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MAPS PRESERVING UNITARY SIMILARITY ON $\mathcal{B}(H)$

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ABSTRACT. Let H be an infinite dimensional complex Hilbert space and denote $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators acting on H. It is proved that a surjective map Φ on $\mathcal{B}(H)$ satisfying that, for any $T, S, R \in \mathcal{B}(H)$ and $\lambda \in \mathbf{C}$, $T + \lambda S \overset{u}{\sim} R \Leftrightarrow \Phi(T) + \lambda\Phi(S) \overset{u}{\sim} \Phi(R)$ is either a unitary isomorphism or a unitary anti-isomorphism multiplied by a scalar.

1. Introduction and statement of the main result. Linear maps preserving similarity have been treated recently in a series of papers. This topic belongs to a broad field of linear preserver problems (see [**1, 6, 12**]). The linear maps preserving similarity on matrix algebras were characterized completely in [**7, 12, 13**]. Similarity preserving linear maps on infinite dimensional operator spaces were studied by Ji, Du, Petek, Semrl and the present authors (see [**2, 9–11, 15, 19**]). Besides linear maps, the additive maps (even the nonlinear maps) were studied and the similarity preserving property was replaced by a weaker assumption of asymptotic similarity preserving (see [**3–5, 8**]).

Hiai, Li and Tsing studied not only similarity preserving linear maps but also unitary similarity preserving ones on finite dimensional spaces. The linear maps preserving unitary similarity in both directions on infinite dimensional operator spaces were discussed in [**16**]. It is clear that if Φ is linear map on $\mathcal{B}(H)$ preserving unitary similarity in both directions, then, for any $T, S, R \in \mathcal{B}(H)$ and $\lambda \in \mathbf{C}$, we have

$$T + \lambda S \overset{u}{\sim} R \Longleftrightarrow \Phi(T) + \lambda\Phi(S) \overset{u}{\sim} \Phi(R). \tag{1.1}$$

The aim of this note is to show that, for a surjective map Φ (no linearity is assumed) on $\mathcal{B}(H)$, the relation (1.1) alone is enough to determine the structure of the map Φ.

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In this paper, H will be a complex infinite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and $\mathcal{B}(H)$ denotes the Banach algebra of all bounded linear operators acting on H. Let $A, B \in \mathcal{B}(H)$. $A \overset{u}{\sim} B$ means that A is unitary similar to B, that is, there is a unitary operator U on H such that $B = UAU^*$. $\mathcal{S}_u(A)$ denotes the unitary similarity orbit of A, i.e., the set of all operators that are unitary similar to A. A subset $\mathcal{M}$ of $\mathcal{B}(H)$ is said to be unitary similarity invariant if $\mathcal{S}_u(A) \subseteq \mathcal{M}$ for every $A \in \mathcal{M}$; if, in addition, $\mathcal{M}$ is a linear subspace, it is called a unitary similarity invariant linear subspace. A map Φ on $\mathcal{B}(H)$ is said to be unitary similarity preserving if $T \overset{u}{\sim} S$ implies that $\Phi(T) \overset{u}{\sim} \Phi(S)$; Φ is said to be unitary similarity preserving in both directions if $T \overset{u}{\sim} S$ if and only if $\Phi(T) \overset{u}{\sim} \Phi(S)$. Denote by $\mathcal{F}(H)$ the set of all finite rank operators on $\mathcal{B}(H)$ and by $\mathcal{F}_0(H)$ the subspace of all finite rank operators X with $\operatorname{tr}(X) = 0$, where $\operatorname{tr}(X)$ denotes the trace of X. We use the notation $x \otimes y$ for a operator of rank not greater than 1 defined by $z \mapsto \langle z, y\rangle x$. It is well known that $x \otimes y$ is a rank one nilpotent if and only if both x and y are nonzero and $\langle x, y\rangle = 0$. Note that the rank one nilpotent operators $x \otimes y$ and $e \otimes f$ are unitary similar if and only if $\|x\|\|y\| = \|e\|\|f\|$.

Our main result is stated as follows.

Main theorem. *Let H be a complex infinite dimensional Hilbert space, and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a surjective map. If, for any $T, S, R \in \mathcal{B}(H)$ and $\lambda \in \mathbf{C}$,*

$$T + \lambda S \overset{u}{\sim} R \Longleftrightarrow \Phi(T) + \lambda\Phi(S) \overset{u}{\sim} \Phi(R),$$

then one of the following holds:

(1) *there exist a non-zero constant c and a unitary operator U such that $\Phi(T) = cUTU^*$ for all $T \in \mathcal{B}(H)$;*

(2) *there exist a non-zero constant c and a conjugate unitary operator U such that $\Phi(T) = cUT^*U^*$ for all $T \in \mathcal{B}(H)$.*

2. Proof of the main result. Before the proof of the main result is given, some lemmas are needed.

Our first lemma is an improvement of [**16**, Lemma 1] and a shorter proof is given here.

Lemma 2.1. *If $\mathcal{M}$ is a non-trivial unitary similarity invariant subspace of $\mathcal{B}(H)$ with $\mathcal{M} \neq \mathbf{C}I$, then $\mathcal{M} \supseteq \mathcal{F}_0(H)$. Here $\mathbf{C}I = \{\lambda I \mid \lambda \in \mathbf{C}\}$.*

Proof. Since M is a unitary similarity invariant subspace, it is enough to show that $\mathcal{M}$ contains a rank one nilpotent operator. Let P be a projection and $A \in \mathcal{M}$; then $U = P + i(I - P)$ is a unitary operator and the commutator $[P, A] = (U^*AU - UAU^*)/(2i)$. So we have $[P, A] \in \mathcal{M}$. Since every operator of $\mathcal{B}(H)$ is a linear combination of projections, we see that $[B, A] \in \mathcal{M}$ for all $B \in \mathcal{B}(H)$. As $\mathcal{M} \neq \mathbf{C}I$, there exist an operator $A \in \mathcal{M}$ and a vector $x \in H$ such that Ax and x are linearly independent. Then, for a fixed nonzero vector $y_0 \in H$, $B = Ax \otimes y_0 - x \otimes A^*y_0 = [A, x \otimes y_0] \in \mathcal{M}$. Pick $x_0, z \in H$ such that $\langle x_0, y_0\rangle = \langle x_0, A^*y_0\rangle = \langle x, z\rangle = 0$, $\langle Ax, z\rangle = 1$. By a simple computation, the rank one nilpotent operator $x_0 \otimes y_0 = [x_0 \otimes z, B] \in \mathcal{M}$. $\square$

The following lemma can be found in [**16**].

