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MORE QUINTIC SURFACES WITH 75 LINES

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ABSTRACT. It is well known that the Fermat quintic surface has 75 lines, which is the only known quintic surface with this property. In this note, we study lines in the non-singular surfaces cut out of the Dwork pencil of 3-folds in $\mathbf{P}^4$ by the symmetric hyperplane. All the singular surfaces in the pencil are found and the types of singularities are determined. After that, all non-singular quintic surfaces containing lines outside the base locus of the pencil are determined. Finally, we count the number of the lines on these surfaces, and show there are four surfaces in the pencil, which contain the same number of lines as the Fermat quintic surface does, but are not isomorphic to the Fermat quintic surface. Hence, no surface in this pencil is isomorphic to the Fermat quintic surface.

1. Introduction. Let $\rho : \chi \to \mathbf{C}$ be the well-known family of Calabi-Yau three-folds, whose fiber $\widetilde{X}_t$ over $t \in \mathbf{C}$ is given in $\mathbf{P}^4$ by the equation $F_t = 0$, where

$$F_t(z_0, z_1, z_2, z_3, z_4) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5tz_0z_1z_2z_3z_4.$$

The above family is known as the Dwork pencil of quintics. This paper studies the Dwork pencil of quintics cut out by the symmetric hyperplane $s_1 := z_0+z_1+z_2+z_3+z_4 = 0$ in $\mathbf{P}^4$ and finds special surfaces (singular or non-singular but containing lines outside the base locus) in the cut pencil. We still call it the Dwork pencil for convenience. Denote by X_t the surface $\widetilde{X}_t \cap \{s_1 = 0\}$ of the pencil. We show the base locus of the pencil always contains 15 lines. For a line on X_t outside the base locus, we call it an *additional line*. Denote by E_t the set of additional lines on a surface X_t; this paper shows $E_t \neq \varnothing$ if and only if $t = 0$, $t = 2$ or $t = 2\tau$ where τ is a root of $\tau^4 + \tau + 1 = 0$. We proved that the surface $X_{2\tau}$ contains 75 lines and is not isomorphic to the quintic Fermat surfaces. Actually, letting $\#E_t$ be the number of lines in E_t, we have $\#E_0 = 20$, $\#E_2 = 40$ and $\#E_{2\tau} = 60$ (Theorem 1.2), i.e., the number of total lines on these surfaces are 35, 55 and 75.

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The general algebraic surface F_n in $\mathbf{P}^3$ of degree n contains no line, if $n \geq 4$. It was Segre who first studied the problem about the special non-singular surface of degree $n \geq 4$, which contains some lines. He gave the upper bound $(n-2)(11n-6)$ for the maximum number of lines lying on a non-singular surface F_n of degree $n \geq 4$ by considering the degree of the flecnodal curve on the surface (see [**4, 5**]) and proved that the number is 64 (less than the bound 76) for $n=4$ by considering the geometry of the quartic surface. In the other direction, Caporaso, Harris and Mazur's work [**2**] gives the lower bound for this number: for degree $n = 4, 6, 8, 12, 20$ the corresponding lower bounds are $64, 180, 256, 864, 1600$; for other $n \geq 4$ the lower bound is $3n^2$ by considering a special surface $F(x,y,z,t) = \phi(x,y) - \psi(z,t)$. The Fermat surface is of that type. To find a sharper bound for the number of lines on a smooth quintic surface, people are interested in looking for new surfaces containing lines. In [**1**], Boissière and Sarti use polyhedral groups to construct examples. However, there is no new surface of fifth degree there.

We say a polynomial is totally symmetric if it is invariant under each coordinate permutation. Let s_i be $z_0^i + z_1^i + z_2^i + z_3^i + z_4^i$ which is totally symmetric. It was Barth's idea to find a special quintic surface in the pencil defined by two equations $s_2 s_3 + \lambda s_5 = 0$ and $s_1 = 0$. von Straten also used a similar pencil to construct a hypersurface with many nodes in $\mathbf{P}^4$ (see [**6**]). To get a full investigation of the pencil of symmetric surfaces in $\mathbf{P}^3$, refer to [**3**]. We will see the pencil defined by $s_2 s_3 + \lambda s_5 = 0$ is the same as the Dwork pencil of three-folds in $\mathbf{P}^4$ cut by the symmetric hyperplane $s_1 = 0$.

We give main results in the following two theorems.

Theorem 1.1. *Any additional line on X_0 is a permutation of* $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & \omega & 1 & \omega^2 \end{pmatrix}$ *and any additional line on X_2 is a permutation of* $\begin{pmatrix} 1 & -1 & \omega & -\omega & 0 \\ 1 & \omega^2 & -1 & 0 & -\omega^2 \end{pmatrix}$ *where ω is a cubic root of* 1. *Any additional line on $X_{2\tau}$ is a permutation of* $\begin{pmatrix} 1 & -1 & \tau & -\tau & 0 \\ 1 & \tau & -1 & 0 & -\tau \end{pmatrix}$ *where τ is a root of* $\tau^4 + \tau + 1 = 0$.

