

**CROSSED PRODUCT C^* -ALGEBRAS
BY FINITE GROUP ACTIONS
WITH THE TRACIAL ROKHLIN PROPERTY**

DAWN ARCHEY

ABSTRACT. Let A be a stably finite simple unital C^* -algebra, and suppose α is an action of a finite group G with the tracial Rokhlin property. Suppose further A has real rank zero and the order on projections over A is determined by traces. Then the crossed product C^* -algebra $C^*(G, A, \alpha)$ also has real rank zero and order on projections over A is determined by traces. Moreover, if A also has stable rank one, then $C^*(G, A, \alpha)$ also has stable rank one.

1. Introduction. The tracial Rokhlin property for finite group actions was introduced in [8] to address the fact that some C^* -algebras can have no actions with the Rokhlin property of certain groups due to K -theoretic obstructions. The tracial Rokhlin property was further developed in [9]. In [2, 8], examples of actions with the tracial Rokhlin property are studied. This paper studies permanence properties of crossed products by actions with the tracial Rokhlin property. The goal of this paper is to prove Theorem 4.2, Theorem 5.1 and Theorem 6.2 which can be collectively summarized in the following theorem.

Theorem. *Let A be an infinite dimensional stably finite simple unital C^* -algebra with real rank zero, and suppose that the order on projections over A is determined by traces. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Then the order on projections over $C^*(G, A, \alpha)$ is determined by traces and $C^*(G, A, \alpha)$ has real rank zero. Moreover, if A also has stable rank one, then $C^*(G, A, \alpha)$ has stable rank one.*

These three theorems are finite group analogs of known results about actions of \mathbf{Z} ; see [7]. The proof techniques are similar to those used there. The most significant changes are in Lemma 3.1 (which is analogous to Lemma 2.5 of [7]). This is the key lemma in the proof

2010 AMS *Mathematics subject classification.* Primary 46L55, Secondary 16S35, 46L35, 46L40.

The author was partially supported by NSF grant DMS 0302401.

Received by the editors on February 7, 2009, and in revised form on May 1, 2009.

DOI:10.1216/RMJ-2011-41-6-1755 Copyright ©2011 Rocky Mountain Mathematics Consortium

of all three theorems. In that proof a different construction of the isomorphism was needed.

It is known that some hypotheses are needed to ensure that the properties of A pass to the crossed product. For the real rank zero situation, consider Example 9 in [3] of an outer action of $\mathbf{Z}/2\mathbf{Z}$ on a simple nuclear C^* -algebra with real rank zero such that the crossed product does not have real rank zero.

However, partial results are known, such as Theorem 2.6 of [9]. This theorem says that if a finite group acts on an infinite dimensional simple unital C^* -algebra with tracial rank zero via an action with the tracial Rokhlin property, then the crossed product has tracial rank zero.

The situation for stable rank one is similar. Various partial results are known such as Theorem 4.6 of [5] which gives cancelation of projections (a condition strictly weaker than stable rank one) under fairly mild hypotheses. Another result along these same lines is Corollary 5.6 of [4]. There, the group must be a finite semidirect product of finite abelian groups, but the action is unrestricted. In that case, if the group acts on a simple unital C^* -algebra with property (SP) and stable rank one in such a way that the crossed product has real rank zero, then the crossed product has stable rank one.

As in the real rank zero situation, it is also known that some conditions will be needed to ensure stable rank one in the crossed product. There is a (nonsimple) unital C^* -algebra A of stable rank one and an action $\mathbf{Z}/2\mathbf{Z}$ on A such that the crossed product does not have stable rank one. See Example 8.2.1 of [1]. However, there is no known example for a simple C^* -algebra.

This paper is organized as follows. In Section 2 we establish notation, give definitions, and prove basic facts about the tracial Rokhlin property. In Section 3 we prove the key lemmas needed for the proofs of the three main theorems, which are proved in Sections 4 (order on projections is determined by traces), 5 (real rank zero) and 6 (stable rank one).

2. The tracial Rokhlin property.

Notation. For any projections p and q in A , we write $p \sim q$ if p is (*Murray-von Neumann*) *equivalent* to q . We write $p \precsim q$ if p is (*Murray-von Neumann*) *subequivalent* to q .