Lemma 2.2. *An operator $N \in \mathcal{F}_0(H)$ is a rank one nilpotent if and only if*

(i) *$N \overset{u}{\sim} \alpha N$ for all $|\alpha| = 1$, and*

(ii) *for every $M \in \mathcal{S}_u(N)$, the following implication holds true*:

If $N + M \overset{u}{\sim} \gamma N$ for some $\gamma \neq 0$, then for every $\beta, |\beta| = 1$, there exists a nonzero δ such that $N + \beta M \overset{u}{\sim} \delta N$.

The next lemma is crucial for our purpose and was proved in [**5**], where the real case and finite dimensional case were also considered.

Lemma 2.3. *Let X be a complex infinite dimensional Banach space, and let $\mathcal{N}_1(X)$ be the set of all rank one nilpotent operators in $\mathcal{B}(X)$. Suppose that $\Phi : \mathcal{N}_1(X) \to \mathcal{N}_1(X)$ is a bijective transformation with the property that*

$$T + S \in \mathcal{N}_1(X) \Longleftrightarrow \Phi(T) + \Phi(S) \in \mathcal{N}_1(X)$$

for all $T, S \in \mathcal{N}_1(X)$. Then there exists an invertible bounded linear *operator or conjugate linear operator $A : X \to X$ such that*

$$\Phi(T) = \lambda_T A T A^{-1} \quad \text{for all } T \in \mathcal{N}_1(X),$$

where λ_T is a scalar depending on T; or there exists an invertible bounded linear operator or conjugate linear operator $A : X^* \to X$ such that

$$\Phi(T) = \lambda_T A T^* A^{-1} \quad \text{for all } T \in \mathcal{N}_1(X),$$

where λ_T is a scalar depending on T.

Proof of Main theorem. We finish the proof by checking several claims.

Claim 1. *$\Phi(0) = 0$ and Φ is injective.*

For any operator $T \in \mathcal{B}(H)$, by the assumption of the main theorem, $T - T \overset{u}{\sim} 0 \Rightarrow \Phi(T) - \Phi(T) \overset{u}{\sim} \Phi(0)$, and so $\Phi(0) = 0$. If $\Phi(T) = \Phi(S)$, then $\Phi(T) - \Phi(S) \overset{u}{\sim} 0 = \Phi(0)$ and $T - S \overset{u}{\sim} 0$. Thus $T = S$, as desired.

Claim 2. *$\Phi(\mathcal{F}_0(H)) = \mathcal{F}_0(H)$.*

Firstly, we prove that $\Phi^{-1}(\mathcal{F}_0(H))$ is a unitary similarity invariant subspace. For any $T, S \in \Phi^{-1}(\mathcal{F}_0(H))$ and $\alpha \in \mathbf{C}$, $\Phi(T), \Phi(S) \in \mathcal{F}_0(H)$ implies $\Phi(T) + \Phi(S)$, $\alpha\Phi(T) \in \mathcal{F}_0(H)$. By the assumption, $T + S, \alpha T \in \Phi^{-1}(\mathcal{F}_0(H))$, that is, $\Phi^{-1}(\mathcal{F}_0(H))$ is a linear subspace. For any operator $T \in \Phi^{-1}(\mathcal{F}_0(H))$ and $R \in \mathcal{S}_u(T)$, using the property of Φ, we have $\Phi(R) \overset{u}{\sim} \Phi(T) \in \mathcal{F}_0(H)$. Since $\mathcal{F}_0(H)$ is a unitary similarity invariant subspace, we get $\Phi(R) \in \mathcal{F}_0(H)$. Hence, $R \in \Phi^{-1}(\mathcal{F}_0(H))$ and thus $\Phi^{-1}(\mathcal{F}_0(H))$ is a unitary similarity invariant subspace. Clearly, $\Phi^{-1}(\mathcal{F}_0(H)) \neq \mathbf{C}I$. By Lemma 2.1, $\Phi^{-1}(\mathcal{F}_0(H)) \supseteq \mathcal{F}_0(H)$, i.e., $\Phi(\mathcal{F}_0(H)) \subseteq \mathcal{F}_0(H)$. The reverse inclusion holds because Φ and Φ^{-1} have the same property.

Claim 3. *Φ preserves the rank one nilpotent operator in both directions and $T+S \in \mathcal{N}_1(H) \Leftrightarrow \Phi(T)+\Phi(S) \in N_1(H)$, where $\mathcal{N}_1(H)$ denotes the collection of all rank one nilpotent operators in $\mathcal{B}(H)$.*

Let $N \in \mathcal{N}_1(H)$. By Claim 2, we have $\Phi(N) \in \mathcal{F}_0(H)$. For $\alpha \in \mathbf{C}$ with $|\alpha| = 1$, $N \overset{u}{\sim} \alpha N$ implies $\Phi(N) \overset{u}{\sim} \alpha\Phi(N)$. Thus $\Phi(N)$ is nilpotent.