Theorem 1.2. *Let E_t be the set of additional lines on the non-singular surface X_t. Then $E_t \neq \varnothing$ if and only if $t = 0, 2$ or 2τ.*

The numbers of the additional lines are $\#E_0 = 20$, $\#E_2 = 40$ *and* $\#E_{2\tau} = 60$. *Furthermore, none of the surfaces* X_0, X_2 *and* $X_{2\tau}$ *is isomorphic to the quintic Fermat surface.*

This article is organized as follows. In Section 2, we collect some concepts such as coordinate plane, base line, additional line, B-line and base line pair. Some useful properties are proved there. In Section 3, we check out all the singular surfaces in the Dwork pencil using the standard procedure. Section 4 is the main part of the paper. We first prove that a general additional line is a B-line, then by considering the action on the base line pairs, we show that the symmetric group S_5 action on the set of additional lines is transitive. Theorem 1.1 is proved there. Finally, in Section 5 we complete the proof of Theorem 1.2.

2. Preliminaries. The symbols denoting lines and planes are given in subsection 2.2. Then we show that the base locus contains 15 lines. In subsection 2.4, we classify the base line pair into skew pair, normal pair and other base line pairs. The transitivity of the skew normal pair (Proposition 4.9) under S_5 action and the property that a skew pair is normalizable (Theorem 4.10) are important to the proof of Theorem 1.1.

2.1. $\mathbf{P}^3$-embedded variety and Dwork pencil of quintic surfaces. Denote by $\mathbf{C}$ the complex number field; let $\mathbf{P}^4$ denote the four-dimensional complex projective space. Denote by s_n the polynomial $z_0^n + z_1^n + z_2^n + z_3^n + z_4^n$. We consider $\mathbf{P}^3$ as the embedded symmetric hyperplane $s_1 = 0$ of $\mathbf{P}^4$. Let I be a set of homogeneous polynomials in $\mathbf{C}[z_0, z_1, z_2, z_3, z_4]$; denote by $\langle I \rangle$ the ideal generated by I in the coordinate ring $\mathbf{C}[z_0, z_1, z_2, z_3, z_4]/\langle s_1 \rangle$ of $\mathbf{P}^3$. A variety defined by an ideal $\langle I \rangle$ in $\mathbf{C}[z_0, z_1, z_2, z_3, z_4]/\langle s_1 \rangle$ is called a $\mathbf{P}^3$-embedded variety.

In particular, the lines, planes and surfaces in $\mathbf{P}^3$ are induced from planes, hyperplanes and the three-folds in $\mathbf{P}^4$. The Dwork pencil is induced by the Dwork pencil of three-folds.

From now on, the surface X_t is defined as a $\mathbf{P}^3$-embedded variety by the polynomial $F_t = s_5 - 5tz_0z_1z_2z_3z_4$. When $t = 1$, the corresponding surface X_1 is

$$F_1(z_0, z_1, z_2, z_3, z_4) = s_5 - 5z_0z_1z_2z_3z_4 = \frac{5}{6}s_2s_3$$

in $\mathbf{C}[z_0, z_1, z_2, z_3, z_4]/\langle s_1 \rangle$. Hence the pencil contains $s_2s_3 = 0$ and is equivalent to the pencil $s_2s_3 + ts_5 = 0$.

2.2. Coordinate plane, base line, residue conic, additional line and B-line. The five coordinate hyperplanes with equation $z_i = 0$ of $\mathbf{P}^4$ induce five planes H_i in $\mathbf{P}^3$. We still call them the coordinate planes.

Let $\{i, j, k, l, m\} = \{0, 1, 2, 3, 4\}$ be the set of indices. The base locus consists of the intersection of X_0 with five coordinate planes. For the coordinate plane H_i, the intersection is given by the following

$$s_5|_{z_i=0} = -5(z_k+z_l)(z_k+z_m)(z_l+z_m)(z_k^2+z_l^2+z_m^2+z_kz_l+z_kz_m+z_lz_m).$$

In this factorization, the linear components are called the base lines on H_i and the quadratic component is called the residue conic C_i on H_i.

There are three base lines on H_i, and $3 \times 5 = 15$ base lines in total.

Denote B_{ij}^k the set of polynomials $B_{ij}^k = \{z_k, z_i + z_j\}$, where i, j, k are distinct indices in $\{0, 1, 2, 3, 4\}$. For $\{i', j'\} = \{0, 1, 2, 3, 4\} - \{i, j, k\}$, the two sets B_{ij}^k and $B_{i'j'}^k$ generate the same ideal $\langle B_{ij}^k \rangle = \langle B_{i'j'}^k \rangle$, which defines a base line.

An additional line is a line on a surface X_t outside the base locus. We denote by E_t the set of additional lines of X_t. Because the base line triangle and the residue conic are the complete intersection of the surface X_0 and a coordinate plane, an additional line intersects the coordinate plane either on a base line or on a residue conic.

