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GENERALIZED BI-CIRCULAR PROJECTIONS ON SPACES OF OPERATORS AND JB^* TRIPLES

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ABSTRACT. We give a characterization of generalized bi-circular projections on spaces of operators $\mathcal{B}(X,Y)$ which support only elementary surjective isometries. We also give a characterization of generalized bi-circular projections for JB^* triples.

1. Introduction. Fosner, Illisevic, and Li in [**7**] have introduced an interesting class of projections on Banach spaces. We refer to these projections as generalized bi-circular projections. The results of Fosner, Illisevic, and Li generalizes earlier results by Stacho and Zalar on bi-circular projections, see [**16, 17**]. Stacho and Zalar call a projection P on a Banach space X a bi-circular projection if $e^{ia}P + e^{ib}(I-P)$ is an isometry for all choices of real numbers a and b. It is easy to show that these projections are norm Hermitian, see [**11**]. Fosner, Illisevic, and Li in [**7**] only require that $P+\lambda(I-P)$ be an isometry for some $\lambda \in \mathbf{T}\setminus\{1\}$. In [**7**], the authors obtained nice results in the finite-dimensional setting and raise the problem of classifying these projections in other Banach spaces. In this paper, we study such projections for spaces $\mathcal{B}(X,Y)$ of bounded operators between pairs of Banach spaces. We also make some observations about generalized bi-circular projections in JB^* triple systems. These results follow from a result of Guerrero and Palacios in [**9**], as well as some results of Friedman and Russo in [**8**].

2. Generalized bi-circular projections on $\mathcal{B}(X,Y)$. In this section, we give a characterization of generalized bi-circular projections on spaces of operators $\mathcal{B}(X,Y)$ supporting only isometries of a special type. Operators of the form $\mathcal{J}(T) = UTV$ on $\mathcal{B}(X,Y)$ with U and V surjective isometries on Y and X, are clearly surjective isometries on

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the $\mathcal{B}(X,Y)$. Isometries of this type are referred to, throughout this paper, as isometries of type I. The isometry group of $\mathcal{B}(X, Y)$ is known to be particularly simple for certain pairs of Banach spaces. It has been shown that several spaces of operators support only isometries of type I, see e.g., [**13, 15**]. The next theorem is our main result for generalized bi-circular projections on spaces of operators $\mathcal{B}(X,Y)$ supporting only isometries of type I.

Theorem 2.1. *If X and Y are complex Banach spaces such that $\mathcal{B}(X,Y)$ supports only isometries of type* I, *then a projection P on $\mathcal{B}(X,Y)$ is a generalized bi-circular projection if and only if*

(i) *$P(T) = P_Y T$ or $P(T) = T P_X$ with P_X and P_Y generalized bi-circular projections on X and Y respectively, or*

(ii) *there exist isometric reflections U_Y and V_X on Y and X respectively so that $P(T) = (T + U_Y T V_X)/2$.*

Proof. If the projection P is as stated in the proposition, then clearly it is a generalized bi-circular projection on $\mathcal{B}(X,Y)$. Conversely, we assume that the operator P is a generalized bi-circular projection. Then the isometry $\mathcal{J} = P + \lambda(Id - P)$ satisfies

$$\mathcal{J}^2 - (\lambda+1)\mathcal{J} + \lambda Id = 0. \tag{2.1}$$

This last equation is equivalent to

$$U^2TV^2 - (\lambda+1)UTV + \lambda T = 0, \tag{2.2}$$

for all $T \in \mathcal{B}(X,Y)$. Given a nontrivial $v \in Y$, we consider the rank one operator T of the form $T(x) = \varphi(x)v$, with $\varphi \in X^*$. We first observe that every x we must have that $\{x, V(x), V^2(x)\}$ is linearly dependent. If for every x $\{x, V(x)\}$ is linearly dependent, then there exists a modulus one constant a, independent of x, such that $V = aId_X$. The equation (2.2) implies that

$$a^2U^2 - (\lambda+1)aU + \lambda Id = 0. \tag{2.3}$$

A theorem from Taylor, see [**18**, page 317], asserts the existence of two projections P_1 and P_2 on Y so that $P_1 + P_2 = Id$, $P_1\,P_2 = P_2\,P_1 = 0$ and