Suppose $M \in \mathcal{S}_u(\Phi(N))$ and $\Phi(N) + M \overset{u}{\sim} \gamma\Phi(N)$ for some nonzero complex number γ. By the assumption, there exists an $R \in \mathcal{S}_u(N)$ such that $N + R \overset{u}{\sim} \Phi^{-1}(\gamma\Phi(N)) \overset{u}{\sim} \gamma N$. By Lemma 2.2, for any complex number β with $|\beta| = 1$, there exists a non-zero complex number δ satisfying $N + \beta R \overset{u}{\sim} \delta N$ and consequently $\Phi(N) + \beta\Phi(R) \overset{u}{\sim} \Phi(\delta N) \overset{u}{\sim} \delta\Phi(N)$. By Lemma 2.2 again, we know that $\Phi(N) \in \mathcal{N}_1(H)$. Suppose $T + S \in \mathcal{N}_1(H)$. Then $\Phi(T) + \Phi(S) \overset{u}{\sim} \Phi(T + S) \in \mathcal{N}_1(H)$. Since the map Φ^{-1} has the same property, Φ must preserve rank one nilpotent operator in both directions and $T + S \in \mathcal{N}_1(H)$ if and only if $\Phi(T) + \Phi(S) \in \mathcal{N}_1(H)$.

Claim 4. *One of the following statements holds*:

(i) *There exists a unitary or conjugate unitary operator U such that $\Phi(T) = \lambda_T UTU^*$ for all $T \in \mathcal{N}_1(H)$, where λ_T is a non-zero complex number depending on T*;

(ii) *There exists a unitary or conjugate unitary operator U such that $\Phi(T) = \lambda_T UT^*U^*$ for all $T \in \mathcal{N}_1(H)$, where λ_T is a non-zero complex number depending on T.*

In fact, by Claim 3 and Lemma 2.3, one of the following must be true:

(a) There exists a bounded linear or conjugate linear invertible operator A such that $\Phi(T) = \lambda_T ATA^{-1}$ for all $T \in \mathcal{N}_1(H)$ with λ_T a non-zero complex number depending on T;

(b) There exists a bounded linear or conjugate linear invertible operator A such that $\Phi(T) = \lambda_T AT^*A^{-1}$ for all $T \in \mathcal{N}_1(H)$ with λ_T a non-zero complex number depending on T.

Assume that (a) holds. For any unit vector x in H, let $P = x \otimes x$. We have

$$x \otimes x\Phi(P)(I - x \otimes x) = x \otimes (\Phi(P)^*x - \langle \Phi(P)^*x, x\rangle x),$$
$$(I - x \otimes x)\Phi(P)x \otimes x = (\Phi(P)x - \langle \Phi(P)x, x\rangle x) \otimes x.$$

Let $y = \Phi(P)^*x - \langle \Phi(P)^*x, x\rangle x$, $z = \Phi(P)x - \langle \Phi(P)x, x\rangle x$. Then $\langle x, y\rangle = \langle x, z\rangle = 0$. For any complex number $\beta \neq 1$ with modulus one, it is easily seen that $W = \beta x \otimes x + (I - x \otimes x)$ is a unitary operator and $W\Phi(P)W^* = \Phi(P) + (\beta - 1)x \otimes y + (\overline{\beta} - 1)z \otimes x$. By the hypothesis of the main theorem, we have $P \overset{u}{\sim} P + \Phi^{-1}((\beta - 1)x \otimes y + (\overline{\beta} - 1)z \otimes x)$. This further yields that there exists a unitary operator V such that $P \overset{u}{\sim}$

$VPV^* + \delta_{(\beta-1)x\otimes y}A^{-1}((\beta-1)x\otimes y)A + \delta_{(\overline{\beta}-1)z\otimes x}A^{-1}((\overline{\beta}-1)z\otimes xA)$. So $\delta_{(\beta-1)x\otimes y}A^{-1}((\beta-1)x\otimes y)A + \delta_{(\overline{\beta}-1)z\otimes x}A^{-1}((\overline{\beta}-1)z\otimes x)A$ is self-adjoint and thus $Ax \in [(A^{-1})^*((\overline{\beta}-1)y), (A^{-1})^*((\beta-1)x)]$. Now $\langle Ax, (A^{-1})^*((\overline{\beta}-1)y)\rangle = 0$ implies that $Ax \in [(A^{-1})^*((\beta-1)x)]$. Hence $A^* = cA^{-1}$ for some scalar c. Since $AA^* = cI$, we see that $c > 0$. Let $U = (1/\sqrt{c})A$. Then U is a unitary or conjugate unitary operator and $\Phi(T) = \lambda_T UTU^*$ for all $T \in \mathcal{N}_1(H)$, i.e., (i) holds. Similarly, if (b) holds, then (ii) must be true.

Define $\Psi : \mathcal{B}(H) \to \mathcal{B}(H)$ by $\Psi(T) = U^*\Phi(T)U$ if Φ takes form (i) in Claim 4, by $\Psi(T) = U^*\Phi(T)^*U$ if Φ takes form (ii) in Claim 4. Obviously the bijection Ψ satisfies that, for any $T, S, R \in \mathcal{B}(H)$ and $\lambda \in \mathbf{C}$,

$$T + \lambda S \overset{u}{\sim} R \Longleftrightarrow \Psi(T) + \tau(\lambda)\Psi(S) \overset{u}{\sim} \Psi(R), \tag{2.1}$$

where $\tau(\lambda) \equiv \lambda$ or $\tau(\lambda) \equiv \overline{\lambda}$. It is also evident that $\Psi(T) = \lambda_T T$ for all $T \in \mathcal{N}_1(H)$.