Notation. Let A be a unital C^* -algebra. We denote by $T(A)$ the set of all tracial states on A , equipped with the weak* topology. For any element of $T(A)$, we use the same letter for its standard extension to $M_n(A)$ for arbitrary n , and to $M_\infty(A) = \bigcup_{n=1}^\infty M_n(A)$ (no closure).

Definition 2.3. Let A be a unital C^* -algebra. We say that the *order on projections over A is determined by traces* if whenever $p, q \in M_\infty(A)$ are projections such that $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then $p \lesssim q$.

Definition 2.4. Let A be an infinite dimensional simple unital C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A . We say that α has the *tracial Rokhlin property* if, for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

- (1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
- (2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- (3) With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x .
- (4) With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

When A is finite, as was shown in [9, Lemma 1.12], condition (4) of Definition 2.4 is not needed.

The following lemma is the finite group analog of Lemma 1.4 in [7] and can be proved using the same argument.

Lemma 2.5. *Let A be an infinite dimensional stably finite simple unital C^* -algebra with real rank zero and such that the order on projections over A is determined by traces. Suppose $\alpha : G \rightarrow A$ is an action of a finite group on A . Then α has the tracial Rokhlin property if and only if for every finite set $F \subset A$ and every $\varepsilon > 0$ there are mutually orthogonal projections $e_g \in A$ for each $g \in G$ such that:*

- (1) $\|\alpha_h(e_g) - e_{gh}\| < \varepsilon$ for $g \in G$.
- (2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- (3) With $e = \sum_{g \in G} e_g$, we have $\tau(1 - e) < \varepsilon$ for all $\tau \in T(A)$.

The next lemma is Lemma 2.4 of [7].

Proposition 2.6. *Let A be a simple unital infinite dimensional C^* -algebra with real rank zero, and assume that the order on projections over A is determined by traces. Let $\alpha: \Gamma \rightarrow \text{Aut}(A)$ be an action of a countable amenable group. Let $p, q \in M_\infty(A)$ be projections such that $\tau(p) < \tau(q)$ for every Γ -invariant tracial state τ on A . (We extend τ to $M_\infty(A)$ as in Notation 2.2.) Then there is an $s \in M_\infty(C^*(\Gamma, A, \alpha))$ such that*

$$s^*s = p, \quad ss^* \leq q, \quad \text{and} \quad ss^* \in M_\infty(A).$$

In particular, $p \precsim q$ in $M_\infty(C^(\Gamma, A, \alpha))$.*

3. The pivotal lemmas. The next two lemmas are used as the key lemmas in the proofs of Theorem 4.2, Theorem 5.1 and Theorem 6.2. Lemma 3.1 is the finite group analog of Lemma 2.5 in [7].

Lemma 3.1. *Let A be an infinite dimensional stably finite simple unital C^* -algebra with real rank zero such that the order on projections over A is determined by traces. Let G be a finite group of order n , and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of G with the tracial Rokhlin property. Let $\iota: A \rightarrow C^*(G, A, \alpha)$ be the inclusion map. Then for every finite set $F \subset C^*(G, A, \alpha)$, every $\varepsilon > 0$, every $N \in \mathbf{N}$ and every nonzero positive element $z \in C^*(G, A, \alpha)$, there exist a projection $e \in A \subset C^*(G, A, \alpha)$, a unital subalgebra $D \subset eC^*(G, A, \alpha)e$, a projection $f \in A$ and an isomorphism $\varphi: M_n \otimes fAf \rightarrow D$, such that:*

(1) *With $(e_{g,h})$ for $g, h \in G$ being a system of matrix units for M_n , we have $\varphi(e_{1,1} \otimes a) = \iota(a)$ for all $a \in fAf$ and $\varphi(e_{g,g} \otimes 1) \in \iota(A)$ for $g \in G$.*

(2) *With $(e_{g,g})$ as in (1), we have $\|\varphi(e_{g,g} \otimes a) - \iota(\alpha_g(a))\| \leq \varepsilon \|a\|$ for all $a \in fAf$.*

(3) *For every $a \in F$ there exist $b_1, b_2 \in D$ such that $\|ea - b_1\| < \varepsilon$, $\|ae - b_2\| < \varepsilon$ and $\|b_1\|, \|b_2\| \leq \|a\|$.*

(4) $e = \sum_{g \in G} \varphi(e_{g,g} \otimes 1)$.

