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MEASURES OF NONCOMPACTNESS AND ASYMPTOTIC STABILITY OF SOLUTIONS OF A QUADRATIC HAMMERSTEIN INTEGRAL EQUATION

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ABSTRACT. In this paper we consider the existence of asymptotically stable solutions of a quadratic Hammerstein integral equation. This equation is investigated in the space of real functions defined, bounded and continuous on the real half-axis. The main tools used in our study are measures of noncompactness and the fixed point theorem of Darbo. An example illustrating the result and comparison with other results are also given.

1. Introduction. The paper discusses the nonlinear quadratic Hammerstein integral equation

$$x(t) = p(t) + f(t, x(t)) \int_0^\infty g(t,\tau)h(\tau, x(\tau))\, d\tau, \quad t \geq 0. \tag{1.1}$$

This equation contains as special cases many functional, integral and functional equations which arise in several papers and monographs both in pure and applied mathematics (cf. [**1–3, 8–11, 13, 19, 21**], for example).

Notice that equation (1.1) contains as a special case the classical functional equation associated with the superposition operator [**3, 18**] which has the form

$$x(t) = f(t, x(t)).$$

Moreover, the classical Hammerstein integral equation on bounded interval

$$x(t) = p(t) + \int_a^b g(t,\tau)h(\tau, x(\tau))\, d\tau \tag{1.2}$$

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and its quadratic counterpart

$$x(t) = p(t) + f(t, x(t)) \int_a^b g(t,\tau)h(\tau, x(\tau))\, d\tau \tag{1.3}$$

are also special cases of equation (1.1).

Note that equations (1.2) and (1.3) arise in various real world phenomena in mathematical physics, mechanics, engineering, biology, vehicular traffic theory, the theory of radiative transfer, the kinetic theory of gases and so on (see [**1, 10, 11, 13, 15, 17, 19, 22**]). In this paper we will study equation (1.1) in the Banach space consisting of real functions which are defined, continuous and bounded on the real half-line. We use measures of noncompactness and the fixed point theorem of Darbo to establish our main result (cf. [**4, 6, 12**]). More precisely, we will use a suitable measure of noncompactness, which enables us to prove that equation (1.1) has solutions which are asymptotically stable.

In the last section of the paper we provide an example illustrating the applicability of our result and we compare our result with results in the literature.

2. Preliminaries and auxiliary facts. In this section we collect some definitions and results. Denote by $\mathbf{R}$ the set of real numbers and by $\mathbf{R}_+$ the interval $[0, \infty)$. Assume that E is an infinite dimensional Banach space with norm $\|\cdot\|$ and zero element θ. Denote by $B(x, r)$ the closed ball centered at x, with radius r. The symbol B_r stands for the ball $B(\theta, r)$. If X is a subset of E then we write $\overline{X}$, $\operatorname{Conv} X$ to denote the closure and convex closure of X, respectively.

If X, Y are subsets of E and $\lambda \in \mathbf{R}$, then we write $X + Y$, λX to denote the usual algebraic operations on sets. Further, denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

We use the following definition of the concept of a measure of noncompactness (cf. [**6**]).

Definition 2.1. A mapping $\mu : \mathfrak{M}_E \to \mathbf{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1^o The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.

2^o $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.

3^o $\mu(\overline{X}) = \mu(X)$.

4^o $\mu(\operatorname{Conv} X) = \mu(X)$.

5^o $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.

6^o If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, \dots)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ described in 1^o is said to be *the kernel of the measure of noncompactness* μ. It is known (see [**6**]) that the intersection set X_∞ from 6^o is a member of the family $\ker\mu$.

Now, we recall the fixed point theorem of Darbo (cf. [**4, 6, 12**]).

Theorem 2.2. *Let Ω be a nonempty, bounded, closed and convex subset of the Banach space E, and let $Q : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that $\mu(QX) \le k\mu(X)$ for any nonempty subset X of Ω. Then Q has a fixed point in the set Ω.*

Remark 2.3. Let us denote by $\operatorname{Fix} Q$ the set of all fixed points of the operator Q on the set Ω. It is known (cf. [6]) that $\operatorname{Fix} Q \in \ker\mu$.

In what follows we will work with the Banach space $BC(\mathbf{R}_+)$ consisting of all real functions defined, continuous and bounded on the interval $\mathbf{R}_+$. This space is equipped with the standard norm

$$||x|| = \sup\{|x(t)| : \ t \in \mathbf{R}_+\}.$$

We will use a measure of noncompactness in the space $BC(\mathbf{R}_+)$ which was introduced in [**6**]. To define this measure let us fix a nonempty subset X of the space $BC(\mathbf{R}_+)$ and a number $T > 0$. Next, for $x \in X$ and $\varepsilon \ge 0$, denote by $\omega^T(x,\varepsilon)$ *the modulus of continuity* of the function x on the interval $[0,T]$, i.e.,

$$\omega^T(x,\varepsilon) = \sup\{|x(t) - x(s)| : \ t, s \in [0,T], \ |t-s| \le \varepsilon\}.$$

Further, let

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : \ x \in X\},$$

$$\omega_0^T(X) = \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon), \qquad \omega_0(X) = \lim_{T\to\infty} \omega_0^T(X).$$

For a fixed number $t \in \mathbf{R}_+$, let

$$X(t) = \{x(t) : \ x \in X\}$$

and

$$\operatorname{diam} X(t) = \sup\{|x(t) - y(t)| : \ x, y \in X\}.$$

Finally, consider the function μ defined on the family $\mathfrak{M}_{BC(\mathbf{R}_+)}$ by the formula

$$\mu(X) = \omega_0(X) + \limsup_{t\to\infty} \operatorname{diam} X(t). \tag{2.1}$$

It is known (cf. [**5, 6**]) that the function μ is a measure of noncompactness in the space $BC(\mathbf{R}_+)$. Moreover, the kernel $\ker \mu$ of this measure consists of all nonempty and bounded sets X such that functions belonging to X are locally equicontinuous on $\mathbf{R}_+$ and the thickness of the bundle formed by graphs of functions from X tends to zero at infinity. This property in conjunction with Remark 2.3 allows us to characterize solutions of equation (1.1) (cf. the next section).