Claim 5. *Let vectors $x, y, f, g \in H$ be such that $\{x, y\}$ and $\{f, g\}$ are linearly independent sets with $\langle x, f\rangle = \langle y, g\rangle = 0$. Then, there exist complex numbers α_i, β_i, $i = 1, 2$, such that $\Psi(x\otimes f + y\otimes g) = (\alpha_1 x + \beta_1 y)\otimes f + (\alpha_2 x + \beta_2 y)\otimes g$.*

Because

$$\Psi(x\otimes f + y\otimes g) - \Psi(x\otimes f) \overset{u}{\sim} \Psi(y\otimes g),$$

there exist vectors w_1, v_1 with $\langle w_1, v_1\rangle = 0$ such that

$$\Psi(x\otimes f + y\otimes g) = \lambda_{x\otimes f} x\otimes f + w_1\otimes v_1.$$

Similarly,

$$\Psi(x\otimes f + y\otimes g) = \lambda_{y\otimes g} y\otimes g + w_2\otimes v_2$$

holds for some vectors $w_2, v_2 \in H$ with $\langle w_2, v_2\rangle = 0$. Thus we have $\lambda_{x\otimes f} x\otimes f - \lambda_{y\otimes g} y\otimes g = w_2\otimes v_2 - w_1\otimes v_1$. Note that x and y as well as f and g are linearly independent, so $w_1, w_2 \in [x, y], v_1, v_2 \in [f, g]$, where $[x, y]$ denotes the linear space spanned by x, y. Write $w_1 = \gamma_1 x + \gamma_2 y$ and $v_1 = \delta_1 f + \delta_2 g$; then $\Psi(x\otimes f + y\otimes g) = \lambda_{x\otimes f} x\otimes f + (\gamma_1 x + \gamma_2 y)\otimes(\delta_1 f + \delta_2 g) = ((\lambda_{x\otimes f} + \gamma_1\overline{\delta_1})x + \gamma_2\overline{\delta_1}y)\otimes f + (\gamma_1\overline{\delta_2}x + \gamma_2\overline{\delta_2}y)\otimes g$. Let

$\alpha_1 = \lambda_{x\otimes f} + \gamma_1\overline{\delta_1}$, $\beta_1 = \gamma_2\overline{\delta_1}$, $\alpha_2 = \gamma_1\overline{\delta_2}$ and $\beta_2 = \gamma_2\overline{\delta_2}$. The desired conclusion follows.

Claim 6. *For every projection P (i.e. $P = P^* = P^2$), there exist $\lambda_P, \mu_P \in \mathbf{C}$ such that $\Psi(P) = \lambda_P P + \mu_P I$.*

For any unit vector x, let $y = \Psi(P)^*x - \langle \Psi(P)^*x, x\rangle x$ and $z = \Psi(P)x - \langle \Psi(P)x, x\rangle x$. Then, for any complex number $\beta \neq 1$ with modulus one, using the methods as in Claim 4, it is easily seen that $P \overset{u}{\sim} P + \Psi^{-1}(F)$, where $F = (\beta-1)x\otimes y + (\overline{\beta}-1)z\otimes x$. This implies that $\Psi^{-1}(F)$ is self-adjoint. Therefore, $\delta_{(\beta-1)x\otimes y}(\beta-1)x\otimes y + \delta_{(\overline{\beta}-1)z\otimes x}(\overline{\beta}-1)z\otimes x$ is also self-adjoint. Here $\delta_T = {\lambda_T}^{-1}$ for $T \in \mathcal{N}_1(H)$. Thus $z = \mu y$ and $\Psi^{-1}((\overline{\beta}-1)z\otimes x) = \overline{\delta_{(\beta-1)x\otimes y}}(\overline{\beta}-1)y\otimes x$. By Claim 5, there exist $\alpha_i, \beta_i, i = 1, 2$ satisfying $P \overset{u}{\sim} P + (\alpha_1 x + \beta_1 y)\otimes x + (\alpha_2 x + \beta_2 y)\otimes y$.

In the following, we will prove $y = 0$ whenever x is in the range of P or the kernel of P. Assume, on the contrary, that $y \neq 0$. Then

$$\Psi^{-1}(F) = \begin{pmatrix} \alpha_1 & \alpha_2\|y\| & 0 \\ \beta_1\|y\| & \beta_2\|y\|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

according the space decomposition $H = [x] \oplus [y/\|y\|] \oplus [x, y/\|y\|]^\perp$. Note that $\Psi^{-1}(F)$ is self-adjoint and $\Psi^{-1}(F) \in \mathcal{F}_0(H)$. So

$$\Psi^{-1}(F) = \begin{pmatrix} \alpha_1 & \overline{\beta}_1\|y\| & 0 \\ \beta_1\|y\| & -\alpha_1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\alpha_1 \in \mathbf{R}$. Now, it is easy to see that

$$\Psi^{-1}(F) \overset{u}{\sim} \begin{pmatrix} 0 & \delta_{(\beta-1)x\otimes y}(\beta-1)\|y\| & 0 \\ \overline{\delta_{(\beta-1)x\otimes y}}(\overline{\beta}-1)\|y\| & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $\gamma = \delta_{(\beta-1)x\otimes y}(\beta-1)\|y\|$; then

$$\Psi^{-1}(F)^2 = (\alpha_1^2 + |\beta_1|^2\|y\|^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{u}{\sim} |\gamma|^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This implies

$$\alpha_1^2 + |\beta_1|^2\|y\|^2 = |\gamma|^2. \tag{2.2}$$

Because

$$\begin{pmatrix} \alpha_1 & \overline{\beta_1}\|y\| - \gamma & 0 \\ \beta_1\|y\| & -\alpha_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \Psi^{-1}(F) - \Psi^{-1}((\beta - 1)x \otimes y) \in \mathcal{N}_1(H),$$

we have

$$\beta_1\|y\|(\overline{\beta_1}\|y\| - \gamma) = -\alpha_1^2. \tag{2.3}$$

Obviously, $\Psi(P) \overset{u}{\sim} \Psi(P) + F$ implies that $P \overset{u}{\sim} P + \Psi^{-1}(F)$ and $(P + \Psi^{-1}(F))^2 = P + \Psi^{-1}(F)$.

