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VALUE DISTRIBUTION OF DIFFERENCES OF MEROMORPHIC FUNCTIONS

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ABSTRACT. Let f be a function transcendental and meromorphic in the plane. Results are proved concerning the existence of zeros of the nth forward difference $\Delta^n f$ and the divided difference $\Delta^n f/f$.

1. Introduction. Let the function f be transcendental and meromorphic in the plane. The forward differences $\Delta^n f$ are defined by [**25**, page52]

$$\begin{aligned} \Delta f(z) &= f(z+1) - f(z), \\ \Delta^{n+1} f(z) &= \Delta^n f(z+1) - \Delta^n f(z), \\ & n = 1, 2, \dots . \end{aligned} \tag{1.1}$$

This paper continues the investigations of [**4**] into the zeros of the forward differences $\Delta^n f$ as defined in (1.1) and the divided differences $\Delta^n f/f$. The work in [**4**] reflects in part the considerable attention given recently to meromorphic solutions in the plane of difference and functional equations [**1, 5, 8, 9, 13, 17**], but the results from [**4**] may also be viewed as discrete analogues of the following sharp theorem [**6, 16**], which uses notation from [**10**].

Theorem 1.1 [**6, 16**]. *Let f be transcendental and meromorphic in the plane with*

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0.$$

Then f' has infinitely many zeros.

The following result was proved in [**4**] using Wiman-Valiron theory [**11**].

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Theorem 1.2 [**4**]. *Let $n \in \mathbf{N}$, and let f be a transcendental entire function of order $\rho < 1/2$, and set*

$$G_n(z) = \frac{\Delta^n f(z)}{f(z)}. \tag{1.2}$$

If G_n is transcendental, then G_n has infinitely many zeros. In particular if f has order less than $\min\{(1/n), (1/2)\}$, then G_n is transcendental and has infinitely many zeros.

Note that if f is an entire function of order less than $1/2$ for which G_n fails to be transcendental for some $n \geq 2$ then f satisfies a homogeneous linear difference equation with rational coefficients and the growth of such solutions was determined in [**17**]. For the first divided difference Theorem 1.2 was extended slightly beyond $\rho = 1/2$ in [**4**].

Theorem 1.3 [**4**]. *There exists $\delta_0 \in (0, 1/2)$ with the following property. Let f be a transcendental entire function with order $\rho(f) < 1/2 + \delta_0$. Then*

$$G(z) = G_1(z) = \frac{\Delta f(z)}{f(z)} = \frac{f(z+1) - f(z)}{f(z)} \tag{1.3}$$

has infinitely many zeros.

The constant δ_0 in Theorem 1.3 is extremely small, but it was conjectured in [**4**] that the conclusion of Theorem 1.3 holds for all entire f with $\rho(f) < 1$. The first result of the present paper extends Theorem 1.3 beyond order $1/2$ for higher divided differences, and broadens the applicability of Theorem 1.2 to meromorphic functions with few poles, even for order $1/2$.

Theorem 1.4. *Let $n \in \mathbf{N}$. Let f be transcendental and meromorphic of order $\rho < 1$ in the plane and assume that G_n as defined by* (1.2) *is transcendental.*

(i) *If G_n has lower order $\mu < \alpha < 1/2$, which holds in particular if $\rho < 1/2$, then*

$$\delta(0, G_n) \leq 1 - \cos \pi\alpha \quad \textit{or} \quad \delta(\infty, f) \leq \frac{\mu}{\alpha}.$$

(ii) *If* $\rho = 1/2$, *then either* G_n *has infinitely many zeros or* $\delta(\infty, f) < 1$.

(iii) *If* f *is entire and* $\rho < 1/2+\delta_0$, *then* G_n *has infinitely many zeros: here* δ_0 *is a small positive absolute constant.*

For meromorphic functions in general the following theorem was proved in [**4**] and addressed a question which represents a natural discrete analogue of Theorem 1.1: if f is transcendental with $\rho(f) < 1$ must Δf have infinitely many zeros?