Let us note that the measure of noncompactness μ defined by (2.1) does not characterize the family of all relatively compact subsets of the space $BC(\mathbf{R}_+)$. Indeed, consider the set in $BC(\mathbf{R}_+)$ consisting of two distinct constant functions. Observe that although this set is relatively compact, it has nonzero measure of noncompactness μ, since the thickness of the bundle formed of its graphs is positive.

In what follows assume that Ω is a nonempty subset of the space $BC(\mathbf{R}_+)$ and Q is an operator defined on Ω with values in $BC(\mathbf{R}_+)$. Consider the operator equation of the form

$$x(t) = (Qx)(t), \quad t \in \mathbf{R}_+. \tag{2.2}$$

Definition 2.4. We say that solutions of equation (2.2) are asymptotically stable if there exists a ball $B(x_0, r)$ in the space $BC(\mathbf{R}_+)$ with $B(x_0, r) \cap \Omega \neq \varnothing$ such that for every $\varepsilon > 0$ there exists a $T > 0$ with the property

$$|x(t) - y(t)| \leq \varepsilon$$

for all solutions x, y of equation (2.2) such that $x, y \in B(x_0, r) \cap \Omega$ and for $t \geq T$.

Note the concept of asymptotic stability described above was introduced in [**9**] (cf. also [**16**]).

3. Main result. In this section we will consider the quadratic Hammerstein integral equation (1.1) assuming that the following hypotheses are satisfied:

(i) $p \in BC(\mathbf{R}_+)$.

(ii) The function $f : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ is continuous and there exists a continuous function $m : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$|f(t,x) - f(t,y)| \leq m(t)|x - y|$$

for all $x, y \in \mathbf{R}$ and for any for any $t \in \mathbf{R}_+$.

(iii) The function $g : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$ is continuous.

(iv) The function $h : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ is continuous and there exist a continuous function $a : \mathbf{R}_+ \to \mathbf{R}_+$ and a continuous and nondecreasing function $b : \mathbf{R}_+ \to \mathbf{R}_+$ with $b(0) = 0$, such that

$$|h(t,x) - h(t,y)| \leq a(t)b(|x - y|)$$

for $t \in \mathbf{R}_+$ and for $x, y \in \mathbf{R}$.

(v) The functions $\tau \to a(\tau)|g(t,\tau)|$, $\tau \to |h(\tau,0)g(t,\tau)|$ are integrable over $\mathbf{R}_+$ for any fixed $t \in \mathbf{R}_+$.

(vi) The functions $G_a,\ G_h,\ F_a,\ F_h,\ M_a,\ M_h :\ \mathbf{R}_+ \to \mathbf{R}_+$ defined by the formulas

$$\begin{aligned}
G_a(t) &= \int_0^\infty a(\tau)|g(t,\tau)|\,d\tau, & \qquad G_h(t) &= \int_0^\infty |h(\tau,0)g(t,\tau)|\,d\tau,\\
F_a(t) &= |f(t,0)|G_a(t), & F_h(t) &= |f(t,0)|G_h(t),\\
M_a(t) &= m(t)G_a(t), & M_h(t) &= m(t)G_h(t)
\end{aligned}$$

are bounded on $\mathbf{R}_+$. Moreover, the functions $F_a(t)$ and $M_a(t)$ vanish at infinity, i.e., $\lim_{t\to\infty} F_a(t) = \lim_{t\to\infty} M_a(t) = 0$.

Observe that, keeping in mind assumption (vi), we may define the finite constant $\overline{G}_a$ by putting

$$\overline{G}_a = \sup\{G_a(t) : \ t \in \mathbf{R}_+\}.$$

Analogously, we define the constants $\overline{G}_h, \ \overline{F}_a, \ \overline{F}_h, \ \overline{M}_a, \ \overline{M}_h$ also to be finite by assumption (vi).

Further, we formulate our last assumptions.

(vii) The following equalities hold:

$$\lim_{T\to\infty}\left\{\sup\left\{|f(t,0)|\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau : \ t \in \mathbf{R}_+\right\}\right\} = 0,$$

$$\lim_{T\to\infty}\left\{\sup\left\{m(t)\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau : \ t \in \mathbf{R}_+\right\}\right\} = 0,$$

$$\lim_{T\to\infty}\left\{\sup\left\{m(t)\int_T^\infty |h(\tau,0)g(t,\tau)|\,d\tau : \ t \in \mathbf{R}_+\right\}\right\} = 0,$$

$$\lim_{T\to\infty}\left\{\sup\left\{|f(t,0)|\int_T^\infty |h(\tau,0)g(t,\tau)|\,d\tau : \ t \in \mathbf{R}_+\right\}\right\} = 0.$$

(viii) There exists a positive solution $r = r_0$ of the inequality

$$||p|| + rb(r)\overline{M}_a + b(r)\overline{F}_a + r\overline{M}_h + \overline{F}_h \le r$$

such that $\overline{M}_a b(r_0) + \overline{M}_h < 1$.

