition $\lim_{t\to\infty} f(x)/x > \alpha^{-1}$ is not satisfied, so Theorem 2.2 fails to work.

4. Proofs. The proof of Theorem 2.1 is based on the following Krasnosel'skii's fixed point theorem, see [**11**].

Lemma 4.1. *Let X be a Banach space and $K \subset X$ a cone in X. Assume Ω_1, Ω_2 are bounded open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$\Gamma : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

be a completely continuous operator such that either

(a) $||\Gamma u|| \le ||u||$ *for any* $u \in K \cap \partial\Omega_1$ *and* $||\Gamma u|| \ge ||u||$ *for any* $u \in K \cap \partial\Omega_2$; *or*

(b) $||\Gamma u|| \geq ||u||$ *for any* $u \in K \cap \partial\Omega_1$ *and* $||\Gamma u|| \leq ||u||$ *for any* $u \in K \cap \partial\Omega_2$.

Then Γ *has a fixed point in* $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$.

We now introduce some related preliminaries. The first one is for the Green's function that we will use to deal with BVP (1.1), (1.2) . It is well known, see for example, Zhang and Wang [**22**], that the function $G_1(t,s)$ given by (2.2) is the Green's function for the BVP

$$\begin{cases} -u'' + \rho^2 u = 0, \ 0 \leq t \leq \omega, \\ u(0) = u(\omega), \ u'(0) = u'(\omega). \end{cases}$$

Lemma 4.2. *For any* $m \geq 1$, *the function* $G_m(t,s)$ *given by* (2.3) *is the Green's function for the* BVP *consisting of the equation*

$$(-1)^m (D^2 - \rho^2)^m u = 0, \quad 0 \leq t \leq \omega, \tag{4.1}$$

and BC (1.2), *i.e., for any* $h(t) \in C[0,\omega]$

$$u(t) = \int_0^\omega G_m(t,s) h(s)\, ds$$

is the unique solution of the BVP consisting of the equation

$$(-1)^m (D^2 - \rho^2)^m u = h(t), \quad 0 \leq t \leq \omega, \tag{4.2}$$

and BC (1.2).

Proof. Obviously the conclusion holds for $m = 1$. Assume it is true for $m = k-1$. When $m = k$, the solution u of BVP (4.2), (1.2) satisfies

$$-(D^2 - \rho^2) u(t) = \int_0^\omega G_{k-1}(t,s) h(s)\, ds.$$

Then by the conclusion for m=1,

$$\begin{aligned} u(t) &= \int_0^\omega G_1(t,\tau) \biggl(\int_0^\omega G_{k-1}(\tau,s) h(s)\, ds \biggr)\, d\tau \\ &= \int_0^\omega \biggl(\int_0^\omega G_1(t,\tau) G_{k-1}(\tau,s)\, d\tau \biggr) h(s)\, ds \\ &= \int_0^\omega G_k(t,s) h(s)\, ds. \end{aligned}$$

This means that $G_m(t,s)$ is the Green's function for BVP (4.1), (1.2). $\square$

Let α be defined by (2.6). Define a cone K in $C[0,\omega]$ by

(4.3)
$$K = \{u \in C[0,\omega] \mid u(t) \geq 0, \quad t \in [0,\omega] \quad \text{and} \quad \min_{t\in[0,\omega]} u(t) \geq \alpha ||u||\}$$

and an operator $\Gamma : \ C[0,\omega] \to C[0,\omega]$ by

(4.4)
$$\Gamma u = \int_0^\omega G_m(t,s)a(s)f(s,u(s))\,ds, \quad t \in [0,\omega].$$

Lemma 4.3. $\Gamma(K) \subset K$ *and* Γ *is completely continuous.*

Proof. For any $u \in K$, $\Gamma u \geq 0$ on $[0,\omega]$. By (2.5) with $i = m$, for any $t \in [0,\omega]$

$$\begin{aligned}(\Gamma u)(t) &\geq \int_0^\omega \omega^{m-1} G^m(\omega/2)a(s)f(s,u(s))\,ds \\ &= \alpha \int_0^\omega \omega^{m-1} G^m(0)a(s)f(s,u(s))\,ds.\end{aligned}$$