(5) *The projection $1 - e$ is Murray-von Neumann equivalent in $C^*(G, A, \alpha)$ to a projection in the hereditary subalgebra of $C^*(G, A, \alpha)$ generated by z .*

(6) *There are N mutually orthogonal projections $f_1, f_2, \dots, f_N \in eDe$, each of which is Murray-von Neumann equivalent in $C^*(G, A, \alpha)$ to $1 - e$.*

Proof. We first note, using the same argument as in the proof of Lemma 2.5 of [7], that it is not necessary to check the estimates $\|b_1\|, \|b_2\| \leq \|a\|$ in condition (3) of the conclusion.

Now we do the main part of the proof. Let $\varepsilon > 0$, and let $F \subset C^*(G, A, \alpha)$ be a finite set. Let $N \in \mathbb{N}$, and let $z \in C^*(G, A, \alpha)$ be a nonzero positive element.

Let u_g for $g \in G$ be the standard unitaries in the crossed product $C^*(G, A, \alpha)$. We regard A as a subalgebra of $C^*(G, A, \alpha)$ in the usual way.

For each $x \in F$ write $x = \sum_{g \in G} a_g u_g$. Let $S \subset A$ be a finite set which contains all the coefficients used for all elements of F . Let $M = 1 + \sup_{a \in S} \|a\|$.

Let $\delta_0 < \varepsilon/16n^2M$. Let δ_1 be such that if p_1, p_2 are projections in a C^* -algebra B and if $a \in B$ is such that $\|a^*a - p_1\| \leq \delta_1$ and $\|aa^* - p_2\| \leq \delta_1$, then there is a partial isometry $s \in B$ such that $s^*s = p_1$, $ss^* = p_2$ and $\|a - s\| \leq \delta_0$. Let $0 < \delta < \min\{\delta_0, \delta_1, (\varepsilon/6n^3), 1\}$.

Since A has real rank zero and (by [9, Lemma 1.5]) α_g is outer for all $g \in G$, Theorem 4.2 of [4] (with $N = \{1\}$) supplies a nonzero projection $q \in A$ which is Murray-von Neumann equivalent in $C^*(G, A, \alpha)$ to a projection in $zC^*(G, A, \alpha)z$. Moreover, Lemma 2.3 of [7] provides nonzero orthogonal Murray-von Neumann equivalent projections $q_0, q_1, \dots, q_{2N} \in qAq$.

Apply the tracial Rokhlin property (Definition 2.4) with δ in place of ε , with S in place of F and with q_0 in place of x . Call the resulting projections e_g for each $g \in G$, and let $e = \sum_{g \in G} e_g$.

Set $f = e_1$, and define $w_{g,h} = u_{gh^{-1}}e_h$. We claim that the elements $(w_{g,h})_{g,h \in G}$ form a δ -approximate system of $n \times n$ matrix units. To prove the claim we compute:

$$\|w_{g,h}^* - w_{h,g}\| = \|u_{gh^{-1}}e_h u_{gh^{-1}}^* - e_g\| = \|\alpha_{gh^{-1}}(e_h) - e_g\| < \delta.$$

Then, using $e_g e_h = \delta_{g,h} e_h$, we find

$$\|w_{g_1, h_1} w_{g_2, h_2} - \delta_{g_2, h_1} w_{g_1, h_2}\| < \delta.$$

For the final condition, since $\|eq_0e\| > 1 - \delta > 0$, the projection e is nonzero, so e_g is nonzero for each $g \in G$. In particular $\|w_{1,1}\| = \|e_1\| = 1 > 1 - \delta$. This proves the claim.