Remark 3.1. Observe that the inequality $\overline{M}_a b(r_0) + \overline{M}_h < 1$ from assumption (viii) is satisfied provided the term

$$||p|| + b(r_0)\overline{F}_a + \overline{F}_h$$

does not vanish identically. Indeed, assume that r_0 is a positive solution of the inequality from (viii), i.e.,

$$||p|| + r_0 b(r_0)\overline{M}_a + b(r_0)\overline{F}_a + r_0\overline{M}_h + \overline{F}_h \le r_0.$$

Then we get

$$\overline{M}_a r_0 b(r_0) + \overline{M}_h r_0 \leq r_0 - (||p|| + b(r_0)\overline{F}_a + \overline{F}_h).$$

Consequently,

$$\overline{M}_a b(r_0) + \overline{M}_h \leq 1 - \frac{1}{r_0}\left(||p|| + b(r_0)\overline{F}_a + \overline{F}_h\right).$$

From this inequality our assertion follows.

Remark 3.2. Let us discuss assumption (vii). In the theory of improper Riemann integral with a parameter one uses the concept of *uniform convergence of improper integral with respect to a parameter* [**14**]. We recall the definition of that concept adapting it to our situation.

So, assume that the function $v(t,\tau) = v : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}$ is such that the integral

$$\int_0^\infty v(t,\tau)\, d\tau \tag{3.1}$$

exists for any $t \in \mathbf{R}_+$. We say that integral (3.1) is *uniformly convergent* with respect to $t \in \mathbf{R}_+$ if

$$\lim_{T\to\infty} \int_0^T v(t,\tau)\, d\tau = \int_0^\infty v(t,\tau)\, d\tau$$

uniformly with respect to $t \in \mathbf{R}$.

Equivalently integral (3.1) is uniformly convergent with respect to $t \in \mathbf{R}_+$ if

$$\lim_{T\to\infty} \left\{ \sup_{t\in\mathbf{R}_+} \int_T^\infty v(t,\tau)\, d\tau \right\} = 0. \tag{3.2}$$

Thus, in the light of definition (3.2) we can formulate equivalently assumption (vii) in the following way: The integrals

$$\int_0^\infty a(\tau)|f(t,0)g(t,\tau)|\,d\tau, \qquad \int_0^\infty m(t)a(\tau)|g(t,\tau)|\,d\tau,$$

$$\int_0^\infty m(t)|h(\tau,0)g(t,\tau)|\,d\tau, \qquad \int_0^\infty |f(t,0)h(\tau,0)g(t,\tau)|\,d\tau$$

are uniformly convergent with respect to $t \in \mathbf{R}_+$.

Let us also provide a few remarks concerning other assumptions formulated above. For example, assumption (ii) implies at most linear growth of the function $f(t,x)$ with respect to the variable x at infinity and at zero. Similarly, assumption (iv) allows such a growth of h as is allowed by (viii) either at infinity or at zero. More precisely, the function b can be assumed to be sublinear at infinity with $\lim_{r\to\infty} b(r)/r < 1$ and $\overline{M}_h < 1$ and $r = r_0$ can be taken sufficiently large or the function b can grow at most linearly close to zero and then we look for $r = r_0$ close to zero rather than infinity (provided the constants are small enough).

Now, we start by formulating our main result.

Theorem 3.3. *Under assumptions* (i)–(viii) *equation* (1.1) *has at least one solution $x = x(t)$ in the space $BC(\mathbf{R}_+)$. Moreover, solutions of equation* (1.1) *are asymptotically stable.*

Proof. Consider the operator H defined on the space $BC(\mathbf{R}_+)$ by the formula

$$(Hx)(t) = p(t) + f(t,x(t)) \int_0^\infty g(t,\tau)h(\tau,x(\tau))\,d\tau, \quad t \in \mathbf{R}_+.$$

In view of imposed assumptions it is easily seen that the function Hx is well defined and continuous for an arbitrary function $x \in BC(\mathbf{R}_+)$. Further, invoking our assumptions, for an arbitrary fixed $t \in \mathbf{R}_+$ we obtain:

$$|(Hx)(t)| \le |p(t)| + |f(t,x(t))| \int_0^\infty |g(t,\tau)||h(\tau,x(\tau))|\,d\tau$$

$$
\begin{aligned}
&\leq |p(t)| + [|f(t,x(t)) - f(t,0)| + |f(t,0)|] \\
&\quad \times \int_0^\infty |g(t,\tau)|[|h(\tau,x(\tau)) - h(\tau,0)| + |h(\tau,0)|]\,d\tau \\
&\leq |p(t)| + [m(t)|x(t)| + |f(t,0)|] \\
&\quad \times \int_0^\infty |g(t,\tau)|[a(\tau)b(|x(\tau)|) + |h(\tau,0)|]\,d\tau \\
&\leq ||p|| + [m(t)||x|| + |f(t,0)|] \\
&\quad \times \int_0^\infty |g(t,\tau)|[a(\tau)b(||x||) + |h(\tau,0)|]\,d\tau \\
&= ||p|| + ||x||b(||x||)M_a(t) + b(||x||)F_a(t) + ||x||M_h(t) + F_h(t) \\
&\leq ||p|| + ||x||b(||x||)\overline{M}_a + b(||x||)\overline{F}_a + ||x||\overline{M}_h + \overline{F}_h.
\end{aligned}
$$

From the above estimate we deduce that the function Hx is bounded on the interval $\mathbf{R}_+$. This allows us to infer that the operator H transforms the space $BC(\mathbf{R}_+)$ into itself. Moreover, this estimate implies the following inequality

$$||Hx|| \leq ||p|| + ||x||b(||x||)\overline{M}_a + b(||x||)\overline{F}_a + ||x||\overline{M}_h + \overline{F}_h.$$

This inequality in conjunction with assumption (viii) ensures the existence of a positive number r_0 such that $\overline{M}_a b(r_0) + \overline{M}_h < 1$ and the operator H transforms the ball B_{r_0} into itself.