Since $\Gamma u \in C[0,\omega]$, there exists a $t_3 \in [0,\omega]$ such that $||\Gamma u|| = (\Gamma u)(t_3)$. Hence, for any $t \in [0,\omega]$

$$||\Gamma u|| = \int_0^\omega G_m(t_3,s)a(s)f(s,u(s))ds \leq \int_0^\omega \omega^{m-1}G^m(0)a(s)f(s,u(s))ds.$$

Therefore, $\min_{t\in[0,\omega]}(\Gamma u)(t) \geq \alpha ||\Gamma u||$ and this implies that $\Gamma(K) \subset K$. The complete continuity of Γ can be shown by a standard argument using the Arzela-Arscoli theorem. We omit the details. $\square$

In the following, for $r > 0$, we let

(4.5)
$$\Omega_r = \{u \in C[0,\omega] \mid ||u|| < r\}.$$

Proof of Theorem 2.1. Let K and Ω_r be defined by (4.3) and (4.5), respectively. For any $u \in K \cap \partial\Omega_{r_*}$, $||u|| = r_*$ and $\alpha r_* \leq u(t) \leq r_*$ on

$[0, \omega]$. From (2.7), $f(t, u(t)) \le \beta^{-1} r_*$ on $[0, \omega]$. For any $t \in [0, \omega]$

$$\begin{aligned}(\Gamma u)(t) &= \int_0^\omega G_m(t, s) a(s) f(s, u(s))\, ds \\ &\le \beta^{-1} r_* \int_0^\omega G_m(t, s) a(s)\, ds \\ &\le \beta^{-1} r_* \beta = r_* = ||u||.\end{aligned}$$

Thus $\|\Gamma u\| \le \|u\|$. For any $u \in K \cap \partial\Omega_{r^*}$, $||u|| = r^*$ and $\alpha r^* \le u(t) \le r^*$ on $[0, \omega]$. From (2.8), $f(t, u(t)) \ge \beta^{-1} r^*$ on $[0, \omega]$. Let $t_1 \in [0, \omega]$ be defined as in (2.14). Then

$$\begin{aligned}(\Gamma u)(t_1) &= \int_0^\omega G_m(t_1, s) a(s) f(s, u(s))\, ds \\ &\ge \beta^{-1} r^* \int_0^\omega G_m(t_1, s) a(s)\, ds \\ &= \beta \beta^{-1} r^* = ||u||.\end{aligned}$$

Thus $\|\Gamma u\| \ge \|u\|$. Therefore, the conclusion follows from Lemma 4.1 (a). □

Proof of Theorem 2.2. (a) If $f^0 < \beta^{-1}$, there exists an $r_* > 0$ such that

$$f(t, x) < \beta^{-1} x \le \beta^{-1} r_*, \quad (t, x) \in [0, \omega] \times [0, r_*].$$

If $f_\infty > (\alpha\beta)^{-1}$, there exists $\hat{r} > r_*$ such that

$$f(t, x) > (\alpha\beta)^{-1} x, \quad (t, x) \in [0, \omega] \times [\widehat{r}, \infty).$$

Then for any r^* with $\alpha r^* \ge \widehat{r}$

$$f(t, x) > (\alpha\beta)^{-1} x \ge \beta^{-1} r^* \quad \text{for all } (t, x) \in [0, \omega] \times [\alpha r^*, r^*].$$

Then the conclusion follows from Theorem 2.1.

(b) The proof is similar to Part (a) and hence is omitted. □

The proofs of Theorems 2.3 and 2.4 are in the same way and hence are omitted.

Proof of Theorem 2.5. (a) If there exists an $r > 0$ such that (2.9) holds, then by the uniform continuity of $f(t,x)/x$ on any compact subset of $[0,\omega] \times [0,\infty)$, there exist r_1 and r_2 such that $r_1 < r < r_2$ and $f(t,x) < \beta^{-1}x$ for all $(t,x) \in [0,\omega] \times [\alpha r_i, r_i]$, $i = 1,2$. By Theorem 2.3 (a) and (b), BVP (1.1), (1.2) has two positive solutions u_1 and u_2 satisfying $||u_1|| \le r_1$ and $||u_2|| \ge r_2$.

Similarly, case (b) follows from Theorem 2.4. $\square$

The proofs of Theorems 2.6 and 2.7 are in the same way and are hence omitted.