Theorem 1.5 [**4**]. *Let* f *be a function transcendental and meromorphic in the plane of lower order* $\lambda(f) < 1$. *Let* $c \in \mathbf{C} \setminus \{0\}$ *be such that at most finitely many poles* z_j, z_k *of* f *satisfy* $z_j - z_k = c$. *Then* $g(z) = f(z+c) - f(z)$ *has infinitely many zeros.*

Clearly all but countably many $c \in \mathbf{C}$ satisfy the hypotheses of Theorem 1.5, but the following construction from [**4**] showed that Theorem 1.5 fails without the hypothesis on c, even for lower order 0, and that if the answer to the above question for meromorphic functions is affirmative, then in contrast to Theorem 1.1 it depends on order and not lower order.

Theorem 1.6 [**4**]. *Let* $\phi(r)$ *be a positive non-decreasing function defined on* $[1, \infty)$ *which satisfies* $\lim_{r\to\infty} \phi(r) = \infty$. *Then there exists a function* f *transcendental and meromorphic in the plane such that* $g(z) = \Delta f(z)$ *has only one zero and*

$$\limsup_{r\to\infty} \frac{T(r,f)}{r} < \infty,$$
$$\liminf_{r\to\infty} \frac{T(r,f)}{\phi(r)\log r} < \infty,$$
$$\limsup_{r\to\infty} \frac{T(r,g)}{\phi(r)\log r} < \infty.$$

The final theorem from [**4**] showed that for transcendental meromorphic functions satisfying the very strong growth restriction $T(r,f) = O(\log r)^2$ as $r \to \infty$, either the first difference or the first divided difference has infinitely many zeros. The proof of this result depended on asymptotic properties of such functions with deficient poles [**2**], but this reliance is dispensed with in the following substantial improvement.

Theorem 1.7. *Let f be a transcendental meromorphic function in the plane, of order less than $1/6$, and define G by (1.3). Then at least one of G and Δf has infinitely many zeros.*

2. Preliminaries for Theorem 1.4. A key role for Theorem 1.4 (iii) will be played by the following result of Miles and Rossi [**23**].

Lemma 2.1 [**23**]. *Let f be a transcendental entire function of order $\rho(f) \leq \rho < \infty$. Let $0 < \gamma < 1$, and for $r > 0$ let*

$$U_r = \left\{ \theta \in [0, 2\pi] : \left| \frac{\mathrm{r}e^{i\theta} f'(\mathrm{r}e^{i\theta})}{f(\mathrm{r}e^{i\theta})} \right| \geq \gamma n(r, 1/f) \right\}. \tag{2.1}$$

Let $M > 3$. Then there exists a set $Q_M \subseteq [1, \infty)$ of lower logarithmic density

$$\underline{\text{logdens}}\, Q_M = \liminf_{r \to \infty} \left(\frac{1}{\log r} \int_{[1,r] \cap Q_M} \frac{dt}{t} \right) \geq 1 - \frac{3}{M}, \tag{2.2}$$

such that

$$m(U_r) > \left(\frac{1-\gamma}{7M(\rho+1)} \right)^2 \quad \textit{for} \quad r \in Q_M, \tag{2.3}$$

in which $m(U_r)$ denotes the Lebesgue measure of U_r.

The next lemma is a version of the celebrated $\cos \pi\lambda$ theorem [**12**, Chapter 6].

Lemma 2.2 [**7**]. *Suppose that g is transcendental and meromorphic in the plane, of lower order $\mu < \alpha < 1$, and define $L(r,g) = \min\{|g(z)| : |z| = r\}$ and*

$$Y_1 = \{r > 1 : \log L(r,g) > \gamma(\cos \pi\alpha + \delta(\infty, g) - 1)T(r,g)\}, \ \gamma = \frac{\pi\alpha}{\sin \pi\alpha}.$$

Then Y_1 has upper logarithmic density at least $1 - \mu/\alpha$.

Lemma 2.3 [**4**]. *Let H be a transcendental entire function of order $\rho_1 < \infty$. For large $r > 0$, define $r\theta(r)$ to be the length of the longest arc of the circle $S(0,r)$ of center 0 and radius r on which $|H(z)| > 1$, with $\theta(r) = 2\pi$ if $|H(z)| > 1$ on all of $S(0,r)$, that is, $L(r,H) > 1$. Then at least one of the following is true*:

(a) *there exists a set $F \subseteq [1,\infty)$ of positive upper logarithmic density such that $L(r,H) > 1$ for $r \in F$;*