If $x \in \operatorname{Rng}(P)$ ($\operatorname{Rng}(P)$ denotes the range of P), then $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & P_{22} & P_{23} \\ 0 & P_{32} & P_{33} \end{pmatrix}$ according to the space decomposition $H = [x] \oplus [y/\|y\|] \oplus [x, y/\|y\|]^{\perp}$. So

$$P + \Psi^{-1}(F) = \begin{pmatrix} 1 + \alpha_1 & \overline{\beta_1}\|y\| & 0 \\ \beta_1\|y\| & P_{22} - \alpha_1 & P_{23} \\ 0 & P_{32} & P_{33} \end{pmatrix}$$

according to the same space decomposition. Comparing $(P + \Psi^{-1}(F))^2$ with $P + \Psi^{-1}(F)$ in the (1,1)-entry, we get

$$|\beta_1|^2\|y\|^2 + \alpha_1^2 = -\alpha_1. \tag{2.4}$$

Combining (2.2) with (2.4) gives

$$\alpha_1 = -|\gamma|^2, \tag{2.5}$$

which reduces relation (2.4) into

$$|\beta_1|^2\|y\|^2 = |\gamma|^2 - |\gamma|^4. \tag{2.6}$$

Combining (2.5) and (2.6) with (2.3) gives $\gamma = 0$ or $\beta_1\|y\| = \overline{\gamma}$. If $\beta_1\|y\| = \overline{\gamma}$, (2.6) also induces $\gamma = 0$. Thus $\Psi^{-1}((\beta - 1)x \otimes y) = \gamma x \otimes (y/\|y\|) = 0$, a contradiction.

If $x \in \ker(P) = \mathrm{Rng}\,(I-P)$, then $P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Q_{22} & Q_{23} \\ 0 & Q_{32} & Q_{33} \end{pmatrix}$ according to the space decomposition $H = [x] \oplus [y/\|y\|] \oplus [x, y/\|y\|]^{\perp}$. This further yields

$$P + \Psi^{-1}(F) = \begin{pmatrix} \alpha_1 & \overline{\beta_1}\|y\| & 0 \\ \beta_1\|y\| & Q_{22} - \alpha_1 & Q_{23} \\ 0 & Q_{32} & Q_{33} \end{pmatrix}.$$

Comparing $(P + \Psi^{-1}(F))^2$ with $P + \Psi^{-1}(F)$ in the (1,1)-entry, we also have

$$|\beta_1|^2\|y\|^2 + \alpha_1^2 = \alpha_1. \tag{2.7}$$

It follows from (2.7) and (2.2) that $\alpha_1 = |\gamma|^2$; this will lead to a contradiction either way.

Thus $y = 0$ and $\Psi(P)x = \langle \Psi(P)x, x\rangle x$ whenever $x \in \mathrm{Rng}\,(P) \cup \ker(P)$. Thus there exist scalars λ and μ such that $\Psi(P) = \lambda P + \mu(I - P)$. Now the conclusion of Claim 6 is obvious.

Claim 7. *There exists a non-zero complex number λ_0 such that $\Psi(I) = \lambda_0 I$ and $\Psi(P) = \lambda_0 P$ for every rank one projection $P \in \mathcal{B}(H)$.*

It is easy to see that $\Psi(I) = \lambda_0 I$ for some nonzero complex number λ_0 by Claim 6. Let $P = x \otimes x$, $\|x\| = 1$. Since H is infinite dimensional, there exists a sub-projection Q of $I - P$ such that $Q + x \otimes x \overset{u}{\sim} Q \overset{u}{\sim} (I - Q)$. By the assumption, $\Psi(Q) \overset{u}{\sim} \Psi(I) - \Psi(Q)$. It follows from Claim 6 that

$$\Psi(Q) = \begin{pmatrix} \xi_1 & 0 \\ 0 & \eta_1 \end{pmatrix}$$

and

$$\Psi(I) - \Psi(Q) = \begin{pmatrix} \lambda_0 - \xi_1 & 0 \\ 0 & \lambda_0 - \eta_1 \end{pmatrix}$$

according to the space decomposition $H = Q(H) \oplus (I - Q)(H)$. Thus we must have $\xi_1 + \eta_1 = \lambda_0$. Since $Q + x \otimes x \overset{u}{\sim} Q$, we have $\Psi(Q) + \Psi(x \otimes x) \overset{u}{\sim} \Psi(Q)$. Using Claim 6 again, we can rewrite

$$\Psi(Q) = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_1 \end{pmatrix}$$

and

$$\Psi(x \otimes x) = \begin{pmatrix} \eta_3 & 0 & 0 \\ 0 & \xi_3 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}$$

according to the space decomposition $Q(H) \oplus P(H) \oplus (I - Q - P)(H)$. By the above matrix representation, in order to complete the proof of Claim 7, we need only to show that $\eta_3 = 0$ and $\xi_3 = \lambda_0$. Assume that $\Psi(Q + x \otimes x) = \begin{pmatrix} \xi_2 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \eta_2 \end{pmatrix}$. Since $Q + x \otimes x \overset{u}{\sim} I - (Q + x \otimes x)$, just like the proof of $\xi_1 + \eta_1 = \lambda_0$, we can get $\xi_2 + \eta_2 = \lambda_0$. From $\Psi(Q + x \otimes x) \overset{u}{\sim} \Psi(Q) + \Psi(x \otimes x)$, i.e.,

$$\begin{pmatrix} \xi_2 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \eta_2 \end{pmatrix} \overset{u}{\sim} \begin{pmatrix} \xi_1 + \eta_3 & 0 & 0 \\ 0 & \eta_1 + \xi_3 & 0 \\ 0 & 0 & \eta_1 + \eta_3 \end{pmatrix}, \tag{2.8}$$