In what follows we show that the operator V is continuous on the ball B_{r_0}. To do this fix a number $\varepsilon > 0$ and take $x, y \in B_{r_0}$ such that $||x - y|| \leq \varepsilon$. Then, invoking assumptions (i)–(vi), for an arbitrarily fixed $t \in \mathbf{R}_+$ we get:

(3.3)

$$
\begin{aligned}
&|(Hx)(t) - (Hy)(t)| \\
&\qquad \leq \bigg| f(t,x(t)) \int_0^\infty g(t,\tau)h(\tau,x(\tau))\,d\tau \\
&\qquad\qquad - f(t,y(t)) \int_0^\infty g(t,\tau)h(\tau,x(\tau))\,d\tau \bigg|
\end{aligned}
$$

$$
\begin{aligned}
&+ \left| f(t, y(t)) \int_0^\infty g(t,\tau) h(\tau, x(\tau))\, d\tau \right. \\
&\qquad \left. - f(t, y(t)) \int_0^\infty g(t,\tau) h(\tau, y(\tau))\, d\tau \right| \\
&\leq |f(t, x(t)) - f(t, y(t))| \\
&\quad \times \int_0^\infty |g(t,\tau)| [|h(\tau, x(\tau)) - h(\tau, 0)| + |h(\tau, 0)|]\, d\tau \\
&\quad + [|f(t, y(t)) - f(t, 0)| + |f(t, 0)|] \\
&\quad \times \int_0^\infty |g(t,\tau)| |h(\tau, x(\tau)) - h(\tau, y(\tau))|\, d\tau \\
&\leq m(t)|x(t) - y(t)| \int_0^\infty |g(t,\tau)| [a(\tau) b(||x||) + |h(\tau, 0)|]\, d\tau \\
&\quad + [m(t)||y|| + |f(t, 0)|] \int_0^\infty |g(t,\tau)| a(\tau) b(|x(\tau) - y(\tau)|)\, d\tau.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
|(Hx)(t) - (Hy)(t)| \leq \varepsilon b(||x||) m(t) &\int_0^\infty a(\tau) |g(t,\tau)|\, d\tau \\
&+ \varepsilon m(t) \int_0^\infty |g(t,\tau) h(\tau, 0)|\, d\tau \\
&+ ||y|| b(\varepsilon) m(t) \int_0^\infty a(\tau) |g(t,\tau)|\, d\tau \\
&+ b(\varepsilon) |f(t, 0)| \int_0^\infty a(\tau) |g(t,\tau)|\, d\tau.
\end{aligned}
$$

This yields

$$
||Hx - Hy|| \leq \varepsilon b(r_0) \overline{M}_a + \varepsilon \overline{M}_h + r_0 b(\varepsilon) \overline{M}_a + b(\varepsilon) \overline{F}_a.
$$

From the above estimate we obtain the desired continuity of the operator H on the ball B_{r_0}.

Next, let us take a nonempty subset X of the ball B_{r_0} and fix arbitrarily $T > 0$ and $\varepsilon > 0$. Choose a function $x \in X$ and $t, s \in [0, T]$ such that $|t - s| \le \varepsilon$. Then, in view of our assumptions we get:

(3.4)