Since $(w_{g,h})_{g,h \in G}$ forms a δ -approximate system of matrix units, $w_{g,1}$ is an approximate partial isometry for each $g \in G$. More specifically,

$$\|w_{g,1}w_{g,1}^* - e_g\| = \|u_g e_1 e_1 u_g^* - e_g\| = \|\alpha_g(e_1) - e_g\| < \delta$$

since the e_g are the tracial Rokhlin projections. Also,

$$\|w_{g,1}^* w_{g,1} - e_1\| = \|e_1 u_g^* u_g e_1 - e_1\| = \|e_1 - e_1\| = 0 < \delta$$

because u_g is a unitary for each g .

Since $\delta < \delta_1$, by the choice of δ_1 there exist partial isometries $z_g \in C^*(G, A, \alpha)$ for each $g \in G$ such that $\|z_g - w_{g,1}\| < \delta_0$ and such that $z_g z_g^* = e_g$ and $z_g^* z_g = e_1$. Moreover, one may check that we may take $z_1 = e_1$.

Let $(e_{g,h})_{g,h \in G}$ be an $n \times n$ system of matrix units for M_n . Define a linear function $\varphi : M_n \otimes e_1 A e_1 \rightarrow C^*(G, A, \alpha)$ by $\varphi(e_{g,h} \otimes a) = z_g a z_h^*$. One can then check in the usual way that φ is a homomorphism.

It is also worth computing at this stage that for $g, h \in G$ and $a \in e_1 A e_1$, we have $\|\varphi(e_{g,h} \otimes a) - w_{g,1} a w_{h,1}^*\| \leq 2\|a\|\delta_0$. Let D be the image of φ , so that φ is clearly surjective as a map from $M_n \otimes e_1 A e_1$ to D . To check that φ is injective we first recall that $\ker(\varphi) \cap (e_{g,h} \otimes e_1 A e_1) = e_{g,h} \otimes I$ where I is an ideal of $e_1 A e_1$ which does not change as g and h vary. But if $0 = \varphi(e_{g,h} \otimes a) = z_g a z_h^*$ for some $a \in e_1 A e_1$; then multiplying on the left by z_g^* and on the right by z_h we see that $e_1 a e_1 = a = 0$, so $I = 0$. Thus φ is injective.

Now $\varphi(e_{1,1} \otimes a) = z_1 a z_1^* = e_1 a e_1 = a$ for any $a \in e_1 A e_1$. Also, $\varphi(e_{g,g} \otimes 1) = z_g e_1 z_g^* = z_g z_g^* z_g z_g^* = e_g \in A$. These two conditions make up (1) of the conclusion.

To verify (2), let $a \in e_1 A e_1$ and estimate

$$\|\varphi(e_{g,g} \otimes a) - \alpha_g(a)\| \leq 2\|a\|\delta_0 + \|u_g e_1 a e_1 u_g^* - \alpha_g(a)\| \leq \varepsilon\|a\|.$$

For (4) we observe

$$\sum_{g \in G} \varphi(e_{g,g} \otimes 1) = \sum_{g \in G} z_g e_1 z_g^* = \sum_{g \in G} e_g = e.$$

Condition (5) holds essentially by construction since $1 - e$ is Murray-von Neumann equivalent to a projection in q_0Aq_0 , but $q_0 \in qAq$ and q is equivalent to a projection in the hereditary subalgebra generated by z . In total this gives $1 - e$ is subequivalent to a projection in the hereditary subalgebra generated by z .

Now for condition (6), since $q_j \sim q_i$, we have $\tau(q_j) < 1/2N$ for $0 \leq j \leq 2N$ and for any $\tau \in T(A)$. In particular, since $1 - e$ is subequivalent to q_0 we have $\tau(1 - e) \leq \tau(q_0) < 1/2N$. This implies $1/2 < \tau/(e)$. Additionally $\tau(q_j) \leq 1/2N$ implies $\tau(\sum_{j=1}^N q_j) < 1/2$. Combining these statements gives $\tau(\sum_{j=1}^N q_j) < \tau(e)$ for all $\tau \in T(A)$. So, since order on projections over A is determined by traces, $\sum_{j=1}^N q_j \lesssim e$. Let $h \in A$ be a projection satisfying $\sum_{j=1}^N q_j \sim h \leq e$, and let s be a partial isometry with $s^*s = \sum_{j=1}^N q_j$ and $ss^* = h$. Let $h_j = sq_js^*$ for $j = 1, \dots, N$. One checks that h_1, \dots, h_N are mutually orthogonal projections summing to h . Furthermore, since $h_j \leq h \leq e$, we have $h_j \leq e$. Furthermore, $h_j \sim g_j$ via the partial isometry sg_j . So now we have $1 - e \lesssim g_j \sim h_j$. Let f_j be a projection such that $1 - e \sim f_j \leq h_j$. Since $f_j \leq h_j$, and the h_j and h_i are orthogonal for $1 \leq i, j \leq N$, we see that f_1, \dots, f_N are mutually orthogonal. Finally $f_j \leq h_j \leq e$ in A and $eAe \subset eDe$, so f_1, \dots, f_N are the projections we desired.