A B-line is an additional line that intersects at least two distinct base lines.

A line in $s_1 = 0$ is of course a line in $\mathbf{P}^4$, hence in the Grassmannian $G(2, 5)$, so we can use that notation for a line L

$$L = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ y_0 & y_1 & y_2 & y_3 & y_4 \end{bmatrix}.$$

A line L of $G(2,5)$ lies in $\mathbf{P}^3$ if $\sum x_i = 0$ and $\sum y_i = 0$. Moreover, for a B-line, each row already has a zero entry, then the other entries are of opposite signs in pairs.

2.3. The symmetric group S_5 operation. A permutation σ of the symmetric group S_5 induces an isomorphism of $\mathbf{P}^4$ as:

$$(x_0 : x_1 : x_2 : x_3 : x_4) \longmapsto (x_{\sigma^{-1}(0)} : x_{\sigma^{-1}(1)} : x_{\sigma^{-1}(2)} : x_{\sigma^{-1}(3)} : x_{\sigma^{-1}(4)});$$

for a polynomial $F(z_0, z_1, z_2, z_3, z_4)$, the pull back is

$$\sigma^*(F) = F(z_{\sigma^{-1}(0)}, z_{\sigma^{-1}(1)}, z_{\sigma^{-1}(2)}, z_{\sigma^{-1}(3)}, z_{\sigma^{-1}(4)})$$

because any σ maps s_1 to s_1, it induces an isomorphism on $\mathbf{C}[z_0, z_1, z_2, z_3, z_4]/\langle s_1 \rangle$; hence the variety defined by $F(z_0, z_1, z_2, z_3, z_4) = 0$ is mapped to the variety defined by $F(z_{\sigma(0)}, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}) = 0$ both in $\mathbf{P}^3$ and $\mathbf{P}^4$. In particular, it maps the set B^k_{ij} to the set $B^{\sigma(k)}_{\sigma(i)\sigma(j)}$, and maps the ideal $\langle B^k_{ij} \rangle$ to the ideal $\langle B^{\sigma(k)}_{\sigma(i)\sigma(j)} \rangle$. It also maps a B-line to a B-line.

For the above representation of L in $G(2,5)$, we have

$$\sigma L = \begin{pmatrix} x_{\sigma^{-1}(0)} & x_{\sigma^{-1}(1)} & x_{\sigma^{-1}(2)} & x_{\sigma^{-1}(3)} & x_{\sigma^{-1}(4)} \\ y_{\sigma^{-1}(0)} & y_{\sigma^{-1}(1)} & y_{\sigma^{-1}(2)} & y_{\sigma^{-1}(3)} & y_{\sigma^{-1}(4)} \end{pmatrix}.$$

2.4. Base line pair, associated set of a base line pair, skew pair, normal pair, equivalence and normalization. Let a_i and b_i be indices satisfying the following,

$$\{a_0, a_1, a_2, a_3, a_4\} = \{b_0, b_1, b_2, b_3, b_4\} = \{0, 1, 2, 3, 4\}.$$

Then we call the unordered pair $(B^{a_2}_{a_0 a_1}, B^{b_2}_{b_0 b_1})$ a base line pair. We denote by $(B^{a_2}_{a_0 a_1}, B^{b_2}_{b_0 b_1})_t$ the set of B-lines intersecting both $\langle B^{a_2}_{a_0 a_1} \rangle$ and $\langle B^{b_2}_{b_0 b_1} \rangle$ in the surface X_t, which can be empty. The set $(B^{a_2}_{a_0 a_1}, B^{b_2}_{b_0 b_1})_t$ is called the associated set of the base line pair $(B^{a_2}_{a_0 a_1}, B^{b_2}_{b_0 b_1})$ over t.

By definition, a permutation σ which maps a base line pair $(B^{a_2}_{a_0 a_1}, B^{b_2}_{b_0 b_1})$ to the image pair $(B^{\sigma(a_2)}_{\sigma(a_0)\sigma(a_1)}, B^{\sigma(b_2)}_{\sigma(b_0)\sigma(b_1)})$ induces a bijection between two associated sets $(B^{a_2}_{a_0 a_1}, B^{b_2}_{b_0 b_1})_t$ and $(B^{\sigma(a_2)}_{\sigma(a_0)\sigma(a_1)}, B^{\sigma(b_2)}_{\sigma(b_0)\sigma(b_1)})_t$ for the same surface X_t.

Proposition 2.1. *Two base lines are coplanar if and only if they both lie on a coordinate plane H_i, or both lie on a plane defined by the equation: $z_i + z_j = 0$ for some $i \neq j \in \{0, 1, 2, 3, 4\}$.*

Proof. Without loss of generality, choose a base line $\langle B^4_{01} \rangle$ on H_4, which intersects a base line on another coordinate plane H_0 or H_1 or H_2 or H_3. If the intersection point P lies on H_3 or H_2, then $P = (1 : -1 : 0 : 0 : 0)$ lies only on $\langle B^3_{01} \rangle$ and $\langle B^2_{01} \rangle$, hence on the same plane $z_0 + z_1 = 0$. If P lies on H_1 or H_0 then $P = (0 : 0 : 1 : -1 : 0)$ lies only on $\langle B^1_{23} \rangle$ and $\langle B^0_{23} \rangle$. Because $\langle B^4_{01} \rangle = \langle B^4_{23} \rangle$, then $\langle B^4_{23} \rangle$, $\langle B^1_{23} \rangle$ and $\langle B^0_{23} \rangle$ are on the same plane $z_2 + z_3 = 0$. □

A skew pair is a base line pair whose base lines are not coplanar.