Proof of Corollary 2.1. From the assumption we see that for sufficiently large i

$$\frac{f(t,x)}{x} < \beta^{-1} \quad \text{for all } (t,x) \in [0,\omega] \times [\alpha r_{2i-1}, r_{2i-1}]$$

and

$$\frac{f(t,x)}{x} > (\alpha\beta)^{-1} \quad \text{for all } (t,x) \in [0,\omega] \times [\alpha r_{2i}, r_{2i}].$$

This shows that for sufficiently large i

$$f(t,x) < \beta^{-1}x \le \beta^{-1}r_{2i-1} \quad \text{for all } (t,x) \in [0,\omega] \times [\alpha r_{2i-1}, r_{2i-1}]$$

and

$$f(t,x) > (\alpha\beta)^{-1}x \ge (\alpha\beta)^{-1}\alpha r_{2i} = \beta^{-1}r_{2i} \qquad \text{for all } (t,x) \in [0,\omega] \times [\alpha r_{2i}, r_{2i}].$$

Therefore, the conclusion follows from Theorem 2.7. $\square$

Proof of Theorem 2.8. (a) Assume BVP (1.1), (1.2) has a positive solution u with $||u|| = r$ for some $r > 0$. Then u is a fixed point of the operator Γ defined by (4.4). From the assumption, $f(t,u(t)) < \beta^{-1}u(t) \le \beta^{-1}r$ on $[0,\omega]$. Thus for any $t \in [0,\omega]$

$$\begin{aligned} u(t) = (\Gamma u)(t) &= \int_0^\omega G_m(t,s)a(s)f(s,u(s))\,ds \\ &< \beta^{-1}r\int_0^\omega G_m(t,s)a(s)\,ds \le r \end{aligned}$$

which contradicts $||u|| = r$. Therefore, BVP (1.1), (1.2) has no positive solutions.

(b) Assume BVP (1.1), (1.2) has a positive solution u with $||u|| = r$. Similar to the proof of Lemma 4.3 we can show that $\Gamma u \in K$ and hence $\alpha r \leq u(t) \leq r$ on $[0, \omega]$. From the assumption, $f(t, u(t)) > (\alpha\beta)^{-1}u(t) \geq \beta^{-1}r$. Let $t_1 \in [0, \omega]$ be defined as in (2.14). Then

$$\begin{aligned} u(t_1) = \Gamma u(t_1) &= \int_0^\omega G_m(t_1, s)a(s)f(s, u(s))\, ds \\ &> \beta^{-1}r \int_0^\omega G_m(t_1, s)a(s)\, ds = r \end{aligned}$$

which contradicts $||u|| = r$. Therefore, BVP (1.1), (1.2) has no positive solutions. $\square$

To prove Theorems 2.9 and 2.10, we need the following well-known result in [**3**] on the fixed point indices for operators on cones.

Lemma 4.4. *Let K and Ω_r be defined by* (4.3) *and* (4.5)*, respectively. Assume $\Gamma : K \cap \overline{\Omega}_r \to K$ is a compact operator such that $\Gamma u \neq u$ for $u \in K \cap \partial\Omega_r$.*

(a) *If $||\Gamma u|| \geq ||u||$ for $u \in K \cap \partial\Omega_r$, then $i(\Gamma, K \cap \Omega_r, K) = 0$.*

(b) *If $||\Gamma u|| \leq ||u||$ for $u \in K \cap \partial\Omega_r$, then $i(\Gamma, K \cap \Omega_r, K) = 1$.*

Based on Lemma 4.4 we derive sufficient conditions for $i(\Gamma, K \cap \Omega_r, K) = 0$.

Lemma 4.5. (a) *Assume $f_0 > \underline{a}^{-1}\rho^{2m}$. Then $i(\Gamma, K \cap \Omega_r, K) = 0$ for all sufficiently small $r > 0$.*

(b) *Assume $f_\infty > \underline{a}^{-1}\rho^{2m}$. Then $i(\Gamma, K \cap \Omega_r, K) = 0$ for all sufficiently large $r > 0$.*