(b) *for each $\tau \in (0,1)$ the set*

$$F_\tau = \{r \geq 1 : \theta(r) > 2\pi(1-\tau)\} \tag{2.4}$$

satisfies

$$\underline{\text{logdens}}\, F_\tau \geq \frac{1 - 2\rho_1(1-\tau)}{\tau}. \tag{2.5}$$

If H has lower order less than $1/2$, which of course is true if $\rho_1 < 1/2$, then case (a) always holds [**3**]. Moreover, if $\rho_1 = 1/2$ then $\theta(r) \to 2\pi$ on a set of positive upper logarithmic density. We outline the standard argument for this assertion, which is obvious if case (a) applies, so assume that H satisfies case (b). Then the right-hand side of (2.5) is 1, and so for each $n \in \mathbf{N}$ the set

$$P_n = \{r \geq 1 : 2\pi - \theta(r) \geq 1/n\}$$

has logarithmic density 0 using (2.4). Hence we may choose a sequence (s_n) increasing to infinity such that

$$\int_{[1,r]\cap P_n} \frac{dt}{t} \leq \frac{\log r}{n}$$

for $r \geq s_n$. Let P be the union of the sets $P_n \cap [s_n, s_{n+1})$. Then $\theta(r) \to 2\pi$ as r tends to infinity outside P. For large r choose n with

$s_n \le r < s_{n+1}$. Then

$$\int_{[1,r]\cap P} \frac{dt}{t} \le \int_{[1,r]\cap P_n} \frac{dt}{t} \le \frac{\log r}{n}$$

and so P has logarithmic density 0. $\square$

Lemma 2.4 [**4**] *Let $n \in \mathbf{N}$. Let f be transcendental and meromorphic of order less than 1 in the plane. Then there exists a set $X_0 \subseteq [1,\infty)$ of finite logarithmic measure such that*

$$G_n(z) = \frac{\Delta^n f(z)}{f(z)} \sim \frac{f^{(n)}(z)}{f(z)} = o(1) \quad \text{as } z \to \infty \text{ with } |z| \notin X_0. \tag{2.6}$$

The proof of the following lemma is related to that of Theorem 4 in [**18**], but the present approach is somewhat simpler.

Lemma 2.5. *Let f be transcendental and meromorphic in the plane, and let $n \in \mathbf{N}$. Let $c > 0$, and let E be an unbounded subset of $[1,\infty)$ with the following property. For each $r \in E$ there exists a compact arc Ω_r of the circle $S(0,r)$, of angular measure at least c, such that*

$$\begin{aligned} &\lim_{r\to\infty, r\in E} r^{2n} M(\Omega_r, f^{(n)}/f) = 0, \quad \text{where} \\ &M(\Omega_r, g) = \max\{|g(z)| : z \in \Omega_r\}. \end{aligned} \tag{2.7}$$

Let $\phi(r)$ be a positive function tending to infinity with $\phi(r) = o(\log r)$ as r tends to infinity. Then for all sufficiently large $r \in E$ we have

$$\left| \frac{zf'(z)}{f(z)} \right| \le n\phi(r) \tag{2.8}$$

for all $z \in \Omega_r$ outside a set of discs having sum of radii at most $(n-1)r/\phi(r)$.

Proof. There is nothing to prove if $n = 1$ so assume that $n \ge 2$. Let $r \in E$ be large and choose $z_r \in \Omega_r$ with

$$|f(z_r)| = M_r = M(\Omega_r, f). \tag{2.9}$$

Now Taylor's formula gives a polynomial P depending on r and of degree at most $n-1$ such that, for $z \in \Omega_r$,

$$f(z) = P(z) + \int_{z_r}^{z} \frac{(z-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt,$$
$$f'(z) = P'(z) + \int_{z_r}^{z} \frac{(z-t)^{n-2}}{(n-2)!} f^{(n)}(t)\,dt.$$

It follows from (2.7) and (2.9) that

$$\begin{gathered} |P(z_r)| = M_r \quad \text{and} \\ |f(z) - P(z)| + |f'(z) - P'(z)| \leq r^{-n} M_r \quad \text{for } z \in \Omega_r. \end{gathered} \tag{2.10}$$

We can write $P(z) = P_1(z)P_2(z)$ where P_1 is the product of the terms $z - c_j$ over all zeros c_j of P with $|c_j| < 2r$, and is 1 if there are no such c_j. Correspondingly, P_2 is a polynomial with all its zeros, if any, lying in $|z| \geq 2r$. Denoting by C positive constants which are independent of r this gives

$$M(\Omega_r, P_2'/P_2) \leq C/r, \quad M^* = M(\Omega_r, P_2) \leq C \min\{|P_2(z)| : z \in \Omega_r\}. \tag{2.11}$$

Also (2.10) yields

$$M_r \leq M(\Omega_r, P) \leq M^* \cdot M(\Omega_r, P_1) \leq M^*(3r)^d, \tag{2.12}$$

where $d \geq 0$ is the degree of P_1.