In order to show (3) we will use the following claim.

Claim. *If $y = \sum_{g \in G} a_g u_g$ with $a_g \in A$ and $\|a_g\| \leq M$, and if $[e_g, a_h] = 0$ for all $g, h \in G$, then there are $d_1, d_2 \in D$ such that $\|ey - d_1\|, \|ye - d_2\| < 8n^2 M \delta_0$.*

Proof of claim. We can write

$$ey = \sum_{g \in G} \sum_{h \in G} e_g a_h u_h = \sum_{g \in G} \sum_{h \in G} (e_g a_h e_g)(e_g u_h)$$

since e_g and a_h commute. Now we make a norm estimate involving one of the factors in the third expression for ey using the fact that z_g is a partial isometry :

$$\begin{aligned} & \left\| \varphi(e_{g,g} \otimes e_1 \alpha_{g^{-1}}(a_h) e_1) - e_g a_h e_g \right\| \\ &= \left\| e_g z_g \alpha_g^{-1}(a_h) z_g^* e_g - e_g a_h e_g \right\| \end{aligned}$$

$$\begin{aligned}
 &\leq \|z_g \alpha_g^{-1}(a_h) z_g^* - z_g \alpha_g^{-1}(a_h) w_{g,1}^*\| \\
 &\quad + \|z_g \alpha_g^{-1}(a_h) w_{g,1}^* - w_{g,1} \alpha_g^{-1}(a_h) w_{g,1}^*\| \\
 &\quad + \|u_g e_1 \alpha_g^{-1}(a_h) e_1 u_g^* - e_g a_h e_g\| \\
 &\leq 2M\delta_0 + 2M\delta.
 \end{aligned}$$

Now we make an estimate involving the other factor:

$$\begin{aligned}
 &\|\varphi(e_{g,h^{-1}g} \otimes e_1) - e_g u_h\| \\
 &\leq \|\varphi(e_{g,h^{-1}g} \otimes e_1) - u_g e_1 u_{h^{-1}g}^*\| + \|u_g e_1 u_{h^{-1}g}^* - e_g u_h\| \\
 &\leq 2\delta_0 + \delta.
 \end{aligned}$$

Let $d_0(g, h) = \varphi(e_{g,g} \otimes e_1 \alpha_{g^{-1}}(a_h) e_1) \varphi(e_{g,h^{-1}g} \otimes e_1)$. Then we have

$$\begin{aligned}
 &\|d_0(g, h) - e_g a_h u_h\| \\
 &\leq \|d_0(g, h) - \varphi(e_{g,g} \otimes e_1 \alpha_{g^{-1}}(a_h) e_1) e_g u_h\| \\
 &\quad + \|\varphi(e_{g,g} \otimes e_1 \alpha_{g^{-1}}(a_h) e_1) e_g u_h - (e_g a_h e_g) (e_g u_h)\| \\
 &\leq \|\varphi(e_{g,g} \otimes e_1 \alpha_{g^{-1}}(a_h) e_1)\| (2\delta_0 + \delta) + 2M\delta_0 + 2M\delta \\
 &\leq 4M\delta_0 + 3M\delta.
 \end{aligned}$$

Now let $d_1 = \sum_{g \in G} \sum_{h \in G} d_0(g, h)$. Then

$$\begin{aligned}
 \|d_1 - ey\| &= \left\| \sum_{g \in G} \sum_{h \in G} d_0(g, h) - \sum_{g \in G} \sum_{h \in G} e_g a_h u_h \right\| \\
 &\leq n^2 M (4\delta_0 + 3\delta) \\
 &< 8n^2 M \delta_0.
 \end{aligned}$$