A normal pair is a base line pair $(B^{a_2}_{a_0 a_1}, B^{b_2}_{b_0 b_1})$ whose indices set $\{a_0, a_1, a_2, b_0, b_1, b_2\}$ is equal to $\{0, 1, 2, 3, 4\}$.

Two base line pairs are equivalent if they contain same base lines, i.e., $(B^{a_2}_{a_0 a_1}, B^{b_2}_{b_0 b_1})$ is equivalent to $(B^{a'_2}_{a'_0 a'_1}, B^{b'_2}_{b'_0 b'_1})$ if the two sets are equal: $\{\langle B^{a_2}_{a_0 a_1} \rangle, \langle B^{b_2}_{b_0 b_1} \rangle\} = \{\langle B^{a'_2}_{a'_0 a'_1} \rangle, \langle B^{b'_2}_{b'_0 b'_1} \rangle\}$.

We say a base line pair can be normalized if it is equivalent to a normal pair. Actually a skew pair can always be normalized (Theorem 4.10).

3. Singular surfaces in the Dwork pencil.

The i-th partial differential of F_t at point P is

$$\partial F_t(P)_i := \frac{\partial F_t}{\partial z_i} = 5z_i^4 - 5tz_0 z_1 \dots \widehat{z_i} \dots z_4 = \frac{5}{z_i}(z_i^5 - tz_0 z_1 z_2 z_3 z_4).$$

Let $\partial F_t(P)$ denote $(\partial F_t(P)_0 : \partial F_t(P)_1 : \partial F_t(P)_2 : \partial F_t(P)_3 : \partial F_t(P)_4)$.

A singular point $P = (x_0 : x_1 : x_2 : x_3 : x_4)$ of the surface X_t satisfies $\partial F_t(P) = (0 : 0 : 0 : 0 : 0)$ or the partials are proportional to $(1 : 1 : 1 : 1 : 1)$, i.e., $\partial F_t(P)_i = 0$ for $i = 0, 1, 2, 3, 4$ or $\partial F_t(P)_i = \partial F_t(P)_j$ for any $i \neq j \in \{0, 1, 2, 3, 4\}$.

The surface X_1 is defined by $s_2 s_3 = 0$, hence the singular locus is the double curve $s_2 = s_3 = 0$. The surface X_0 defined by $s_5 = 0$ is non-singular because the only prime ideal over $\langle z_0^4 - z_1^4, z_1^4 - z_2^4, z_2^4 - z_3^4, z_3^4 - z_4^4, s_1 \rangle$ is $\langle z_0, z_1, z_2, z_3, z_4 \rangle$.

3.1. Singular points on H_i. If P lies on a base line, because of the symmetries, we can assume $P = (1 : -1 : a : -a : 0)$, then we have the $\partial F_t(P)$ as a homogeneous coordinate with non-zero entries:

$$\partial F_t(P) = (1 : 1 : a^4 : a^4 : -ta^2);$$

hence, $a^4 = 1$ and $t = -a^2 = \pm 1$. When $t = 1$ we have P on the double curve $s_2 = s_3 = 0$ of X_1. When $t = -1$ we have a singularity P as $(1 : -1 : 1 : -1 : 0)$ up to permutation.

If P lies on a residue conic, for example C_4, assume $P = (1 : x : y : -(1 + x + y) : 0)$, satisfies $x^2 + y^2 + 1 + x + y + xy = 0$,

$$\partial F_t(P) = (1 : x^4 : y^4 : (1 + x + y)^4 : txy(1 + x + y))$$

if $x = 1$, then $(2+y)^4 = 1$, so we have $y = -1$, which causes P not to be on the residue conic C_4; if $x = \sqrt{-1}$ then $y = -\sqrt{-1}$ and $t = 1$, which causes a concurrent point of base line, residue conic and the double curve $s_2 = s_3 = 0$ of X_1.

3.2. Singular points in general. We assume $x_i \neq 0$ for $i = 0, 1, 2, 3, 4$. If $\partial F_t(P) = (0 : 0 : 0 : 0 : 0)$ we have $x_0^5 = x_1^5 = x_2^5 = x_3^5 = x_4^5$ which is not on $s_1 = 0$.