Let $z \in \Omega_r$ lie outside the union of the discs of center c_j and radius $r/\phi(r)$. Then (2.11) and (2.12) give

$$|P_1(z)| \geq \frac{r^d}{\phi(r)^d}, \quad |P(z)| = |P_1(z)P_2(z)| \geq \frac{M^* r^d}{C\phi(r)^d} \geq \frac{M_r}{C\phi(r)^d},$$

which on combination with (2.10) and (2.11) leads to

$$\begin{aligned} \left|\frac{zf'(z)}{f(z)}\right| &= \left|\frac{zP'(z) + o(|P(z)|)}{P(z)(1+o(1))}\right| \\ &= \left|\frac{zP'(z)}{P(z)}(1+o(1)) + o(1)\right| \\ &\leq (1+o(1)\left|\frac{zP_1'(z)}{P_1(z)}\right| \\ &\quad + (1+o(1))\left|\frac{zP_2'(z)}{P_2(z)}\right| \\ &\quad + o(1) \leq ((n-1)+o(1))\phi(r). \qquad \square \end{aligned}$$

3. Deficiencies and the logarithmic derivative. We need the following lemma, which is a combination of [**15**, Lemma 4] (see also [**14**]) and Lemma 9 from [**18**].

Lemma 3.1. *Let f be transcendental and meromorphic of finite order in the plane, and set*

$$h(r) = r\frac{d}{dr}(T(r,f)) = \frac{1}{2\pi}\int_0^{2\pi} n(r, e^{i\phi}, f)\, d\phi. \tag{3.1}$$

For each large r, let L_r be any measurable subset of $[0, 2\pi)$ such that the Lebesgue measure of L_r tends to 0 as $r \to \infty$. Then there exists a subset E_0 of $[1, +\infty)$ of logarithmic density 1 such that, as $r \to \infty$ in E_0,

$$\int_{L_r} \left|\mathrm{Re}\left(\frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})}\right)\right| d\theta = o(h(r)). \tag{3.2}$$

Suppose next that

$$\delta(\infty, f) > 1 - \sigma, \quad 0 < \sigma < 1,\ K > 1,\ \sigma K < 1. \tag{3.3}$$

Then there exists a subset E_1 of $(1, +\infty)$, having lower logarithmic density $1 - 1/K$, such that for r in E_1 we have

$$(1 - K\sigma)h(r) \le I(r) = \frac{1}{2\pi}\int_0^{2\pi} \left|\mathrm{Re}\left(\frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})}\right)\right| d\theta. \tag{3.4}$$

Note that (3.1) is of course the classical Cartan formula [**10**, page 8] and that $h(r)$ tends to infinity since f is transcendental. □

4. Proof of Theorem 1.4. Let f be a transcendental meromorphic function in the plane of order $\rho < 1$, let $n \in \mathbf{N}$ and let G_n be defined by (1.2), and assume that G_n is transcendental. Lemma 2.4 gives a set $X_0 \subseteq [1, \infty)$ of finite logarithmic measure such that (2.6) holds. Let the positive function $\phi(r)$ tend to infinity on $[1, \infty)$, and satisfy

$$\phi(r) = o(\log r), \quad \phi(r) = o(h(r)), \quad \phi(r) = o(n(r, f) + n(r, 1/f)), \tag{4.1}$$

where $h(r)$ is defined by (3.1). This is certainly possible since f is transcendental of order less than 1. For each large r, set

$$V_r = \left\{ \theta \in [0, 2\pi] : \left| \frac{\mathrm{re}^{i\theta} f'(re^{i\theta})}{f(\mathrm{re}^{i\theta})} \right| > n\phi(r) \right\}. \tag{4.2}$$

Let N be a large positive integer. For large $r > 0$ let $r\beta(r)$ be the length of the longest arc of the circle $S(0, r)$ of center 0 and radius r on which $|z^N G_n(z)| < 1$, with $\beta(r) = 2\pi$ if $|z^N G_n(z)| < 1$ on all of $S(0, r)$. We begin with the following lemma.