Proof. (a) Let $0 < l < 1$ be fixed, and define $\Gamma_1 : C[0, \omega] \to C[0, \omega]$ by

$$(\Gamma_1 u)(t) = \int_0^\omega G_m(t, s)u^l(s)\, ds.$$

Similar to the proof of Lemma 4.3, we can show that Γ_1 is compact and $\Gamma_1 K \subset K$. Fix $t_4 \in (0,\omega)$ and define $r_1 = (\alpha^l \int_0^\omega G_m(t_4,s)\,ds)^{1/(1-l)}$. Then for $0 < r \le r_1$ and $u \in K \cap \partial\Omega_r$, $u^l(t) \ge (\alpha r)^l$ on $[0,\omega]$, and hence

$$\begin{aligned} ||\Gamma_1 u|| \ge (\Gamma_1 u)(t_4) &= \int_0^\omega G_m(t_4,s)u^l(s)\,ds \\ &\ge (\alpha r)^l \int_0^\omega G_m(t_4,s)\,ds = (\alpha r)^l \alpha^{-l} r_1^{1-l} \\ &\ge (\alpha r)^l \alpha^{-l} r^{1-l} = r = ||u||. \end{aligned} \tag{4.6}$$

By Lemma 4.4 (a), $i(\Gamma_1, K\cap\Omega_r, K) = 0$.

Define a homotopy operator $H : [0,1]\times K \to K$ by

$$H(s,u) = (1-s)\Gamma u + s\Gamma_1 u.$$

Then $H(s,\cdot)$ is compact for $0 \le s \le 1$. Since $f_0 > \underline{a}^{-1}\rho^{2m}$, we can choose $\varepsilon > 0$ and $0 < r_2 \le r_1$ such that for $(t,x) \in [0,\omega]\times[0,r_2]$

$$a(t)f(t,x) \ge (\rho^{2m}+\varepsilon)x \quad \text{and} \quad x^l \ge (\rho^{2m}+\varepsilon)x.$$

Let $0 < r \le r_2$. We now show that $H(s,u) \ne u$ for all $0 \le s \le 1$ and $u \in K \cap \partial\Omega_r$. Assume the contrary, i.e., there exist an $s_1 \in [0,1]$ and $u_1 \in K\cap\partial\Omega_r$ with $H(s_1,u_1) = u_1$. Then u_1 satisfies

$$(-1)^m(D^2-\rho^2)^m u_1(t) = (1-s_1)a(t)f(t,u_1(t)) + s_1 u_1^l(t) \tag{4.7}$$

and BC (1.2). Hence

$$\begin{aligned} &\int_0^\omega (-1)^m(D^2-\rho^2)^m u_1(t)\,dt \\ &\qquad = \int_0^\omega [(1-s_1)a(t)f(t,u_1(t)) + s_1 u_1^l(t)]\,dt. \end{aligned}$$

By (1.2) we have

$$\begin{aligned}\rho^{2m}\int_0^\omega u_1(t)\,dt &= \int_0^\omega (-1)^m(D^2-\rho^2)^m u_1(t)\,dt\\ &= \int_0^\omega [(1-s_1)a(t)f(t,u_1(t))+s_1u_1^l(t)]\,dt\\ &\geq (1-s_1)\int_0^\omega (\rho^{2m}+\varepsilon)u_1(t)\,dt\\ &\quad + s_1\int_0^\omega (\rho^{2m}+\varepsilon)u_1(t)\,dt\\ &= (\rho^{2m}+\varepsilon)\int_0^\omega u_1(t)\,dt\end{aligned}$$

which is a contradiction since $u_1(t)>0$ on $[0,\omega]$. Hence,

$$\begin{aligned}i(\Gamma,K\cap\Omega_r,K) &= i(H(0,\cdot),K\cap\Omega_r,K)\\ &= i(H(1,\cdot),K\cap\Omega_r,K) = i(\Gamma_1,K\cap\Omega_r,K)=0.\end{aligned}$$

(b) The proof is similar to Part (a) and hence is omitted. □

Proof of Theorem 2.9. (a) If $f^0<\beta^{-1}$, then the proofs of Theorems 2.1 and 2.2 imply that $||\Gamma u||<||u||$ for $u\in K\cap\partial\Omega_r$ with $r>0$ small enough. By Lemma 4.4 (b), $i(\Gamma,K\cap\Omega_r,K)=1$.