If $\partial F_t(P) = (1 : 1 : 1 : 1 : 1)$, any x_i has to be a root of the following quintic equation.

$$z^5 - z + c = 0, \qquad c = -tx_0x_1x_2x_3x_4$$

P **has five distinct coordinates**, the set $\{x_0, x_1, x_2, x_3, x_4\}$ consists of the five roots of the above equation; hence $c = -x_0x_1x_2x_3x_4$ and we get $t = 1$, the singularity lies on the double curve of X_1.

P **has four distinct coordinates**, we assume x_0, x_1, x_2, x_3 to be the distinct roots of the above equation; then the fifth root is tx_4, hence $x_0 + x_1 + x_2 + x_3 + tx_4 = 0$. But P is on $s_1 = 0$ then $t = 1$.

P **has three distinct coordinates**, if $P = (x : x : y : y : -2(x+y))$ we have

$$x^4 + 2txy^2(x + y) = y^4 + 2tx^2y(x + y) = 16(x + y)^4 - tx^2y^2.$$

This gives no solution for three distinct coordinates. If $P = (x : x : x : y : -(3x + y))$ we have

$$x^4 + tx^2y(3x + y) = y^4 + tx^3(3x + y) = (3x + y)^4 - tx^3y,$$

giving only one solution containing three distinct coordinates (up to permutation)

$$t = 3, \qquad P = (2 : 2 : 2 : -3 + \sqrt{-7} : -3 - \sqrt{-7})$$

P **has two distinct coordinates**, there are two possibilities up to permutation. If $P = (1 : 1 : 1 : 1 : -4)$ we have $t = 51$ and if $P = (2 : 2 : 2 : -3 : -3)$ we have $t = -13/12$.

3.3. Conclusions concerning singular points. The above results can be organized as this table (originally by Barth).

t	number of singularities	coordinates up to permutation	Type
1	∞	$s_2 = s_3 = 0$	non-isolated
3	20	$(2:2:2:-3+\sqrt{-7}:-3-\sqrt{-7})$	node
51	5	$(1:1:1:1:-4)$	node
$-13/12$	10	$(2:2:2:-3:-3)$	node
-1	15	$(1:1:-1:-1:0)$	cusp

The only new information are the types of the surface singularities, which can be determined in the standard way: Let $f(x, y, z) = F_t(1, x, y, z, -(1 + x + y + z)) = 0$ be the affine equation of the surface X_t. Then we can get the analytic germ at the singular point. For example, on X_{51}, we have $f(x + 1, y + 1, z + 1)$ be the analytic germ whose tangent cone is an irreducible quadratic form $-375\,z^2 + 250\,yz - 375\,y^2 + 250\,xz + 250\,xy - 375\,x^2$; hence the point $(1 : 1 : 1 : 1 : -4)$ is of A_1 type on X_{51}. So are the surfaces X_3 and $X_{-13/12}$. For X_{-1}, the tangent cone degenerates into $5\,(y - z + x)(-y + z + x)$. After the coordinate change $X = y - z + x, Y = -y + z + x, Z = z$, the tangent cone is XY and the cubic part is $10\,Z^3 + 5\,ZY^2 + 15\,Z^2X - (15/2)\,YZX + (25/4)\,XY^2 + (25/2)\,ZX^2 + (15/4)\,X^3$ intersecting X and Y transversally, so the singularities must be of A_2 type.

4. Additional lines on X_t. We now prove Theorem 1.1 and the first part of Theorem 1.2 here. In $\mathbf{P}^3$, a general line in the Grassmannian

$G(2,5)$ needs six indeterminate parameters, while a B-line needs only two parameters to determine itself. We first prove every additional line belongs to a base line pair in subsection 4.1. In subsection 4.2, we get the additional lines on X_0 (Proposition 4.6) and show that for $X_t (t \neq 0)$ every additional line belongs to a skew pair. In subsection 4.3, we discuss the group action. Although the transitivity of the additional lines is not easy to be achieved directly, we can prove the transitivity of the skew normal pairs (Proposition 4.9). We also show a skew pair is normalizable (Theorem 4.10) and hence prove Theorem 4.11. By this theorem, we need only to compute in the skew normal pair: $(B^4_{01}, B^3_{02})_t$. In subsection 4.4, the computation shows the additional lines in the pair can be transformed into each other by permutations and the matrix forms in Theorem 1.1 are determined. Moreover, it shows that only $t = 2$ and $t = 2\tau$ give non-empty $(B^4_{01}, B^3_{02})_t$, which becomes the first part of Theorem 1.2.

4.1. Every additional line is a B-line. Recall the definition in subsection 2.2: a B-line is an additional line which intersects at least two distinct base lines. Hence we only need to prove that an additional line must intersect at least two different base lines.