Lemma 4.1. *Suppose that $\beta(r) \to 2\pi$ on a set Y_1 of upper logarithmic density $\lambda \in (0, 1)$. Then $\delta(\infty, f) \leq 1 - \lambda$.*

Proof. It may be assumed that the intersection of Y_1 with the exceptional set X_0 of (2.6) is empty, since this does not reduce the upper logarithmic density. Then by (2.6), (4.2), Lemma 2.5 and the fact that N is large, the Lebesgue measure of V_r satisfies $m(V_r) = o(1)$ on Y_1. Let

$$L_r = V_r \quad \text{if } r \in Y_1, \qquad L_r = \varnothing \quad \text{if } r \notin Y_1.$$

It may be assumed further that $Y_1 \subseteq E_0$, where E_0 is as in Lemma 3.1, again since this does not reduce the upper logarithmic density. Thus Lemma 3.1 gives, using (4.1) and (4.2),

$$\int_0^{2\pi} \left| \mathrm{Re} \left(\frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} \right) \right| d\theta \leq O(\phi(r)) + o(h(r)) = o(h(r)) \tag{4.3}$$

for large $r \in Y_1$.

Now assume that $\delta(\infty, f) > 1 - \sigma > 1 - \lambda$. Then $\sigma > 0$ and we may choose $K > 1$ with $\sigma < 1/K < \lambda$. Hence (3.3) is satisfied and Lemma 3.1 implies that (3.4) holds on a set E_1 of lower logarithmic density at least $1 - 1/K > 1 - \lambda$, so that there must exist an arbitrarily large $r \in Y_1 \cap E_1$. But for these r inequalities (3.4) and (4.3) give $(1 - K\sigma)h(r) \leq o(h(r))$, which is a contradiction. This proves Lemma 4.1. □

We first prove part (i) of Theorem 1.4, and to this end we assume that G_n has lower order $\mu < \alpha < 1/2$. This certainly holds if $\rho < 1/2$, because in this case f may be written as a quotient of entire functions of order less than $1/2$ and a simple argument shows that the same is true of G_n. Assume further that $\delta(0, G_n) > 1 - \cos \pi\alpha$. Then by Lemma 2.2 there exists a subset Y_1 of $[1, \infty)$ having upper logarithmic density at least $1 - \mu/\alpha$ such that

$$\lim_{\substack{r \to \infty \\ r \in Y_1}} r^N M(r, G_n) = 0,$$

which of course gives $\beta(r) = 2\pi$ for large r in Y_1. Thus Lemma 4.1 implies at once that $\delta(\infty, f) \leq \mu/\alpha$, which completes the proof of part (i).

Parts (ii) and (iii) will now be proved together, so assume either that $\rho = 1/2$ and $\delta(\infty, f) = 1$, or that f is entire of order ρ with $\rho - 1/2$ small and positive, and in both cases that G_n has finitely many zeros. Then there exists a rational function R_0 with at most a pole of order $N - 1$ at infinity such that

$$H(z) = \frac{1}{2z^N}\left(\frac{1}{G_n(z)} - R_0(z)\right) \tag{4.4}$$

is entire and transcendental, of order $\rho_1 \leq \rho$, and there exists $r_1 > 0$ such that

$$|z^N G_n(z)| < 1 \quad \text{for } |z| \geq r_1,\ |H(z)| > 1. \tag{4.5}$$

Let $\theta(r)$ be defined as in Lemma 2.3.

Suppose first that $\theta(r) \to 2\pi$ on a set Y_1 of upper logarithmic density $\lambda \in (0, 1)$. This certainly holds under the hypotheses of part (ii), by the remarks following Lemma 2.3 and also applies for part (iii) if H satisfies case (a) of Lemma 2.3. Then by (4.5) the hypotheses of Lemma 4.1 are satisfied, and so we have $\delta(\infty, f) \leq 1 - \lambda < 1$, which is a contradiction.