we know that one of the following three equations holds: $\xi_1 + \eta_3 = \eta_1 + \xi_3$, $\xi_1 + \eta_3 = \eta_1 + \eta_3$ and $\eta_1 + \xi_3 = \eta_1 + \eta_3$. If $\xi_1 + \eta_3 = \eta_1 + \eta_3$, then $\xi_1 = \eta_1, \Psi(Q) = \eta_1 I$, a contradiction. Similarly, $\eta_1 + \xi_3 = \eta_1 + \eta_3$ will lead to a contradiction, too. So we have $\xi_1 + \eta_3 = \eta_1 + \xi_3$. It also follows from (2.7) that either

$$\begin{cases} \xi_1 + \eta_3 = \eta_1 + \xi_3 = \xi_2 \\ \eta_1 + \eta_3 = \eta_2 \end{cases} \tag{2.9}$$

or

$$\begin{cases} \xi_1 + \eta_3 = \eta_1 + \xi_3 = \eta_2 \\ \eta_1 + \eta_3 = \xi_2 \end{cases}. \tag{2.10}$$

If (2.10) holds, then $\xi_1 + \eta_1 + 2\eta_3 = \xi_2 + \eta_2$. As $\xi_1 + \eta_1 = \xi_2 + \eta_2 = \lambda_0$, we see that $\eta_3 = 0$. So $\xi_1 = \eta_2$, $\eta_1 = \xi_2$ and $\Psi(Q + x \otimes x) = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \xi_1 \end{pmatrix}$. Now, it follows from $Q + x \otimes x - Q \overset{u}{\sim} x \otimes x$ that $\Psi(Q + x \otimes x) - \Psi(Q) \overset{u}{\sim} \Psi(x \otimes x)$, i.e.,

$$\begin{pmatrix} \eta_1 - \xi_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi_1 - \eta_1 \end{pmatrix} \overset{u}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi_3 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

a contradiction. Therefore, (2.9) must be true. Using $\xi_1+\eta_1 = \xi_2+\eta_2 = \lambda_0$, we also have $\eta_3 = 0$ and so $\xi_1 = \xi_2$, $\eta_1 = \eta_2$. It is easy to see that $\xi_1 = (\lambda_0 + \xi_3)/2$ and $\eta_2 = (\lambda_0 - \xi_3)/2$. Thus

$$\Psi(Q) = \begin{pmatrix} \frac{\lambda_0+\xi_3}{2} & 0 \\ 0 & \frac{\lambda_0-\xi_3}{2} \end{pmatrix},$$

according to the space decomposition $H = Q(H) \oplus (I - Q)(H)$. Choose two sub-projections Q_1, Q_2 of Q such that $Q_1 + Q_2 = Q$ and $Q_1 \overset{u}{\sim} Q_2 \overset{u}{\sim} (I - Q_1) \overset{u}{\sim} (I - Q_2)$. Then $\Psi(Q_1) + \Psi(Q_2) \overset{u}{\sim} \Psi(Q)$ since $Q_1 + Q_2 \overset{u}{\sim} Q$. It follows that

$$\begin{aligned} \frac{\lambda_0 + \xi_3}{2}Q_1 + \frac{\lambda_0 - \xi_3}{2}(I - Q_1) + \frac{\lambda_0 + \xi_3}{2}Q_2 + \frac{\lambda_0 - \xi_3}{2}(I - Q_2) \\ \overset{u}{\sim} \frac{\lambda_0 + \xi_3}{2}Q + \frac{\lambda_0 - \xi_3}{2}(I - Q), \end{aligned}$$

i.e., $\xi_3 Q + (\lambda_0 - \xi_3)I \overset{u}{\sim} \xi_3 Q + [(\lambda_0 - \xi_3)/2]I$. Hence we must have $\lambda_0 = \xi_3$.

Claim 8. *Ψ preserves rank one operators in both directions.*

Since Ψ and Ψ^{-1} have the same properties, we need only to check that Ψ maps rank one operators to rank one operators. Let $x, y \in H$ be nonzero vectors. Note that $\Psi(x \otimes y) \overset{u}{\sim} \|x\|\Psi((x/\|x\|) \otimes y)$. Thus we may assume $\|x\| = 1$. By Claim 4 and Claim 7, we need only to treat with the case that x and y are linearly independent and $\langle x, y\rangle \neq 0$. So $y = \langle y, x\rangle x + z$ for some nonzero $z \in [x]^\perp$ and $\Psi(x \otimes y) = \Psi(\langle x, y\rangle x \otimes x + x \otimes z)$. By (2.1), $x \otimes y \overset{u}{\sim} \langle x, y\rangle x \otimes x + x \otimes z$ implies $\Psi(x \otimes y) \overset{u}{\sim} \tau(\langle x, y\rangle)\Psi(x \otimes x) + \Psi(x \otimes z)$. Using Claim 4 and Claim 7 again, we have

$$\begin{aligned} \Psi(x \otimes x) &= \lambda_0 x \otimes x, \\ \Psi(x \otimes z) &= \lambda_{x\otimes z} x \otimes z, \end{aligned}$$

and so $\Psi(x \otimes y) \overset{u}{\sim} \tau(\langle x, y\rangle)\lambda_0 x \otimes x + \lambda_{x\otimes z} x \otimes z$. Hence $\Psi(x \otimes y)$ is a rank one operator.