Similarly, if $f_\infty>(\alpha\beta)^{-1}$, then the proofs of Theorems 2.1 and 2.2 imply that $||\Gamma u||>||u||$ for $u\in K\cap\partial\Omega_R$ with R large enough. By Lemma 4.4 (a), $i(\Gamma,K\cap\Omega_R,K)=0$. If $f_\infty>\underline{a}^{-1}\rho^{2m}$, then the same conclusion follows from Lemma 4.5 (b). Therefore,

$$\begin{aligned}i(\Gamma,K\cap(\Omega_R\setminus\overline{\Omega}_r),K) &= i(\Gamma,K\cap\Omega_R,K)-i(\Gamma,K\cap\Omega_r,K)\\ &= 0-1\neq 0,\end{aligned}$$

and hence $\Gamma u=u$ has a positive solution.

The other parts of the theorem can be proved similarly. Note that by Lemma 4.4 and the proof of Theorem 2.1, if there exists $r_*>0$ such that (2.7) holds and $\Gamma u\neq u$ on $K\cap\partial\Omega_{r_*}$, then $i(\Gamma,K\cap\Omega_{r_*},K)=1$; and if there exists an $r^*>0$ such that (2.8) holds and $\Gamma u\neq u$ on $K\cap\partial\Omega_{r^*}$, then $i(\Gamma,K\cap\Omega_{r^*},K)=0$. □

Proof of Theorem 2.10. (a) As in the proof of Theorem 2.9 we see that (2.9) implies $i(\Gamma, K\cap\Omega_r, K) = 1$, and from the assumptions $f_0 > \min\{(\alpha\beta)^{-1}, \underline{a}^{-1}\rho^{2m}\}$ and $f_\infty > \min\{(\alpha\beta)^{-1}, \underline{a}^{-1}\rho^{2m}\}$, there exist $0 < R_1 < r < R_2$ such that $i(\Gamma, K\cap\Omega_{R_1}, K) = 0$ and $i(\Gamma, K\cap\Omega_{R_2}, K) = 0$ for $0 < R_1 < r < R_2$. Therefore, $i(\Gamma, K\cap(\Omega_{R_2}\setminus\overline{\Omega}_r), K) \neq 0$ and $i(\Gamma, K\cap(\Omega_r\setminus\overline{\Omega}_{R_1}), K) \neq 0$, and hence $\Gamma u = u$ has at least two positive solutions.

(b) The proof is similar to Part (a) and hence is omitted. □

The proofs of Theorems 2.11 and 2.12 are in the same direction as those of Theorems 2.9 and 2.10 based on the following lemma.

Lemma 4.6. (a) *Assume $f_0 > \nu_0$. Then $i(\Gamma, K\cap\Omega_r, K) = 0$ for all sufficiently small $r > 0$.*

(b) *Assume $f^0 < \nu_0$. Then $i(\Gamma, K\cap\Omega_r, K) = 1$ for all sufficiently small $r > 0$.*

(c) *Assume $f_\infty > \nu_0$. Then $i(\Gamma, K\cap\Omega_r, K) = 0$ for all sufficiently large $r > 0$.*

(d) *Assume $f^\infty < \nu_0$. Then $i(\Gamma, K\cap\Omega_r, K) = 1$ for all sufficiently large $r > 0$.*

Proof. (a) Let $0 < l < 1$ be fixed, and define $\Gamma_2 : C[0,\omega] \to C[0,\omega]$ by

$$(\Gamma_2 u)(t) = \int_0^\omega G_1(t,s)a(s)u^l(s)\,ds.$$

Similar to the proof of Lemma 4.3, we can show that Γ_2 is compact and $\Gamma_2 K \subset K$. Fix $t_5 \in (0,\omega)$ and define $r_3 = (\alpha^l \int_0^\omega G_1(t_5,s)a(s)ds)^{1/(1-l)}$. Then for $0 < r \le r_3$ and $u \in K\cap\partial\Omega_r$, $u^l(t) \ge (\alpha r)^l$ on $[0,\omega]$, and hence

$$\begin{aligned}||\Gamma_2 u|| \ge (\Gamma_2 u)(t_5) &= \int_0^\omega G_1(t_5,s)a(s)u^l(s)\,ds \\ &\ge (\alpha r)^l \int_0^\omega G_1(t_5,s)a(s)\,ds \\ &= (\alpha r)^l \alpha^{-l} r_3^{1-l} \\ &\ge (\alpha r)^l \alpha^{-l} r^{1-l} = r = ||u||.\end{aligned} \tag{4.8}$$

By Lemma 4.4 (a), $i(\Gamma_2, K\cap\Omega_r, K)=0$.