$$\begin{aligned}
|(Hx)(t) - (Hx)(s)| &\le |p(t) - p(s)| \\
&\quad + \left| f(t, x(t)) \int_0^\infty g(t,\tau) h(\tau, x(\tau))\, d\tau \right. \\
&\quad \left. - f(s, x(s)) \int_0^\infty g(t,\tau) h(\tau, x(\tau))\, d\tau \right| \\
&\quad + \left| f(s, x(s)) \int_0^\infty g(t,\tau) h(\tau, x(\tau))\, d\tau \right. \\
&\quad \left. - f(s, x(s)) \int_0^\infty g(s,\tau) h(\tau, x(\tau))\, d\tau \right| \\
&\le \omega^T(p, \varepsilon) + |f(t, x(t)) - f(s, x(s))| \\
&\quad \times \int_0^\infty |g(t,\tau)| [|h(\tau, x(\tau)) - h(\tau, 0)| + |h(\tau, 0)|]\, d\tau \\
&\quad + |f(s, x(s))| \int_0^\infty |g(t,\tau) - g(s,\tau)| [|h(\tau, x(\tau)) - h(\tau, 0)| \\
&\quad + |h(\tau, 0)|]\, d\tau \\
&\le \omega^T(p, \varepsilon) + [|f(t, x(t)) - f(t, x(s))| + |f(t, x(s)) - f(s, x(s))|] \\
&\quad \times \int_0^\infty |g(t,\tau)| [a(\tau) b(|x(\tau)|)\, d\tau \\
&\quad + |h(\tau, 0)|]\, d\tau + [|f(s, x(s)) - f(s, 0)| + |f(s, 0)|] \\
&\quad \times \int_0^\infty |g(t,\tau) - g(s,\tau)| [a(\tau) b(|x(\tau)|) + |h(\tau, 0)|]\, d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le \omega^T(p,\varepsilon) + [m(t)|x(t)-x(s)| + \omega^T_{r_0}(f,\varepsilon)] \\
&\quad \times \left\{ b(r_0)\int_0^\infty a(\tau)|g(t,\tau)|\,d\tau + \int_0^\infty |g(t,\tau)h(\tau,0)|\,d\tau \right\} \\
&\quad + [m(s)|x(s)| + (f(s,0))] \\
&\quad \times \left\{ \int_0^\infty b(r_0)a(\tau)|g(t,\tau)-g(s,\tau)|\,d\tau \right. \\
&\qquad\qquad\qquad \left. + \int_0^\infty |g(t,\tau)-g(s,\tau)||h(\tau,0)|\,d\tau \right\} \\
&\le \omega^T(p,\varepsilon) + b(r_0)\omega^T(x,\varepsilon)m(t)\int_0^\infty a(\tau)|g(t,\tau)|\,d\tau \\
&\quad + b(r_0)\omega^T_{r_0}(f,\varepsilon)\int_0^\infty a(\tau)|g(t,\tau)|\,d\tau \\
&\quad + \omega^T(x,\varepsilon)m(t)\int_0^\infty |g(t,\tau)h(\tau,0)|\,d\tau \\
&\quad + \omega^T_{r_0}(f,\varepsilon)\int_0^\infty |g(t,\tau)h(\tau,0)|\,d\tau \\
&\quad + [m(s)r_0 + |f(s,0)|]b(r_0)\int_0^\infty a(\tau)|g(t,\tau)-g(s,\tau)|\,d\tau \\
&\quad + [m(s)r_0 + |f(s,0)|]\int_0^\infty |g(t,\tau)-g(s,\tau)||h(\tau,0)|\,d\tau \\
&\le \omega^T(p,\varepsilon) + b(r_0)\overline{M}_a\omega^T(x,\varepsilon) + b(r_0)\overline{G}_a\omega^T_{r_0}(f,\varepsilon) \\
&\quad + \overline{M}_h\omega^T(x,\varepsilon) + \overline{G}_h\omega^T_{r_0}(f,\varepsilon) \\
&\quad + [m(s)r_0 + |f(s,0)|]b(r_0)\left\{ \int_0^T a(\tau)|g(t,\tau)-g(s,\tau)|\,d\tau \right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \int_T^\infty a(\tau)[|g(t,\tau)| + |g(s,\tau)|]\, d\tau \Big\} \\
&\quad + [m(s)r_0 + |f(s,0)|] \Big\{ \int_0^T |g(t,\tau) - g(s,\tau)||h(\tau,0)|\, d\tau \\
&\qquad\qquad + \int_T^\infty [|g(t,\tau)| + |g(s,\tau)|]|h(\tau,0)|\, d\tau \Big\} \\
&\le \omega^T(p,\varepsilon) + \left(b(r_0)\overline{M}_a + \overline{M}_h\right) \omega^T(x,\varepsilon) \\
&\quad + \left(b(r_0)\overline{G}_a + \overline{G}_h\right) \omega^T_{r_0}(f,\varepsilon) \\
&\quad + (M_T r_0 + F_T) b(r_0) \omega_1^T(g,\varepsilon) \int_0^T a(\tau)\, d\tau \\
&\quad + (M_T r_0 + F_T) \omega_1^T(g,\varepsilon) \int_0^T |h(\tau,0)|\, d\tau \\
&\quad + [m(s)r_0 + |f(s,0)|] b(r_0) \int_T^\infty a(\tau)|g(s,\tau)|\, d\tau \\
&\quad + [m(s)r_0 + |f(s,0)|] \int_T^\infty |g(s,\tau)h(\tau,0)|\, d\tau \\
&\quad + [m(s)r_0 + |f(s,0)|] b(r_0) \int_T^\infty a(\tau)|g(t,\tau)|\, d\tau \\
&\quad + [m(s)r_0 + |f(s,0)|] \int_T^\infty |g(t,\tau)h(\tau,0)|\, d\tau,
\end{aligned}
$$

where we denoted

$$
\begin{aligned}
\omega^T_{r_0}(f,\varepsilon) &= \sup\{|f(t,x) - f(s,x)| : \ t,s \in [0,T],\ |t-s| \le \varepsilon, \\
&\qquad\qquad\qquad\qquad\qquad\qquad x \in [-r_0, r_0]\}, \\
\omega_1^T(g,\varepsilon) &= \sup\{|g(t,\tau) - g(s,\tau)| : \ t,s,\tau \in [0,T],\ |t-s| \le \varepsilon\}, \\
F_T &= \sup\{|f(t,0)| : \ t \in [0,T]\},
\end{aligned}
$$

$$M_T = \sup\{m(t):\ t \in [0,T]\}.$$

Now, observe that in view of our assumptions we have the following estimates:

$$
\begin{aligned}
(3.5)\quad m(s)\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau
&\le [|m(s)-m(t)| + m(t)]\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau\\
&\le \omega^T(m,\varepsilon)\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau + m(t)\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau\\
&\le \omega^T(m,\varepsilon)\int_0^\infty a(\tau)|g(t,\tau)|\,d\tau + m(t)\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau\\
&\le \overline{G}_a\omega^T(m,\varepsilon) + m(t)\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau.
\end{aligned}
$$

In a similar way we obtain:

(3.6)

$$
\begin{aligned}
|f(s,0)|\int_T^\infty a(\tau)|g(t,\tau)|d\tau
&\le [|f(s,0)-f(t,0)| + |f(t,0)|]\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau\\
&\le \overline{\omega}^T(f,\varepsilon)\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau + |f(t,0)|\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau\\
&\le \overline{G}_a\overline{\omega}^T(f,\varepsilon) + |f(t,0)|\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau,
\end{aligned}
$$

$$
\begin{aligned}
(3.7)\quad m(s)\int_T^\infty |g(t,\tau)h(\tau,0)|\,d\tau
&\le \omega^T(m,\varepsilon)\int_0^\infty |g(t,\tau)h(\tau,0)|\,d\tau \\
&\quad + m(t)\int_T^\infty |g(t,\tau)h(\tau,0)|\,d\tau \\
&\le \overline{G}_h\omega^T(m,\varepsilon) + m(t)\int_T^\infty |g(t,\tau)h(\tau,0)|\,d\tau,
\end{aligned}
$$

$$
\begin{aligned}
(3.8)\quad &|f(s,0)|\int_0^\infty |g(t,\tau)h(\tau,0)|\,d\tau \\
&\le \overline{\omega}^T(f,\varepsilon)\int_0^\infty |g(t,\tau)h(\tau,0)|\,d\tau + |f(t,0)|\int_T^\infty |g(t,\tau)h(\tau,0)|\,d\tau \\
&\le \overline{G}_h\overline{\omega}^T(f,\varepsilon) + |f(t,0)|\int_T^\infty |g(t,\tau)h(\tau,0)|\,d\tau,
\end{aligned}
$$

where we denoted

$$\overline{\omega}^T(f,\varepsilon) = \sup\{|f(t,0)-f(s,0)| :\ t,s\in[0,T],\ |t-s|\le\varepsilon\}.$$