Claim 9. *There exists nonzero scalar c such that $\Psi(F) = cF$ for every $F \in \mathcal{F}(H)$, where $\mathcal{F}(H)$ denotes the collection of all finite rank operators in $\mathcal{B}(H)$.*

For every $T, S \in \mathcal{B}(H)$, if $\operatorname{rank}(T-S)=1$, using $\Psi(T)-\Phi(S) \overset{u}{\sim} \Psi(T-S)$ and Claim 8, we have $\operatorname{rank}(\Psi(T)-\Psi(S))=1$. Since Ψ and Ψ^{-1} have the same property, we see that $\operatorname{rank}(T-S)=1 \Leftrightarrow \operatorname{rank}(\Psi(T)-\Psi(S))=1$. The next easy observation is that Ψ preserves all rank distances, that is,

$$\operatorname{rank}(T-S)=\operatorname{rank}(\Psi(T)-\Psi(S)), \quad T, S \in \mathcal{F}(H).$$

Indeed, it is easy to see that if $\operatorname{rank}(T-S)=m$ implies the existence of operators $T=T_o, T_1, \ldots T_{m-1}, T_m=S$ such that $\operatorname{rank}(T_i-T_{i-1})=1$, $i=1, \ldots, m$. It follows that $\operatorname{rank}(\Psi(T_i)-\Psi(T_{i-1}))=1$. By the triangle inequality, we have $\operatorname{rank}(\Psi(T)-\Psi(S)) \leq m=\operatorname{rank}(T-S)$. Considering Ψ^{-1} instead of Ψ, we get the reverse inequality. Thus $\Psi: \mathcal{F}(H) \to \mathcal{F}(H)$ is a bijective map preserving adjacency in both directions. By [**17**, Theorems 1.5 and 1.6] and $\Psi(0)=0$, there exists an ring automorphism $\sigma: \mathbf{C} \to \mathbf{C}$ and σ-linear bijective maps $A, B: H \to H$ such that either

$$\Psi(x \otimes y)=Ax \otimes By \quad \text{for all } x, y \in H$$

or

$$\Psi(x \otimes y)=By \otimes Ax \quad \text{for all } x, y \in H.$$

Note that $\Psi(T)=\lambda_T T$ for every rank one nilpotent operator T. Thus both A and B are scalar operators. Hence, there is a scalar c so that $\Psi(x \otimes y)=cx \otimes y$ holds for all $x, y \in H$. Furthermore, using [**17**, Theorems 1.5 and 1.6], we see that Ψ is additive on $\mathcal{F}(H)$ and therefore $\Psi(F)=cF$ holds for every $F \in \mathcal{F}(H)$.

Claim 10. *For every finite rank operator F, the main theorem holds.*

By Claim 9 and the definition of Ψ, one of the following is true:

(i) there exist a constant c and a unitary operator U such that $\Phi(F)=cUFU^*$ for every $F \in \mathcal{F}(H)$;

(ii) there exist a constant c and a conjugate unitary operator U such that $\Phi(F)=cUF^*U^*$ for every $F \in \mathcal{F}(H)$;

(iii) there exist a constant c and a conjugate unitary operator U such that $\Phi(F)=cUFU^*$ for every $F \in \mathcal{F}(H)$;

(iv) there exist a constant c and a unitary operator U such that $\Phi(F) = cUF^*U^*$ for every $F \in \mathcal{F}(H)$.

We need to check that neither case (iii) nor case (iv) holds. Otherwise, for every $F \in \mathcal{F}(H)$ and $\lambda \in \mathbf{C}$, $\Phi(\lambda F) = \overline{\lambda}\Phi(F)$. On the other hand, by our assumption on Φ we have $\Phi(\lambda F) \overset{u}{\sim} \lambda\Phi(F)$. Thus $\overline{\lambda}\Phi(F) \overset{u}{\sim} \lambda\Phi(F)$ for every finite rank operator $\Phi(F)$ and every complex number λ, which is impossible.

By Claim 10, in the following, we may, without loss of generality, assume that $\Phi(F) = F$ for every $F \in \mathcal{F}(H)$. To finish the proof, we need to show that $\Phi(T) = T$ holds for every $T \in \mathcal{B}(H)$.

Claim 11. *For every unitary operator $U \in \mathcal{B}(H)$, there exists a complex number δ_U such that $\Phi(U) = U + \delta_U I$.*

Let x be a unit vector. Denote $y = \Phi(U)^*x - \langle \Phi(U)^*x, x\rangle x$ and $z = \Phi(U)x - \langle \Phi(U)x, x\rangle x$. Then, according to the space decomposition $H = [x] \oplus [x]^{\perp}$, we have

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}, \qquad \Phi(U) = \begin{pmatrix} W_{11} & x \otimes y \\ z \otimes x & W_{22} \end{pmatrix}.$$

Firstly, we check that $U_{21} = 0$ if and only if $z = 0$. Indeed for any complex number β with modulus one, we have $\Phi(U) \overset{u}{\sim} \Phi(U) + F$, where $F = (\beta - 1)x \otimes y + (\overline{\beta} - 1)z \otimes x$. This implies that $U \overset{u}{\sim} U + F$ and hence $U + F$ is a unitary operator. It follows from $UU^* = I$ and $U_{21} = 0$ that $U_{22}U_{22}^* = I_{[x]^{\perp}}$. Thus, $(U + F)(U + F)^* = I$ implies that $|\beta - 1|^2\|x\|^2 z \otimes z + U_{22}U_{22}^* = I_{[x]^{\perp}}$. This details that $z = 0$. Since Φ and Φ^{-1} has the same property, $z = 0$ implies $U_{21} = 0$. In order to complete the proof of Claim 11, we need only prove that $(\Phi(U) - U)x \in [x]$ holds for every unit vector $x \in H$. If $z = 0$, then $U_{21} = 0$. In this case we have $\Phi(U)x = \langle \Phi(U)x, x\rangle x$ and $Ux = \langle Ux, x\rangle x$. Thus $(\Phi(U) - U)x \in [x]$. If $z \neq 0$, then

$$(\beta - 1)U_{21}x \otimes z + (\overline{\beta} - 1)z \otimes U_{21}x + |\beta - 1|^2 z \otimes z = 0, \tag{2.11}$$

since $UU^* = (U + F)(U + F)^* = I$. So there exists some scalar δ such that $U_{21}x = \delta z$, which reducing relation (2.11) into

$$((\beta - 1)\delta + (\overline{\beta} - 1)\overline{\delta} + |\beta - 1|^2) z \otimes z = 0.$$

Hence,

$$(\beta-1)\delta+(\overline{\beta}-1)\overline{\delta}+|\beta-1|^2=0$$

holds for every $\beta\in\mathbf{C}$ with $|\beta|=1$. Picking $\beta=i$, we get

$$2\mathrm{Re}\,((i-1)\delta)+|i-1|^2=0. \tag{2.12}$$

Choosing $\beta=-i$, we get

$$2\mathrm{Re}\,((-i-1)\delta)+|-i-1|^2=0. \tag{2.13}$$

Combining (2.12) and (2.13) gives $\delta=1$, i.e., $U_{21}x=Ux-\langle Ux,x\rangle x=z=\Phi(U)x-\langle\Phi(U)x,x\rangle x$. So we still have $(\Phi(U)-U)x\in[x]$, as desired.