We now turn our attention to the construction of d_2 . We can write

$$ye = \sum_{g \in G} \sum_{h \in G} a_h \alpha_h(e_g) u_h.$$

We note that

$$\left\| \sum_{g \in G} \sum_{h \in G} a_h \alpha_h(e_g) u_h - \sum_{g \in G} \sum_{h \in G} a_h e_{hg} u_h \right\| < n^2 \delta.$$

But

$$\sum_{g \in G} \sum_{h \in G} a_h e_{hg} u_h = \sum_{h \in G} \sum_{g \in G} e_{hg} a_h u_h = \sum_{h \in G} \sum_{k \in G} e_k a_h u_h$$

by making the change of variables, $k = hg$. This last is of the same form as ey , so using the argument above there is an element $d_2 \in D$ such that

$$\left\| \sum_{h \in G} \sum_{k \in G} e_k a_h u_h - d_2 \right\| \leq n^2 M(4\delta_0 + 3M\delta).$$

Thus,

$$\|ye - d_2\| \leq n^2 M(4\delta_0 + 3M\delta) + n^2 \delta < 8n^2 M\delta_0.$$

We are now in a position to prove (3). Let $x \in F$, and choose $b_g \in S$ such that $x = \sum_{g \in G} b_g u_g$. Define

$$a_g = (1 - e)b_g(1 - e) + \sum_{h \in G} e_h b_g e_h.$$

Now we get

$$b_g - a_g = \sum_{h \in G} [(1 - e)b_g e_h + e_h b_g(1 - e)] + \sum_{h \in G} \sum_{\substack{k \in G \\ k \neq h}} e_k b_g e_h.$$

Because $b_g \in S$ we have, $\|[b_g, e_h]\| < \delta$. Then

$$\begin{aligned} \|b_g - a_g\| &\leq \sum_{h \in G} \left[\|(1 - e)[b_g, e_h]\| \right. \\ &\quad + \|(1 - e)e_h b_g\| + \|[b_g, e_h](1 - e)\| + \|b_g e_h(1 - e)\| \\ &\quad \left. + \sum_{\substack{k \in G \\ k \neq h}} \|e_k [b_g, e_h]\| + \sum_{\substack{k \in G \\ k \neq h}} \|e_k e_h b_g\| \right] \\ &= (2n + n^2)\delta \\ &< 3n^2\delta. \end{aligned}$$

Set $y = \sum_{g \in G} a_g u_g$. Then

$$\|x - y\| \leq \sum_{g \in G} \|(b_g - a_g)u_g\| \leq \sum_{g \in G} \|b_g - a_g\| \leq n(3n^2\delta) = 3n^3\delta.$$

One easily checks that $[a_g, e_k] = 0$ for all $g, k \in G$. Thus the claim applies to y and provides $d_1 \in D$ such that $\|ey - d_1\| < 8n^2M\delta_0$. Therefore,

$$\|ex - d_1\| \leq \|ex - ey\| + \|ey - d_1\| \leq 3n^3\delta + 8n^2M\delta_0 < \varepsilon$$

by the choice of δ and δ_0 .

Similarly, the claim provides $d_2 \in D$ such that $\|ye - d_2\| < 8n^2M\delta_0$ which implies that d_2 satisfies $\|xe - d_2\| < \varepsilon$. \square

Given objects satisfying part (1) of the conclusion of Lemma 3.1, we can make a useful homomorphism into $C^*(G, A, \alpha)$ which should be thought of as a kind of twisted inclusion of A . The following lemma is stated in terms of an arbitrary unital C^* -algebra B , but we note it applies when $B = C^*(G, A, \alpha)$ and ι is the standard embedding.