Further, let us observe that linking (3.4)–(3.8) and taking into account the uniform continuity of the functions $p(t)$, $m(t)$ on the interval $[0,T]$ and the uniform continuity of the functions $f(t,x)$, $g(t,\tau)$ on the sets $[0,T]\times[-r_0,r_0]$, $[0,T]\times[0,T]$ respectively, we obtain the following inequality

$$
\begin{aligned}
\omega_0^T(HX) &\le \left(\overline{M}_a b(r_0) + \overline{M}_h\right)\omega_0^T(X) \\
&\quad + r_0 b(r_0)\left\{ m(s)\int_T^\infty a(\tau)|g(s,\tau)|\,d\tau + m(t)\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau \right\}
\end{aligned}
$$

$$
\begin{aligned}
&+b(r_0)\left\{|f(s,0)|\int_T^\infty a(\tau)|g(s,\tau)|d\tau + |f(t,0)|\int_T^\infty a(\tau)|g(t,\tau)|d\tau\right\}\\
&+r_0\left\{m(s)\int_T^\infty |g(s,\tau)h(\tau,0)|d\tau + m(t)\int_T^\infty |g(t,\tau)h(\tau,0)|d\tau\right\}\\
&+\left\{|f(s,0)|\int_T^\infty |g(s,\tau)h(\tau,0)|d\tau + |f(t,0)|\int_T^\infty |g(t,\tau)h(\tau,0)|d\tau\right\}.
\end{aligned}
$$

From the above inequality and assumption (vii) we deduce the following estimate

$$
\omega_0(HX) \le (\overline{M}_a b(r_0) + \overline{M}_h)\omega_0(X). \tag{3.9}
$$

Now, let us take arbitrary functions $x, y \in X$ and fix $t \in \mathbf{R}_+$. Then, using estimate (3.3), in view of assumptions (ii)–(vi) we get:

$$
\begin{aligned}
|(Hx)(t) - (Hy)(t)| \le{}& m(t)\operatorname{diam} X(t) b(r_0)\int_0^\infty a(\tau)|g(t,\tau)|\,d\tau\\
&+ m(t)\operatorname{diam} X(t)\int_0^\infty |g(t,\tau)h(\tau,0)|\,d\tau\\
&+ r_0 m(t)\int_0^\infty a(\tau)|g(t,\tau)|b(\operatorname{diam} X(\tau))\,d\tau\\
&+ |f(t,0)|\int_0^\infty a(\tau)|g(t,\tau)|b(\operatorname{diam} X(\tau))\,d\tau\\
={}& b(r_0)M_a(t)\operatorname{diam} X(t) + M_h(t)\operatorname{diam} X(t)\\
&+ r_0 b(2r_0)M_a(t) + b(2r_0)F_a(t)\\
\le{}& b(r_0)M_a(t)\operatorname{diam} X(t) + \overline{M}_h\operatorname{diam} X(t)\\
&+ r_0 b(2r_0)M_a(t) + b(2r_0)F_a(t).
\end{aligned}
$$

Hence we obtain the following estimate:

$$
\begin{aligned}
\operatorname{diam}(HX)(t) \le{}& (b(r_0)M_a(t) + \overline{M}_h)\operatorname{diam} X(t)\\
&+ r_0 b(2r_0)M_a(t) + b(2r_0)F_a(t).
\end{aligned}
$$

This estimate in conjunction with assumption (vi) yields

$$\limsup_{t\to\infty} \operatorname{diam}(HX)(t) \le \overline{M}_h \limsup_{t\to\infty} \operatorname{diam} X(t). \tag{3.10}$$

Now, combining (3.9), (3.10) and taking into account the definition of the measure of noncompactness μ given by formula (2.1), we obtain

$$\mu(HX) \le (\overline{M}_a b(r_0) + \overline{M}_h)\mu(X).$$

Hence, in view of assumption (viii) and Theorem 2.2 we conclude that the operator H has at least one fixed point x in the ball B_{r_0} which is a solution of equation (1.1). Moreover, keeping in mind Remark 2.3 and Definition 2.4 we infer that solutions of equation (1.1) are asymptotically stable. This completes the proof. □

4. An example and remarks. This section gives an example of our theory. We provide also a few remarks concerning the assumptions imposed in Theorem 3.3. Moreover, we compare our result from Theorem 3.3 with those obtained in other papers.

To date no results have appeared in the literature on the existence of asymptotically stable solutions of integral equations of the type (1.1). In the papers [**7, 8**] we discussed the existence of solutions of equation (1.1) which vanish at infinity. One of the assumptions imposed in those papers requires that the function $h = h(t,x)$ satisfies the inequality

$$|h(t,x)| \le a(t)b(|x|) \tag{4.1}$$

for $t \in \mathbf{R}_+$ and $x \in \mathbf{R}$. Observe that the corresponding assumption in Theorem 3.3 (assumption (iv)) is more general than (4.1). Also in [**7, 8**] we required that the function $p = p(t)$ vanish at infinity while in Theorem 3.3 we assumed its boundedness only.