Lemma 4.1. *The ideal of the residue conic C_l on the coordinate plane H_l is generated by any one of the equations $z_i^2 + z_j^2 + z_k^2 + z_iz_j + z_iz_k + z_jz_k = 0$ in the coordinate ring $\mathbf{C}[z_0, z_1, z_2, z_3, z_4]/\langle s_1\rangle$, where i, j, k are distinct indices in $\{0, 1, 2, 3, 4\} - \{l\}$.*

Proof. Let z_i, z_j, z_k, z_l, z_m be the five coordinates; by $z_l = 0$ we have $z_k = -(z_i + z_j + z_m)$ and $z_m = -(z_i + z_j + z_k)$. Then

$$\begin{aligned} z_i^2 + z_j^2 + z_k^2 + z_iz_j + z_iz_k + z_jz_k \\ &= z_i^2 + z_j^2 + z_iz_j + z_k(z_k + z_i + z_j) \\ &= z_i^2 + z_j^2 + z_iz_j + (z_i + z_j + z_m)z_m \\ &= z_i^2 + z_j^2 + z_m^2 + z_iz_j + z_iz_m + z_jz_m; \end{aligned}$$

therefore, we can choose any three indices in $\{0, 1, 2, 3, 4\} - \{l\}$. □

Corollary 4.2. *If C_i and C_j are two distinct residue conics, then $C_iC_j = 2$.*

Proof. By Lemma 4.1, we have $C_i \cap C_j = \{(z_0, z_1, z_2, z_3, z_4) | s_1 = 0, z_i = z_j = 0, z_k^2 + z_l^2 + z_m^2 + z_k z_l + z_k z_m + z_l z_m = 0\}$ where $\{i, j, k, l, m\} = \{0, 1, 2, 3, 4\}$; hence, $C_i C_j = 2$. □

Proposition 4.3. *If the surface is non-singular, an additional line of the surface intersects at most three residue conics.*

Proof. Let L be an additional line on a non-singular surface X_t of the pencil, K the canonical divisor and C_0, C_1, C_2, C_3, C_4 the residue conics on X_t. Let $n = L(C_0 + C_1 + C_2 + C_3 + C_4)$ and $D = L + \sum_{i=0}^{4} C_i$. By the adjunction formula, we have $C_i^2 + C_i K = -2$, implying $C_i^2 = -4$. And $L^2 = -3$ by the same reason. Now

$$\begin{aligned} D^2 &= L^2 + \sum_{i=0}^{4} C_i^2 + 2L \sum_{i=0}^{4} C_i + 2 \sum_{0 \le i < j \le 4} C_i C_j \\ &= -3 - 20 + 2n + 40 = 17 + 2n. \end{aligned}$$

Let H be a hyperplane divisor; then

$$Z = \begin{pmatrix} H^2 & HD \\ HD & D^2 \end{pmatrix} = \begin{pmatrix} 5 & 11 \\ 11 & 2n + 17 \end{pmatrix}.$$

Suppose that $n \ge 4$; then $\det(Z) = 5(2n + 17) - 121 \ge 125 - 121 = 4$, a contradiction to the Hodge index theorem. □

We illustrated in Figure 1 the configurations of the additional lines, base lines and the residue conics on the five coordinate planes (of course the five planes intersect each other in the projective space, but we didn't draw that explicitly because our main concern is the intersection between the lines and conics). Then the following is the result of this subsection.

Theorem 4.4. *An additional line is a* B*-line, and the set E_t of additional lines on surface X_t is actually the set of* B*-lines on X_t.*

4.2. For $t \neq 0$, any B-line belongs to the associated set of a skew pair. First we assert that a B-line intersects a skew pair on

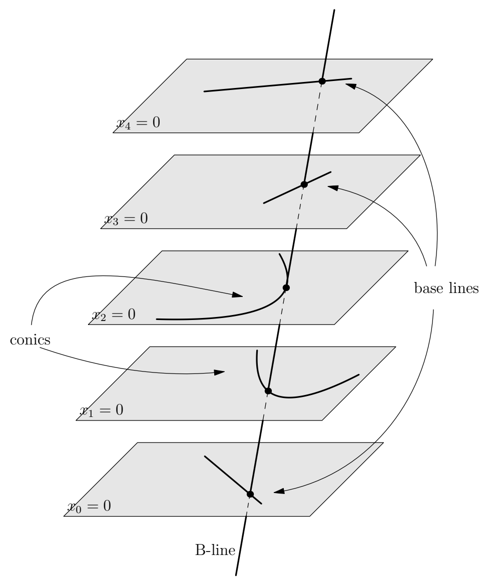

FIGURE 1. An additional line intersects five coordinate planes.

X_t for $t \neq 0$, then discuss the B-lines on X_0. The third lemma is technical. It describes the form of a skew pair, which is useful in the next subsection.

Lemma 4.5. *There is no* B*-line on any coordinate plane H_i. When $t \neq 0$, there is no* B*-line on the plane defined by equation $z_i + z_j = 0$.*

Proof. The complete intersection of H_i and X_t is a triangle and a residue conic, containing no B-lines. For the remainder, we consider the intersection of the plane $z_0 + z_1 = 0$ with X_t, substitute $z_1 = -z_0$ and $z_4 = -(z_2 + z_3)$; we have

$$F_t = -5z_2z_3(z_2 + z_3)(tz_0^2 + z_2^2 + z_2z_3 + z_3^2)$$

being a conic and three base lines B_{01}^2, B_{01}^3 and B_{01}^4 which meet at the point $(1 : -1 : 0 : 0 : 0)$. $\square$

By Proposition 2.1, if the associated set $(B^{a_2}_{a_0a_1}, B^{b_2}_{b_0b_1})_t$ for a base line pair $(B^{a_2}_{a_0a_1}, B^{b_2}_{b_0b_1})$ is non-empty on some surface X_t where $t \neq 0$, the two base lines are skew.