It therefore remains only to prove part (iii) in the case where the entire function H satisfies conclusion (b) of Lemma 2.3. Let $M > 3$ and choose positive γ and τ such that γ is small and

$$\eta = \left(\frac{1 - \gamma}{7M(\rho + 1)}\right)^2 - 2\pi\tau > 0 \tag{4.6}$$

but η is small. Since f is entire in this case we may apply Lemma 2.1. This gives a subset Q_M of $[1, \infty)$ satisfying (2.2) and (2.3), and there is no loss of generality in assuming that $Q_M \cap X_0 = \varnothing$, where X_0 is as in (2.6), as this assumption does not reduce the lower logarithmic density.

Let F_τ be as in Lemma 2.3. Then for large $r \in F_\tau \setminus X_0$ we have $\theta(r) > 2\pi(1-\tau)$ and $m(V_r) \leq 2\pi\tau + o(1)$, using (2.6), (4.2), (4.5), Lemma 2.5 and the fact that N is large. By (2.1) and (4.1) we also have $m(U_r) \leq 2\pi\tau + o(1)$ for these r. Hence (2.3) and (4.6) show that the intersection $Q_M \cap F_\tau$ is bounded, which by (2.2) and (2.5) forces

$$1 - 2\rho_1(1-\tau) \leq \frac{3\tau}{M}$$

and

$$2\rho - 1 \geq 2\rho_1 - 1 \geq \frac{\tau}{1-\tau}\left(1 - \frac{3}{M}\right) \geq \tau\left(1 - \frac{3}{M}\right).$$

Since $\rho < 1$ and γ is small, while η is small in (4.6), it follows that ρ must satisfy

$$2\rho - 1 \geq \frac{1}{2\pi}\left(\frac{1}{14M}\right)^2\left(1 - \frac{3}{M}\right) = q(M).$$

As noted in [**4**] the right hand side $q(M)$ in the last inequality has a maximum relative to the interval $(3, \infty)$ at $M = 9/2$, with $q(9/2) = 1/23814\pi$. This proves Theorem 1.4. □

5. Lemmas needed for Theorem 1.7. We need the following lemma from [**20**]. The result is closely related to [**19**, Lemma 2.4] and the method of proof is essentially the same.

Lemma 5.1 [**20**]. *Let h be transcendental and meromorphic in the plane, of order less than $\rho < \infty$, and with finitely many poles. Let (z_j) be a sequence in $\{z \in \mathbf{C} : |z| > 2\}$ such that $z_j \to \infty$ without repetition, and with exponent of convergence less than ρ. Let $M_1, M_2 \in \mathbf{R}$ be such that*

$$\rho + M_1 < 1, \quad M_2 \leq M_1 - 4\rho. \tag{5.1}$$

For $m = 1, 2$, let H_m be the union of the closures of the discs $B(z_j, |z_j|^{M_m})$.

Next, let R_1 be large and positive, such that

$$h^{-1}(\{\infty\}) \subseteq B(0, R_1/2), \quad M(R_1, h) = \max\{|h(z)| : |z| = R_1\} > e^4, \tag{5.2}$$

and

$$\log |h(z)| \leq \left|\frac{z}{2}\right|^\rho \quad \text{for } |z| \geq \frac{1}{2}R_1, \tag{5.3}$$

as well as

$$\left(\frac{1}{2}R_1\right)^{M_1 - M_2} > 4, \qquad \sum_{|z_j| > R_1/2} 26|z_j|^{\rho + (M_2 - M_1)/2} < 1. \tag{5.4}$$

Let w_0 lie outside H_1, with

$$|w_0| > R_1, \quad |h(w_0)| > T_1^2, \quad T_1 > M(R_1, h)^2, \tag{5.5}$$

and let C_0 be the component of the set $\{z \in \mathbf{C} \setminus H_2 : |h(z)| > T_1\}$ in which w_0 lies. Then C_0 is unbounded.

Note that (5.2), (5.3) and (5.4) hold for all sufficiently large R_1. Moreover, it follows from (5.1) and the fact that the sequence (z_j) has exponent of convergence less than ρ that the set of $r \geq 1$ for which the circle $S(0, r)$ meets H_1 has finite logarithmic measure. Hence there exist arbitrarily large $w_0 \notin H_1$ satisfying (5.5). □

Lemma 5.2. *Let f be a transcendental meromorphic function in the plane, of order less than $1/6$. Assume that G as defined by (1.3) has finitely many zeros. Then there exist a non-zero complex number b and a set $E_0 \subseteq [1, \infty)$ of lower logarithmic density greater than $2/3$, such that $f(z) \sim b$ for all large z with $|z| \in E_0$.*