Let us observe that assumptions (vi) and (vii) in Theorem 3.3 seem to be more complicated than corresponding assumptions in [**7**] (cf. Theorem 3.4). However, if we assume that inequality (4.1) is satisfied then we conclude that $|h(t,0)| \le Ba(t)$ for $t \in \mathbf{R}_+$, where $B = b(0)$ is a constant. This shows that in such a situation we can reduce these parts of assumptions (vi) and (vii), where the function $\tau \to h(\tau,0)$

is involved. More precisely, the requirements concerning the functions $G_h(t)$, $F_h(t)$ and $M_h(t)$ are automatically satisfied and we can dispense with them in assumption (vi). Similarly, we can dispense with the two last equalities in assumption (vii). Summing up we see that the result contained in Theorem 3.3 generalizes those from [**7, 8**]. In order to compare our result with those contained in other papers first we provide an example.

Example 4.1. Consider the quadratic Hammerstein integral equation of the form

$$\begin{aligned} x(t) &= \sin(t^2+1) \\ &\quad + \frac{1}{20}\left[\sqrt[3]{t}x(t)\arctan x(t) + 5\sqrt[4]{t+1}\right] \\ &\quad \times \int_0^\infty \frac{\tau}{1+t^2+\tau^4}\left(\frac{\tau^3}{1+\tau^2} + \tau\sqrt[3]{x^2(\tau)}\right) d\tau, \end{aligned} \tag{4.2}$$

where $t \in \mathbf{R}_+$. Observe that this equation is a special case of equation (1.1), where $p(t) = \sin(t^2+1)$ and

$$\begin{aligned} f(t,x) &= \frac{1}{20}\left[\sqrt[3]{t}x\arctan x + 5\sqrt[4]{t+1}\right], \\ g(t,\tau) &= \frac{\tau}{1+t^2+\tau^4}, \\ h(t,x) &= \frac{t^3}{t^2+1} + t\sqrt[3]{x^2}. \end{aligned}$$

Obviously the function $p(t)$ satisfies assumption (i) with $||p|| = 1$ and the function $g(t,\tau)$ satisfies assumption (iii). Moreover, the functions $f(t,x)$ and $h(t,x)$ are continuous on $\mathbf{R}_+ \times \mathbf{R}$ and for arbitrary $x, y \in \mathbf{R}$ and $t \in \mathbf{R}_+$ we obtain:

$$\begin{aligned} |f(t,x) - f(t,y)| &\le \frac{\pi+1}{40}\sqrt[3]{t}|x-y|, \\ |h(t,x) - h(t,y)| &\le t\sqrt[3]{|x-y|^2}. \end{aligned}$$

Hence we infer that the function $f(t,x)$ satisfies assumption (ii) with $m(t) = (\pi+1)/40\sqrt[3]{t}$ and the function $h(t,x)$ satisfies (iv) with $a(t) = t$

and $b(r) = r^{2/3}$. Moreover, $f(t,0) = \sqrt[4]{t+1}/4$ and $h(t,0) = t^3/(t^2+10)$.

Further, notice that the function $\tau \to a(\tau)|g(t,\tau)|$ is integrable over $\mathbf{R}_+$ for any fixed $t \in \mathbf{R}_+$, and we have

$$G_a(t) = \int_0^\infty a(\tau)|g(t,\tau)|\,d\tau = \int_0^\infty \frac{\tau^2}{1+t^2+\tau^4}\,d\tau = \frac{\sqrt{2}\pi}{4}\frac{1}{\sqrt[4]{t^2+1}}. \tag{4.3}$$

Hence we get that $\overline{G}_a = \sqrt{2}\pi/4$. In a similar way we obtain

$$|g(t,\tau)h(\tau,0)| = \frac{\tau}{1+t^2+\tau^4}\frac{\tau^3}{\tau^2+1} \le \frac{\tau^2}{1+t^2+\tau^4}.$$

Linking the above estimate with (4.3) we conclude that the function $\tau \to |g(t,\tau)h(\tau,0)|$ is integrable over $\mathbf{R}_+$ for each fixed $t \in \mathbf{R}_+$. Moreover, we get

$$G_h(t) \le \frac{\sqrt{2}\pi}{4}\frac{1}{\sqrt[4]{t^2+1}},$$
$$\overline{G}_h \le \frac{\sqrt{2}\pi}{4}.$$

Further, we obtain

$$F_a(t) = |f(t,0)|G_a(t) = \frac{\sqrt{2}\pi}{16}\sqrt[4]{\frac{t+1}{t^2+1}}.$$

Hence we infer that $F_a(t) \to 0$ as $t \to \infty$ and

$$\overline{F}_a \le \frac{\pi}{16(2-\sqrt{2})}. \tag{4.4}$$

Analogously, we get

$$F_h(t) = |f(t,0)|G_h(t) \le \frac{\sqrt{2}\pi}{16}\sqrt[4]{\frac{t+1}{t^2+1}}.$$

Thus the function $F_h(t)$ is bounded and we have:

$$\overline{F}_h \le \frac{\pi}{16(2-\sqrt{2})}. \tag{4.5}$$

Arguing in the same way we obtain:

$$M_a(t) = m(t)G_a(t) \le \frac{\sqrt{2}\pi(\pi+1)}{160} \frac{\sqrt[3]{t}}{\sqrt[4]{t^2+1}},$$
$$M_h(t) = m(t)G_h(t) \le \frac{\sqrt{2}\pi(\pi+1)}{160} \frac{\sqrt[3]{t}}{\sqrt[4]{t^2+1}}.$$

This shows that the functions $M_a(t)$ and $M_h(t)$ are bounded on $\mathbf{R}_+$. Moreover, we get:

$$\overline{M}_a \le \frac{\sqrt{2}\pi(\pi+1)}{160}, \qquad \overline{M}_h \le \frac{\sqrt{2}\pi(\pi+1)}{160}. \tag{4.6}$$

In view of the above we infer that assumptions (v) and (vi) of Theorem 3.3 hold.