Proposition 4.6. *The complete intersection of the surface X_0 and the plane $z_i + z_j = 0$ consist of five lines, and any additional line is a permutation of* $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & \omega & 1 & \omega^2 \end{pmatrix}$.

Proof. From the proof of Lemma 4.5, there are two B-lines intersecting coplanar base lines on the plane $z_i + z_j = 0$. They can be written as the following up to permutation

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & \omega & 1 & \omega^2 \end{pmatrix},$$

where ω is a primitive cubic root of 1. □

Lemma 4.7. *For a skew pair $(B^{a_2}_{a_0a_1}, B^{b_2}_{b_0b_1})$, the indices satisfy* (i) *the two superscripts can't be equal,*

(ii) *the sets of subscripts can't be equal,*

(iii) *the set of subscripts of one entry can't be the others' three indices' complement with respect to $\{0 \dots 4\}$.*

Proof. For (i), if the two superscripts are the same, then the two base lines lie on the same coordinate plane $z_{a_2} = 0$. For (ii), if the sets of subscripts are equal: $\{a_0, a_1\} = \{b_0, b_1\}$; then the two base lines lie on the same plane $z_{a_0} + z_{a_1} = 0$. For (iii), if $\{a_0, a_1\}$ is the complement of $\{b_0, b_1, b_2\}$, then $\langle B^{b_2}_{b_0b_1}\rangle = \langle B^{b_2}_{a_0a_1}\rangle$, which is impossible by (ii). □

4.3. Normalization of a skew pair.

Proposition 4.8. *A normal pair which is skew must be $(B^{a_2}_{a_0a_1}, B^{a_4}_{a_0a_3})$, where*

$$\{a_0, a_1, a_2, a_3, a_4\} = \{0, 1, 2, 3, 4\}.$$

Proof. If the skew pair $(B^{a_2}_{a_0a_1}, B^{b_2}_{b_0b_1})$ is normal, we have

$$\{a_0, a_1, a_2\} \cup \{b_0, b_1, b_2\} = \{0, 1, 2, 3, 4\} = \{a_0, a_1, a_2, a_3, a_4\};$$

hence, there is only one duplicated index

$$\#(\{a_0, a_1, a_2\} \cap \{b_0, b_1, b_2\}) = 1.$$

We prove the duplicated index merely appears as the subscripts. By (i) in Lemma 4.7, The duplicated indices can't both appear as superscripts. If the duplicated indices consist of one superscript and one subscript, we assume $a_0 = b_2$, then by the definition of normal, $\{b_0, b_1\} = \{a_3, a_4\}$ a contradiction to (iii) of Lemma 4.7. □

For a normal pair $(B^{a_2}_{a_0a_1}, B^{a_4}_{a_0a_3})$, all of its equivalent pairs are the following

$$(B^{a_2}_{a_0a_1}, B^{a_4}_{a_1a_2}), \qquad (B^{a_2}_{a_3a_4}, B^{a_4}_{a_0a_3}), \qquad (B^{a_2}_{a_3a_4}, B^{a_4}_{a_1a_2}),$$

which are not normal; hence, it is the unique normal pair in its equivalent class of base line pairs.

Proposition 4.9. *Any two skew normal pairs can be mapped to each other by a permutation in* S_5; *hence,* S_5 *is transitive on all the skew normal pairs.*

Proof. By Proposition 4.8, we can assume the two normal pairs are $(B^{a_2}_{a_0a_1}, B^{a_4}_{a_0a_3})$ and $(B^{b_2}_{b_0b_1}, B^{b_4}_{b_0b_3})$ where the indices satisfy

$$\{a_0, a_1, a_2, a_3, a_4\} = \{b_0, b_1, b_2, b_3, b_4\} = \{0, 1, 2, 3, 4\};$$

then the permutation σ defined by $\sigma(a_i) = b_i$ maps $(B^{a_2}_{a_0a_1}, B^{a_4}_{a_0a_3})$ to $(B^{b_2}_{b_0b_1}, B^{b_4}_{b_0b_3})$. □

Theorem 4.10. *A skew pair can be normalized to a normal pair.*

Proof. Let a_i and b_i be the indices which satisfy the following condition

$$\{a_0, a_1, a_2, a_3, a_4\} = \{b_0, b_1, b_2, b_3, b_4\} = \{0, 1, 2, 3, 4\}.$$