Define a homotopy operator $\widetilde{H}:[0,1]\times K\to K$ by

$$\widetilde{H}(s,u)=(1-s)\Gamma u+s\Gamma_2 u,$$

where Γ is defined by (4.4) with $m=1$. Then $\widetilde{H}(s,\cdot)$ is compact for $0\le s\le 1$. Since $f_0>\nu_0$, we can choose $\varepsilon>0$ and $0<r_4\le r_3$ such that, for $(t,x)\in[0,\omega]\times[0,r_4]$,

$$f(t,x)\ge(\nu_0+\varepsilon)x \quad\text{and}\quad x^l\ge(\nu_0+\varepsilon)x.$$

Let $0<r\le r_4$. We now show that $\widetilde{H}(s,u)\ne u$ for all $0\le s\le 1$ and $u\in K\cap\partial\Omega_r$. Assume the contrary, i.e., there exists $s_2\in[0,1]$ and $u_2\in K\cap\partial\Omega_r$ with $\widetilde{H}(s_2,u_2)=u_2$. Then u_2 satisfies

$$-u_2''+\rho^2u_2=(1-s_2)a(t)f(t,u_2)+s_2a(t)u_2^l \tag{4.9}$$

and BC (1.2). Without loss of generality assume $v_0(t)$ be the eigenfunction of SLP (2.17) associated with ν_0 such that $v_0(t)>0$ on $(0,\omega)$. Multiplying both sides of (4.9) by v_0 and integrating the resiting equality over the interval $[0,\omega]$, we have

$$\begin{aligned}\int_0^\omega &[-u_2''(t)+\rho^2u_2(t)]v_0(t)\,dt\\ &=\int_0^\omega[(1-s_2)a(t)f(t,u_2(t))+s_2a(t)u_2^l(t)]v_0(t)\,dt.\end{aligned}$$

Using integration by parts and BC (1.2), we obtain that

$$\begin{aligned}\nu_0\int_0^\omega &a(t)u_2(t)v_0(t)\,dt\\ &=\int_0^\omega[-u_2''(t)+\rho^2u_2(t)]v_0(t)\,dt\\ &=\int_0^\omega[(1-s_2)a(t)f(t,u_2(t))+s_2a(t)u_2^l(t)]v_0(t)\,dt\\ &\ge(1-s_2)\int_0^\omega(\nu_0+\varepsilon)a(t)u_2(t)v_0(t)\,dt\\ &\quad+s_2\int_0^\omega(\nu_0+\varepsilon)a(t)u_2(t)v_0(t)\,dt\\ &=(\nu_0+\varepsilon)\int_0^\omega a(t)u_2(t)v_0(t)\,dt\end{aligned}$$

which is a contradiction since $\int_0^\omega a(t)u_2(t)v_0(t)\,dt > 0$. Hence,

$$\begin{aligned} i(\Gamma, K \cap \Omega_r, K) &= i(\widetilde{H}(0,\cdot), K \cap \Omega_r, K) \\ &= i(\widetilde{H}(1,\cdot), K \cap \Omega_r, K) = i(\Gamma_2, K \cap \Omega_r, K) = 0. \end{aligned}$$

The proofs of (b), (c) and (d) are similar to Part (a) and hence are omitted. □

Proof of Theorem 2.11. (a) If $f^0 < \nu_0$, then by Lemma 4.6 (b), $i(\Gamma, K \cap \Omega_r, K) = 1$ for some sufficiently small r. Similarly, if $f_\infty > \nu_0$, by Lemma 4.6 (a), $i(\Gamma, K \cap \Omega_R, K) = 0$ for sufficiently large R. Therefore,

$$\begin{aligned} &i(\Gamma, K \cap (\Omega_R \setminus \overline{\Omega}_r), K) \\ &\qquad = i(\Gamma, K \cap \Omega_R, K) - i(\Gamma, K \cap \Omega_r, K) = 0 - 1 \neq 0, \end{aligned}$$

and hence $\Gamma u = u$ has a positive solution.