In what follows let us notice that with help of standard calculations we obtain:

$$\begin{aligned}\int_T^\infty a(\tau)|g(t,\tau)|\,d\tau &= \int_T^\infty \frac{\tau^2}{1+t^2+\tau^4}\,d\tau \\ &= \frac{1}{\sqrt[4]{t^2+1}}\bigg\{\frac{\sqrt{2}\pi}{4} \\ &\quad - \frac{1}{4\sqrt{2}}\ln\left(\frac{T^2-\sqrt{2}T+1}{T^2+\sqrt{2}T+1}\right) - \frac{\sqrt{2}}{4}[\arctan(\sqrt{2}T+1) \\ &\quad + \arctan(\sqrt{2}T-1)]\bigg\},\end{aligned}$$

where $T > 0$ is arbitrarily fixed. This yields the estimate

$$\begin{aligned}&\sup_{t\in\mathbf{R}_+} |f(t,0)| \int_T^\infty a(\tau)|g(t,\tau)|\,d\tau \\ &\quad \le \frac{\sqrt{2}}{2(2-\sqrt{2})}\bigg\{\frac{\sqrt{2}\pi}{4} - \frac{1}{4\sqrt{2}}\ln\left(\frac{T^2-\sqrt{2}T+1}{T^2+\sqrt{2}T+1}\right) \\ &\qquad - \frac{\sqrt{2}}{4}[\arctan(\sqrt{2}T+1) + \arctan(\sqrt{2}T-1)]\bigg\}.\end{aligned}$$

From this estimate we conclude that the first equality from assumption (vii) is satisfied. In a similar way we can check that the other equalities appearing in (vii) hold. We omit the easy details.

Finally, let us consider the inequality from assumption (viii) which has the form

$$||p|| + rb(r)\overline{M}_a + b(r)\overline{F}_a + r\overline{M}_h + \overline{F}_h \leq r. \tag{4.7}$$

In view of the estimates (4.4)–(4.6) we infer that each positive solution of the inequality

$$1+\frac{\sqrt{2}\pi(\pi+1)}{160}r^{5/3}+\frac{\pi}{16(2-\sqrt{2})}r^{2/3}+\frac{\sqrt{2}\pi(\pi+1)}{160}r+\frac{\pi}{16(2-\sqrt{2})}\leq r \tag{4.8}$$

also satisfies inequality (4.7).

It is easily seen that the number $r_0 = 4$ satisfies inequality (4.8). Hence from Theorem 3.3 we deduce that equation (4.2) has at least one solution in the space $BC(\mathbf{R}_+)$ belonging to the ball B_4 (cf. also Remark 3.1). Moreover, solutions of equation (4.2) are asymptotically stable.

It seems that the most important and general result in this field was obtained in the paper [**20**], where the author studied a special case of equation (1.1) with $f(t,x) = 1$ and $p = 0$, i.e., the following Hammerstein integral equation

$$x(t) = \int_0^\infty g(t,\tau)h(\tau,x(\tau))\,d\tau. \tag{4.9}$$

Let us mention that assumptions concerning equation (4.9) which were imposed in [**20**] are parallel to our assumptions (ii), (iii), (v) and (vi) but two of the assumptions in [**20**] are different.

For our purposes let us quote those assumptions in the form adapted to our situation. Namely, they have the form:

(ix) There exists a function $z \in BC(\mathbf{R}_+)$ such that for each $M > 0$ we have

$$\lim_{t\to\infty} \{\sup \{|h(t,x) - z(t)| : \ |x| \leq M\}\} = 0.$$

(x)

$$\lim_{|x|\to\infty}\left\{\sup\left\{\frac{|h(t,x)|}{|x|}\,:\ t\in\mathbf{R}_+\right\}\right\}<\frac{1}{L},$$

where

$$L=\sup\left\{\int_0^\infty |g(t,\tau)|\,d\tau\,:\ t\in\mathbf{R}_+\right\}.$$

In order to show that these assumptions are more restrictive than those imposed in Theorem 3.3, let us consider a special case of equation (4.2), where we put $f(t,x)=1$ and $p=0$. Namely, we consider the following equation

$$x(t)=\int_0^\infty \frac{\tau}{1+t^2+\tau^4}\left(\frac{\tau^3}{1+\tau^2}+\tau\sqrt[3]{x^2(\tau)}\right)d\tau. \tag{4.10}$$

Observe that the function $h(t,x)=t^3/(1+t^2)+t\sqrt[3]{x^2}$ appearing in equation (4.10) does not satisfy assumption (ix). Indeed, for any fixed $x\in\mathbf{R}$ we have

$$\lim_{t\to\infty}|h(t,x)-z(t)|=\lim_{t\to\infty}\left|t\sqrt[3]{x^2}+\frac{t^3}{1+t^2}-z(t)\right|=\infty.$$

Similarly we can show that assumption (x) is also not satisfied.

Finally let us remark that we can redefine the functions g and h appearing in equation (4.10) in such a way that we do not change this equation but the theory developed in [**20**] is applicable. To this end it is sufficient to multiply the function g by $1+t^{1+\varepsilon}$ and to divide the function h by $1+t^{1+\varepsilon}$, where ε is an arbitrarily fixed number from the interval $(0,1)$. Observe that in such a situation the result obtained in [**20**] can be applied with $z\equiv 0$.

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