Proof. Let N be a large positive integer and choose ρ with $\rho(f) < \rho < 1/6$. Since G is transcendental [**4**, Lemma 2.1] and has finitely many zeros and order less than ρ, it follows from the $\cos \pi\rho$ theorem [**12**, Chapter 6] that there exists $E_0 \subseteq [1, \infty)$, with lower logarithmic density greater than $2/3$, such that

$$\lim_{r \to \infty, r \in E_0} r^N M(r, G) = 0. \tag{5.6}$$

Now define h by

$$h(z) = \frac{1}{z^N G(z)}. \tag{5.7}$$

By Lemma 2 of [**22**] (see also [**21**, Lemma 4.1]) there exist arbitrarily large T_1 such that the length $L(r, T_1, h)$ of the level curves $|h(z)| = T_1$ lying in $B(0, r)$ satisfies

$$L(r, T_1, h) = O(r^2) \quad \text{as } r \to \infty. \tag{5.8}$$

Here T_1 may be chosen so that, for additional convenience, the level curves $|h(z)| = T_1$ do not pass through the origin and have no multiple points. Hence these level curves may be parametrized locally in terms of $\arg h$, and for any given $w \in \mathbf{C}$ the stationary points of $\arg z$ and $\log |z - w|$ on these level curves form a discrete set. If this is not the case, then either $|h(z)| \equiv T_1$ on a ray passing through the origin, which contradicts the choice of T_1, or $|h(z)| \equiv T_1$ on a circle of center w, which is impossible since h is transcendental.

Next, let (z_j) be the set of all distinct zeros and poles of f' with $r_j = |z_j| > 2$, and choose σ and M_1, M_2 satisfying

$$\rho < \sigma < \frac{1}{6}, \quad M_1 = \sigma + \frac{2}{3}, \; M_2 = \sigma. \tag{5.9}$$

This choice may be made so that no circle $S(z_j, r_j^\sigma)$ is tangent to a level curve $|h(z)| = T_1$. For $m = 1, 2$, let H_m be the union of the closures of the discs $B(z_j, |z_j|^{M_m})$. We assert that

$$\frac{f''(u)}{f'(u)} = o(1) \quad \text{for } |u - z| \le 1 \tag{5.10}$$

and for large $z \notin H_2$. To prove (5.10) let R_0 be large and positive, and let $z \in \mathbf{C} \setminus H_2$ with $|z| > 2R_0 + 1$. Then the series expansion for f''/f' gives, for $|u - z| \le 1$,

$$\left| \frac{f''(u)}{f'(u)} \right| \le \frac{n(R_0, f') + n(R_0, 1/f')}{|u| - R_0} + \sum_{|v_j| > R_0} \frac{1}{|u - v_j|},$$

where the v_j are simply the z_j but with repetition according to multiplicity. Since z lies outside H_2 this yields

$$|u - v_j| \geq |z - v_j| - 1 \geq \frac{|v_j|^\sigma}{2}$$

for $|v_j| > R_0$ and so

$$\left|\frac{f''(u)}{f'(u)}\right| \leq \frac{n(R_0, f') + n(R_0, 1/f')}{R_0} + 2 \sum_{|v_j| > R_0} \frac{1}{|v_j|^\sigma} \to 0$$

as $R_0 \to \infty$, using (5.9). This proves (5.10), from which it follows that

$$f(z+1) - f(z) = \int_z^{z+1} f'(v)\, dv = \int_z^{z+1} f'(z)(1 + o(1))\, dv \sim f'(z)$$

and

$$\frac{f'(z)}{f(z)} \sim G(z) \tag{5.11}$$

for large $z \notin H_2$. Next, (5.9) implies that there exists a set X_1 of finite logarithmic measure with $S(0,r) \cap H_1 = \varnothing$ for $r \notin X_1$. In particular, (5.11) holds for large z with $|z| \notin X_1$. It may be assumed that $X_1 \cap E_0 = \varnothing$, and it follows at once from (5.6) and (5.11) that

$$\lim_{r \to \infty, r \in E_0} \int_{S(0,r)} \left|\frac{f'(z)}{f(z)}\right| |dz| = 0. \tag{5.12}$$

The function h is transcendental of order less than ρ with finitely many poles, and the sequence (z_j) has exponent of convergence less than ρ. Thus Lemma 5.1 may be applied to h, with M_1, M_2 given by (5.9) and hence satisfying (5.1). Let R_1 be large and positive, so large that (5.2), (5.3) and (5.4) hold, which is possible by (5.9). Choose T_1 and $w_0 \notin H_1$ as in (5.5), and such that (5.8) also holds. Let C_0 be the component determined in Lemma 5.1: then C_0 is unbounded.