Lemma 3.2. *Let A be any simple unital C^* -algebra, let B be a unital C^* -algebra, and let $\iota : A \rightarrow B$ be a unital injective homomorphism.*

Let $e, f \in A$ be projections, and let $n \in \mathbf{N}$. Assume that there is an injective unital homomorphism $\varphi : M_n \otimes fAf \rightarrow \iota(e)B\iota(e)$ such that, with $(e_{j,k})$ being the standard system of matrix units for M_n , we have $\varphi(e_{1,1} \otimes a) = \iota(a)$ for all $a \in fAf$. Then there is a corner $A_0 \subset M_{n+1} \otimes A$ which contains

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in (1 - e)A(1 - e) \text{ and } b \in M_n \otimes fAf \right\}$$

as a unital subalgebra, and an injective unital homomorphism $\psi : A_0 \rightarrow B$ such that

$$\psi \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \iota(a) + \varphi(b)$$

for $a \in (1 - e)A(1 - e)$ and $b \in M_n \otimes fAf$.

Moreover, if $\alpha : G \rightarrow \text{Aut}(A)$ is an action of a finite group on A , $B = C^*(G, A, \alpha)$, and ι is the standard inclusion, then for every α -invariant tracial state τ on A there is a tracial state σ on $C^*(G, A, \alpha)$ such that the extension $\bar{\tau}$ of τ to $M_{n+1} \otimes A$ satisfies $\bar{\tau}|_{A_0} = \sigma \circ \psi$.

Proof. For the main part of the lemma we simply observe that the proof of Lemma 2.6 in [7] works in this generality.

For the part about traces, the proof of Lemma 2.6 in [7] also works if one replaces the conditional expectation there with the conditional expectation $E(\sum_{g \in G} a_g u_g) = a_1$. \square

4. Traces and order on projections in crossed products. In this section, we prove that if A is a simple unital C^* -algebra with real rank zero such that the order on projections over A is determined by traces, and if $\alpha : G \rightarrow \text{Aut}(A)$ is an action of a finite group G with the tracial Rokhlin property, then the order on projections over $C^*(G, A, \alpha)$ is determined by traces.

We begin with a comparison lemma for projections in crossed products by actions with the tracial Rokhlin property.

Lemma 4.1. *Assume the hypotheses of Lemma 3.2 with $B = C^*(G, A, \alpha)$, and assume in addition that A has real rank zero and that the order on projections over A is determined by traces. Let $\psi : A_0 \rightarrow C^*(G, A, \alpha)$ be as in the conclusion of Lemma 3.2. Suppose that $p, q \in \psi(A_0)$ are projections such that $\tau(p) < \tau(q)$ for all tracial states τ on $C^*(G, A, \alpha)$. Then there exists a projection $r \in \psi(A_0)$ such that $r \leq q$ and r is Murray-von Neumann equivalent to p in $C^*(G, A, \alpha)$.*

Proof. The same proof used for Lemma 3.1 of [7] works by changing the group from \mathbf{Z} to the finite group G . \square

Theorem 4.2. *Let A be an infinite dimensional simple unital C^* -algebra with real rank zero, and suppose that the order on projections over A is determined by traces. Let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Then the order on projections over $C^*(G, A, \alpha)$ is determined by traces.*

Proof. We first observe that the hypotheses on A imply that A is finite, but $M_n(A)$ satisfies all the same hypotheses, so A is in fact stably finite.

A special case of Lemma 3.9 of [9] implies that $\text{id}_{M_n} \otimes \alpha$ as an action on $M_n \otimes A$ has the tracial Rokhlin property for any $n \in \mathbf{N}$. Therefore, it suffices to consider projections in $C^*(G, A, \alpha)$.

The rest of the proof can be done in the same way as the proof of Theorem 3.5 of [7]. The only changes are to use Lemma 3.1 instead of Lemma 2.5 of [7] and Lemma 3.2 instead of Lemma 2.6 of [7]. \square

5. Real rank of crossed products. The main theorem of this section is 5.1.

Theorem 5.1. *Let A be an infinite dimensional stably finite simple unital C^* -algebra with real rank zero. Suppose that the order on projections over A is determined by traces and $\alpha : G \rightarrow \text{Aut}(A)$ is an action of a finite group with the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has real rank zero.*

Proof. Set $B = C^*(G, A, \alpha)$.

As in the proof of Theorem 4.2, the other hypotheses imply that A is stably finite.