$U = \overline{a}\lambda P_1 + \overline{a}P_2$. Consequently $\mathcal{J}(T) = (\lambda P_1 + P_2)\,T$ and $P(T) = P_2\,T$. We observe that P_2 is a generalized bi-circular projection on Y. If there exists an x_0 so that $\{x_0,\, V(x_0)\}$ is linearly independent, then $V^2(x_0) = ax_0 + b\,V(x_0)$. A convenient choice of $\varphi \in X^*$ so that $\varphi(x_0) = 0$ and $\varphi(V(x_0)) = 1$ reduces equation (2.2) to $b\,U^2 v - (\lambda+1)Uv = 0$, for all $v \in Y$. If $b \neq 0$, then $U = \gamma Id_Y$, for a constant γ of modulus 1. In this case, equation (2.2) becomes $\gamma^2 V^2 - \gamma(\lambda+1)V + \lambda Id_X = 0$ and Taylor's theorem asserts the existence of projections Q_1 and Q_2 on X such that $V = \overline{\gamma}\lambda Q_1 + \overline{\gamma}Q_2$. This implies that $\mathcal{J}(T) = T\,[\lambda Q_1 + Q_2]$ and thus $P(T) = T\,Q_2$, with Q_2 a generalized bi-circular projection on X. If $b = 0$, it follows that $\lambda = -1$ and equation (2.2) implies that $U^2 T V^2 = T$, for every $T \in B(X,Y)$. Hence, $U^2 = aId_Y$ and $V^2 = \overline{a}Id_X$, with a a modulus 1 complex number of the form $a = e^{i\theta}$. We set $U_Y = e^{-i\theta/2}U$ and $V_X = e^{i\theta/2}V$. Therefore $P(T)$ must be as in (ii). □

Remark 2.2. (1) Grząślewicz, in [**10**], showed that the surjective isometries on $\mathcal{B}(l^p,\, l^r)$, with $1/p + 1/r \neq 1$ and $p, r \in (1, \infty)$, are of type I. For $r \neq 2$, the generalized bi-circular projections on l^r are just the average of the identity and an isometric reflection. This holds true for symmetric sequence spaces with 1-symmetric basis, as a consequence of Arazy's characterization of isometries on such spaces, see [**1**].

(1) Pairs of Banach spaces $(X,\, Y)$ for which $\mathcal{B}(X,Y)$ supports only surjective isometries of type I are called ideal pairs and have been studied by Khalil and Saleh, [**13**]. Thus $X = l^p$ and $Y = l^r$, with $p, r \in (1,\infty)$ and $1/p + 1/r \neq 1$, are an ideal pair, see [**1**].

(2) It is known that generalized bi-circular projections are bi-contractive, see [**9, 14**]. This raises the question as to whether the bi-contractive projections on $\mathcal{B}(X,Y)$ are generalized bi-circular projections in the case that $(X,\, Y)$ form an ideal pair?

3. Generalized bi-circular projections on JB^*-triples. In this section we note that the characterization of generalized bi-circular projections follows easily in some settings such as JB^* triples. This is because of the connection between bi-contractive projections and [**9**].

Lemma 3.1. *Let P be a non-zero linear projection on a Banach space X, and $\alpha \in \mathbf{C}$ such that $I + (\alpha I - P) = U$ (an isometry). Then $|\alpha| = 1$. Moreover we have*:

(i) *P is bi-contractive whenever $\alpha \neq 1$ and*

(ii) *P is Hermitian whenever the argument of α is irrational modulo π.*

It follows immediately as a consequence of this lemma that every generalized bi-circular projection is bi-contractive. Combining this fact together with the following result of Friedman and Russo yields a characterization of generalized bi-circular projections on JB^* triples. We refer the reader to the paper of Friedman and Russo, [**8**] and the references therein, for the background on JB^* triples. Their theorem is the following.

Theorem 3.2. *Let P be a bi-contractive projection on a JB^*-triple U. Then there is a surjective isometry θ on U of order 2 such that $P = (I + \theta)/2$.*

Since every operator on U of the form $P = (I + \theta)/2$, with θ a surjective isometry of order 2 is a generalized bi-circular projection, we have the following corollary.

Corollary 3.3. *P is a generalized bi-circular projection on a JB^* triple U if and only if there is a surjective isometry θ of order 2 such that $P = (I + \theta)/2$.*

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