The other parts of the theorem can be proved similarly. Note that by Lemma 4.4 and the proof of Theorem 2.1, if there exists $r_* > 0$ such that (2.7) holds and $\Gamma u \neq u$ on $K \cap \partial\Omega_{r_*}$, then $i(\Gamma, K \cap \Omega_{r_*}, K) = 1$; and if there exists $r^* > 0$ such that (2.8) holds and $\Gamma u \neq u$ on $K \cap \partial\Omega_{r^*}$, then $i(\Gamma, K \cap \Omega_{r^*}, K) = 0$. □

Proof of Theorem 2.12. (a) As in the proof of Theorem 2.11 we see that (2.9) implies $i(\Gamma, K \cap \Omega_r, K) = 1$, and from the assumptions $f_0 > \nu_0$ and $f_\infty > \nu_0$, there exist $0 < R_1 < r < R_2$ such that $i(\Gamma, K \cap \Omega_{R_1}, K) = 0$ and $i(\Gamma, K \cap \Omega_{R_2}, K) = 0$ for $0 < R_1 < r < R_2$. Therefore, $i(\Gamma, K \cap (\Omega_{R_2} \setminus \overline{\Omega}_r), K) \neq 0$ and $i(\Gamma, K \cap (\Omega_r \setminus \overline{\Omega}_{R_1}), K) \neq 0$, and hence $\Gamma u = u$ has at least two positive solutions.

(b) The proof is similar to Part (a) and hence is omitted. □

5. Nonlinear eigenvalue problem. In the last section, we apply our results in Section 2 to the eigenvalue problem consisting of the equation

$$(5.1) \qquad (-1)^m (D^2 - \rho^2) u = \lambda a(t) f(t, u), \quad 0 \leq t \leq \omega,$$

and BC (1.2), where $\rho > 0$ and $\lambda > 0$. Let β and ζ be defined as in (2.6) and (2.12), respectively. By simply applying Theorems 2.9, 2.10, Corollary 2.1 and Theorem 2.8 with f replaced by λf we obtain the following results immediately.

Theorem 5.1. BVP (5.1), (1.2) *has at least one positive solution if* $\lambda > 0$ *satisfies either*

(a) $\lambda f^0 < \beta^{-1}$ *and* $\lambda f_\infty > \zeta$;

(b) $\lambda f_0 > \zeta$ *and* $\lambda f^\infty < \beta^{-1}$;

(c) $\lambda f_0 > \zeta$ *or* $\lambda f_\infty > \zeta$, *and there exists* $r \in (0, \infty)$ *such that*

$$\lambda \le \lambda_* := \frac{r}{\beta \max_{(t,x)\in[0,\omega]\times[\alpha r, r]} f(t,x)}; \tag{5.2}$$

or

(d) $\lambda f^0 < \beta^{-1}$ *or* $\lambda f^\infty < \beta^{-1}$, *and there exists* $r \in (0, \infty)$ *such that*

$$\lambda \ge \lambda^* := \frac{r}{\beta \min_{(t,x)\in[0,\omega]\times[\alpha r, r]} f(t,x)}. \tag{5.3}$$

Theorem 5.2. BVP (5.1), (1.2) *has at least two positive solutions if* $\lambda > 0$ *satisfies either*

(a) $\lambda f_0 > \zeta$ *and* $\lambda f_\infty > \zeta$, *and there exists* $r \in (0, \infty)$ *such that* $\lambda < \lambda_*$ *with* λ_* *defined in* (5.2); *or*

(b) $\lambda f^0 < \beta^{-1}$ *and* $\lambda f^\infty < \beta^{-1}$, *and there exists* $r \in (0, \infty)$ *such that* $\lambda > \lambda^*$ *with* λ^* *defined in* (5.3).

Theorem 5.3. *Let* $\{r_i\}_{i=1}^\infty \subset \mathbf{R}$ *be such that* $0 < r_1 < r_2 < r_3 < \cdots$. *Let* $E_1 = \cup_{i=1}^\infty [\alpha r_{2i-1}, r_{2i-1}]$ *and* $E_2 = \cup_{i=1}^\infty [\alpha r_{2i}, r_{2i}]$. *Assume*

$$\limsup_{E_1 \ni x \to \infty} \max_{t\in[0,\omega]} \frac{f(t,x)}{x} = 0 \quad \text{and} \quad \liminf_{E_2 \ni x \to \infty} \min_{t\in[0,\omega]} \frac{f(t,x)}{x} = \infty.$$

Then for any $\lambda > 0$, BVP (5.1), (1.2) *has an infinite number of positive solutions.*

Theorem 5.4. BVP (5.1), (1.2) *has no positive solutions if* $\lambda > 0$ *satisfies either*

(a) $\lambda f(t,x)/x < \beta^{-1}$ *for all* $(t,x) \in [0,\omega] \times (0,\infty)$, *or*

(b) $\lambda f(t,x)/x < \beta^{-1}$ *for all* $(t,x) \in [0,\omega] \times (0,\infty)$.