Now choose a sequence (s_m) such that

$$2s_m \leq s_{m+1} \leq s_m^3, \quad s_m \in E_0, \tag{5.13}$$

this being possible since E_0 has lower logarithmic density greater than $2/3$, and since $S(0, s_m)$ does not meet H_2 we may assume using (5.6) and (5.7) that $S(0, s_m) \subseteq C_0$ for all m. Now the part Y_m of ∂C_0 lying in $s_m \leq |z| \leq s_{m+1}$ is contained in the union of the level set $|h(z)| = T_1$ and the circles $S(z_j, r_j^\sigma)$. The number of such circles which meet Y_m is $O(s_{m+1})^\rho$ and their radii have sum $O(s_{m+1})^{\rho+\sigma} = o(s_{m+1})$. Hence the arc length of Y_m is $O(s_{m+1})^2$ using (5.8).

We form a path γ_m in the closure of C_0 joining $S(0, s_m)$ to $S(0, s_{m+1})$ as follows. First take a radial segment λ given by $\arg z = \theta, s_m \leq |z| \leq s_{m+1}$, with θ chosen so that this segment is never tangent to any level curve $|h(z]| = T_1$, which may be done by the remark following (5.8), nor to any of the circles $S(z_j, r_j^\sigma)$. By construction Y_m consists of a union of closed curves lying in $s_m < |z| < s_{m+1}$. Hence any arc of λ which lies outside the closure of C_0 may be replaced by an arc of Y_m. Using (5.7), (5.11), (5.13) and the fact that N is large, we obtain

$$\begin{aligned}\int_{\gamma_m \cup S(0,s_m)} \left| \frac{f'(z)}{f(z)} \right| |dz| &\leq \int_{\gamma_m \cup S(0,s_m)} \frac{2}{|z|^N T_1} |dz| \\ &= O(s_{m+1}^2 s_m^{-N}) = O(s_m^{-1}).\end{aligned}$$

Hence from the union of the curves γ_m and circles $S(0, s_m)$ a simple curve γ may be constructed which tends to infinity and satisfies, by (5.13) again,

$$\int_\gamma \left| \frac{f'(z)}{f(z)} \right| |dz| < \infty.$$

Thus there exists a non-zero complex number b such that $f(z) \to b$ as $z \to \infty$ on γ, which on combination with (5.12) gives the conclusion of the lemma. □

It is perhaps worth remarking that the condition $\rho(f) < 1/6$ seems unlikely to be sharp in Theorem 1.7 but is required in our method in order to deduce Lemma 5.2 from Lemma 5.1. We need $M_2 > \rho(f)$ in order to achieve (5.10) for large z outside H_2, so that in (5.1) the second inequality forces $5\rho(f) < M_1$ and the first inequality then requires $6\rho(f) < 1$.

6. Proof of Theorem 1.7. Let f and G be as in the statement of the theorem, and assume that G has finitely many zeros. Then it follows from Lemma 5.2 that there exist a non-zero complex number b and a set $E_0 \subseteq [1, \infty)$ of lower logarithmic density greater than $2/3$, such that $f(z) \sim b$ for all large z with $|z| \in E_0$. Let

$$F = \frac{\Delta f}{f - b}.$$

Then F must have infinitely many zeros, because otherwise Lemma 5.2 may also be applied to $f - b$ to give a non-zero constant b_1 and a set $E_1 \subseteq [1, \infty)$, again of lower logarithmic density greater than $2/3$, such that $f(z) - b \sim b_1$ for all large z with $|z| \in E_1$, which is evidently impossible.

Let z be large and a zero of F. Then z is not a pole of f because otherwise writing

$$G = F \cdot \frac{f - b}{f}$$

shows that z is also a zero of G, contrary to the assumption that G has finitely many zeros. But $\Delta f = F \cdot (f - b)$, and hence z is a zero of Δf. This proves Theorem 1.7. $\square$

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