This proof is very similar to the proof of Theorem 4.5 of [7] except simpler. Therefore, instead of giving the details of the proof we will describe how the two proofs differ. The norm estimates come out a bit different (the estimates can be improved), but we will not discuss that.

To justify the fact that the crossed product is simple, one should use Corollary 1.6 of [9], instead of Corollary 1.14 of [7]. Similarly, in the proof of the integer case, Osaka and Phillips applied Lemma 2.5 of [7] to obtain e , p , and $f \in A$ projections, integers $n, m > 0$, a unital subalgebra $D \subset eBe$ and an isomorphism $\varphi : D \rightarrow M_n \otimes fAf$. We apply Lemma 3.1 to obtain a projection $e \in A \subset C^*(G, A, \alpha)$, a unital subalgebra $D \subset eC^*(G, A, \alpha)e$, a projection $f \in A$ and an isomorphism $\varphi : M_n \otimes fAf \rightarrow D$.

We use our e to replace both e and p in the proof of Theorem 4.5 of [7]. Let $q = 1 - e$. Since $a_0e = ea_0$, there is no need to replace p by a smaller projection approximately commuting with a_0 . This is the source of the improved estimates later on. \square

Corollary 5.2. *Let A be an infinite dimensional stably finite simple unital C^* -algebra with real rank zero, and suppose that the order on projections over A is determined by traces. Let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Then the restriction map is a bijection from the tracial states of $C^*(G, A, \alpha)$ to the α -invariant tracial states of A .*

Proof. This follows from Proposition 2.2 of [6], since $C^*(G, A, \alpha)$ has real rank zero by Theorem 5.1. \square

6. Stable rank of crossed products. In this section, we prove that if A is an infinite dimensional simple unital C^* -algebra with real rank zero and stable rank one, such that the order on projections over A is determined by traces, and if $\alpha : G \rightarrow \text{Aut}(A)$ is an action of a finite group with the tracial Rokhlin property, then $C^*(G, A, \alpha)$ has stable rank one.

Lemma 6.1. *Let A be an infinite dimensional simple unital C^* -algebra with real rank zero and such that the order on projections over A is determined by traces. Let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Let $q_1, \dots, q_n \in C^*(G, A, \alpha)$ be nonzero projections, let $a_1, \dots, a_m \in C^*(G, A, \alpha)$ be arbitrary, and let $\varepsilon > 0$. Then there exists a unital subalgebra $A_0 \subset C^*(G, A, \alpha)$ which is stably isomorphic to A , a projection $p \in A_0$, nonzero projections $r_1, \dots, r_n \in pA_0p$, and elements $b_1, \dots, b_m \in C^*(G, A, \alpha)$, such that:*

- (1) $\|q_k r_k - r_k\| < \varepsilon$ for $1 \leq k \leq n$.
- (2) For $1 \leq k \leq n$ there is a projection $g_k \in r_k A_0 r_k$ such that $1 - p \sim g_k$ in $C^*(G, A, \alpha)$.
- (3) $\|a_j - b_j\| < \varepsilon$ for $1 \leq j \leq m$.
- (4) $pb_j p \in pA_0 p$ for $1 \leq j \leq m$.

Proof. The same argument used in the proof of Lemma 5.2 of [7] works here as well. As in the proof of 5.1, when we apply Lemma 3.1, we set $p = e$ in order to use the proof of Lemma 5.2 of [7]. \square

Theorem 6.2. *Let A be an infinite dimensional simple unital C^* -algebra with real rank zero and stable rank one, and such that the order on projections over A is determined by traces. Let $\alpha : G \rightarrow \text{Aut}(A)$ be an action of a finite group with the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has stable rank one.*

Proof. The same proof used to show Theorem 5.3 of [7] can be used to prove this theorem as well. The only difference is, instead of using Lemma 5.2 of [7], we must use Lemma 6.1. \square

Acknowledgments. The author is indebted to the Fields Institute for Research in Mathematical Sciences for its hospitality during the Thematic Program on Operator Algebras in the fall of 2007 and to the Center for Advanced Studies in Mathematics at Ben Gurion University. This research is part of the author's Ph.D. thesis at the University of Oregon, completed under the direction of N. Christopher Phillips.

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