REFERENCES

1. F.M. Atici and G.Sh. Guseinov, *On the existence of positive solutions for nonlinear differential equations with periodic boundary conditions*, J. Comput. Applied Math. **132** (2001), 341–356.

2. P. Bailey, N. Everitt and A. Zettl, *SLEIGN2 package for Sturm-Liouville problems*, `www.math.niu.edu/SL2/`.

3. K. Deimling, *Nonlinear functional nalysis*, Springer-Verlag, New York, 1985.

4. L. Erbe, *Eigenvalue criteria for existence of positive solutions to nonlinear boundary value problems*, Math. Computer Modelling **32** (2000), 529–539.

5. J.R. Graef and L. Kong, *Existence results for nonlinear periodic boundary value problems*, Proc. Edinburgh Math. Soc. **52** (2009), 79–95.

6. J.R. Graef, L. Kong, and H. Wang, *Existence, multiplicity, and dependence on a parameter for a periodic boundary value problem*, J. Differential Equations **245** (2008), 1185–1197.

7. Q. Kong, *Existence and nonexistence of solutions of second-order nonlinear boundary value problems*, Nonlinear Anal. **66** (2007), 2635–2651.

8. Q. Kong, H. Wu and A. Zettl, *Dependence of the n-th Sturm-Liouville eigenvalue on the problem*, J. Differential Equations **156** (1999), 328–356.

9. ———, *Dependence of eigenvalues on the problem*, Math. Nachr. **188** (1997), 173–201.

10. Q. Kong and A. Zettl, *Eigenvalues of regular Sturm-Liouville problems*, J. Differential Equations **131** (1996), 1–19.

11. M.A. Krasnosel'skii, *Topological methods in the theory of nonlinear integral equations*, Pergamon Press, Oxford, 1964.

12. M.K. Kwong and J.S.W. Wong, *The shooting method and non-homogeneous multi-point BVPs of second-order ODE*, Bound. Value Probl. 2007, Art. ID 64012, 16 pp.

13. K.Q. Lan, *Multiple positive solutions of Hammerstein integral equations and applications to periodic boundary value problems*, Applied Math. Comp. **154** (2004), 531–542.

14. Y. Li, *Positive solutions of fourth-order periodic boundary value problems*, Nonlinear Anal. **54** (2003), 1069–1078.

15. ———, *Existence and uniqueness for higher order periodic boundary value problems under spectral separation conditions*, J. Math. Anal. Appl. **322** (2006), 530–539.

16. F. Li, Y. Li and Z. Liang, *Existence and multiplicity of solutions to 2mth-order ordinary differential equations*, J. Math. Anal. Appl. **331** (2007), 958–977.

17. Y. Naito and S. Tanaka, *On the existence of multiple solutions of the boundary value problem for nonlinear second-order differential equations*, Nonlinear Anal. **56** (2004), 919–935.

18. I. Rachunková, M. Tvrdý and I. Vrkoč, *Existence of nonnegative and non-positive solutions for second order periodic boundary value problems*, J. Differential Equations **176** (2001), 445–469.

19. P.J. Torres, *Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem*, J. Differential Equations **190** (2003), 643–662.

20. Q. Yao, *Positive solutions of nonlinear second-order periodic boundary value problems*, Appl. Math. Lett. **20** (2007), 583–590.

21. E. Zeidler, *Nonlinear functional analysis and its applications* I: *Fixed-point theorems*, Springer-Verlag, New York, 1986.

22. Z. Zhang and J. Wang, *On existence and multiplicity of positive solutions to periodic boundary value problems for singular nonlinear second order differential equations*, J. Math. Anal. Appl. **281** (2003), 99–107.

Department of Mathematics, Northern Illinois University, DeKalb, IL 60115
Email address: kong@math.niu.edu

Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403
Email address: min-wang@utc.edu