Claim 12. $\Phi(T)=T$ *for every* $T\in\mathcal{B}(H)$.

Firstly, we will prove $\Phi(S)=S+\delta_S I$ for every self-adjoint operator S, where δ_S is a scalar depending on S. Let S be a self-adjoint operator; then S can be written in the form $S=\eta(U+V)$, where U,V are unitary operators and η is a scalar. Thus $\Phi(S)\overset{u}{\sim}\eta(\Phi(U)+\Phi(V))=\eta(U+\delta_U I+V+\delta_V I)=S+\delta_S I$. Therefore, there exists unitary operator W_S such that $\Phi(S)=W_SSW_S^*+\delta_S I$. Choosing arbitrarily a finite rank self-adjoint operator F, we have

$$S+F+\delta_{S+F}I\overset{u}{\sim}\Phi(S+F)\overset{u}{\sim}\Phi(S)+\Phi(F)=W_SSW_S^*+F+\delta_S I.$$

So $\sigma_e(S+\delta_{F+S}I)=\sigma_e(W_SSW_S^*+\delta_S I)$, where $\sigma_e(\cdot)$ denotes the essential spectrum of operator. This further yields $\delta_{F+S}=\delta_S I$, and hence $F+S\overset{u}{\sim}F+W_SSW_S^*$. Let us check that $S=W_SSW_S^*$. For any vector $u\in H$, define analytic functions ϕ_u and ψ_u by $\phi_u(\lambda)=\langle(\lambda-S)^{-1}u,u\rangle$ and $\psi_u(\lambda)=\langle(\lambda-S)^{-1}W_S^*u,W_S^*u\rangle$, $|\lambda|>\|S\|$, respectively. If $S\neq W_SSW_S^*$, there exist u_0 and real number λ_0 such that $\phi_{u_0}(\lambda_0)\neq\psi_{u_0}(\lambda_0)$. Assume that $\phi_{u_0}(\lambda_0)\neq 0$, and let $\alpha=(1/\phi_{u_0}(\lambda_0))$. Then $\alpha\in\mathbf{R}$ and $\alpha u_0\oplus u_0$ is self-adjoint. It is obvious that $\lambda_0\in\sigma(S+\alpha u_0\otimes u_0)$ while $\lambda_0\notin\sigma(W_SSW_S^*+\alpha u_0\otimes u_0)$, which contradicting to the fact that $S+\alpha u_0\otimes u_0\overset{u}{\sim}W_SSW_S^*+\alpha u_0\otimes u_0$. Therefore, we must have $\Phi(S)=S+\delta_S I$.

For every $T\in\mathcal{B}(H)$, we can write T in the form $T=S_1+iS_2$ with S_1,S_2 self-adjoint. Clearly, $\Phi(T)\overset{u}{\sim}\Phi(S_1)+i\Phi(S_2)=T+\delta_T I$. So there exists a unitary operator W_T such that $\Phi(T) = W_T^* T W_T + \delta_T I$. For any rank one operator $x \otimes y \in \mathcal{B}(H)$, we have $\Phi(x \otimes y + T) \overset{u}{\sim} \Phi(x \otimes y) + \Phi(T)$, that is,

$$x \otimes y + T + \delta_{x \otimes y + T} I \overset{u}{\sim} x \otimes y + W_T^* T W_T + \delta_T I.$$

This implies that $\sigma_e(T + \delta_{x \otimes y + T} I) = \sigma_e(W_T^{-1} T W_T + \delta_T I)$. Hence, $\delta_{x \otimes y + T} = \delta_T I$ and $x \otimes y + T \overset{u}{\sim} x \otimes y + W_T^{-1} T W_T$ holds for all rank one operator $x \otimes y$. Now a similar argument as that in the previous paragraph shows that $T = W_T^* T W_T$ and hence $\Phi(T) = T + \delta_T I$.

It remains to check that $\delta_T = 0$ for every T. For any $T, S \in \mathcal{B}(H)$ and $\alpha \in \mathbf{C}$, $S + \alpha T \overset{u}{\sim} (S + \alpha T)$ implies that $\Phi(S) + \alpha \Phi(T) \overset{u}{\sim} \Phi(S + \alpha T)$, that is, $S + \delta_S I + \alpha T + \alpha \delta_T I \overset{u}{\sim} S + \alpha T + \delta_{S + \alpha T} I$. Comparing the spectra, it is easy to see that $\delta_{S + \alpha T} = \delta_S + \alpha \delta_T$, i.e., δ is a linear functional on $\mathcal{B}(H)$. Let N be a square-zero operator; then $N \overset{u}{\sim} \beta N$ holds for every complex number β with modulus one. Thus we have $N + \delta_N I \overset{u}{\sim} \beta N + \beta \delta_N I$ as δ is linear. It follows that $\delta_N = \beta \delta_N$ and hence $\delta_N = 0$. Note that every operator on H can be written as a sum of at most five square zero operators [**14**, Theorem 2]. So we must have $\delta_T = 0$ for every operator T, finishing the proof. $\square$

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