If there is a skew pair $(B^{a_2}_{a_0 a_1}, B^{b_2}_{b_0 b_1})$ which is not normalizable, by definition of normalization, the indices satisfy the following

$$\{a_0, a_1, a_2\} \cup \{b_0, b_1, b_2\} \neq \{0, 1, 2, 3, 4\},$$

and because $\langle B^{a_2}_{a_0 a_1}\rangle = \langle B^{a_2}_{a_3 a_4}\rangle$, we have

$$\{a_3, a_4, a_2\} \cup \{b_0, b_1, b_2\} \neq \{0, 1, 2, 3, 4\}.$$

By (i) of Lemma 4.7, b_2 is not equal to a_2; without loss of generality, we can assume $b_2 = a_0$. Then

$$\{a_3, a_4, a_2\} \cup \{b_0, b_1, a_0\} = \{a_0, a_2, a_3, a_4, b_0, b_1\} \neq \{0, 1, 2, 3, 4\};$$

hence $a_1 \in \{b_3, b_4\}$, which induces $\{a_3, a_4, a_2\} \cup \{b_3, b_4, a_0\} = \{0, 1, 2, 3, 4\}$, a contradiction. □

Theorem 4.11. *The set E_t of additional lines on the surface X_t, where $t \neq 0$, satisfies the following*

$$E_t = \bigcup_{\sigma \in S_5} \sigma(B^4_{01}, B^3_{02})_t$$

for E_0,

$$E_0 = \bigcup_{\sigma \in S_5} \sigma(B^4_{01}, B^3_{02})_0 \bigcup E,$$

where E is the set of B-lines on plane $z_i + z_j = 0$ in Proposition 4.6. *The set E contains* 20 B*-lines.*

Proof. For E_t where $t \neq 0$, use transitivity of the skew normal pairs.

For E_0, by Proposition 4.6, there are 2 B-lines on the plane $z_i + z_j = 0$, and there are 10 such planes, hence 20 B-lines in E. □

4.4. Non-empty associated sets $(B^4_{01}, B^3_{02})_t$. Let L be a B-line in $(B^4_{01}, B^3_{02})_t$ as follows:

$$\begin{pmatrix} 1 & -1 & A & -A & 0 \\ 1 & B & -1 & 0 & -B \end{pmatrix}.$$

The point of L whose coordinates are $(1+z, -1+zB, A-z, -A, -zB)$ lies on X_t if z is any one root of the polynomial

$$\begin{aligned}\frac{1}{5}&F_t(1+z, -1+zB, A-z, -A, -zB)\\ &= \phi_t(B,A)z^4 + \psi_t(B,A)z^3 + \psi_t(A,B)z^2 + \phi_t(A,B)z\end{aligned}$$

where $\phi_t(x,y)$ and $\psi_t(x,y)$ are the following polynomials:

$$\phi_t(x,y) = 1-x^4+y+tx^2y, \qquad \psi_t(x,y) = 2(x^3-y^2+1)+txy(x-1-xy).$$

If $t=0$, the four polynomials $\phi_0(x,y)$, $\phi_0(y,x)$, $\psi_0(x,y)$ and $\psi_0(y,x)$ have no common root. Let t be non-zero. By eliminating t, we have two surfaces $\phi_t(x,y)=0$ and $\phi_t(y,x)=0$ intersecting at the space curve whose coordinates (t,x,y) satisfy the following equations.

$$t = \frac{x^4-y-1}{x^2y}, \qquad (x-y)(x^2y^2+xy(x^2+y^2)+x+y+1)=0,$$

and two surfaces $\psi_t(x,y)=0$ and $\psi_t(y,x)=0$ intersect at the space curve

$$t = \frac{2(y^3-x^2+1)}{xy(xy+1-y)}, \qquad (x-y)(x^2y^2+xy(x^2+y^2)+x+y+1)=0.$$

Let $x=y$; we have two solutions, one is $x=y=\tau$, $t=2\tau$ where τ is a root of $\tau^4+\tau+1=0$, while the other is $x=y=(1\pm\sqrt{5})/2, t=1$ which lies on the singular surface X_1.

Now, always assume τ to be a root of $\tau^4+\tau+1=0$. Use the following MAPLE instructions:

```
phi := (x, y)- > 1 - x^4 + y + t * x^2 * y;
psi := (x, y)- > 2 * (x^3 - y^2 + 1) + t * x * y * (x - 1 - x * y);
solve({phi(x, y), phi(y, x), psi(x, y), psi(y, x)});
```

It gives the following solutions (we omit the solution on X_1 and the solutions representing the base locus):

1. $x=\omega$, $y=\omega^2$ and $t=2$ where ω is a primitive cubic root of 1.
2. x is a root of $x^4-x^3+1=0$, $y=-x^3$; hence,

$$t = \frac{x^4-y-1}{x^2y} = \frac{x^4+x^3-1}{-x^5} = -\frac{2x^4}{x^5} = 2\Big(-\frac{1}{x}\Big).$$

Observe that $-1/\tau$ is a root of $x^4 - x^3 + 1 = 0$; we have $t = 2\tau$, $x = -1/\tau$ and $y = 1/\tau^3$.

3. x is a root of $x^4+(x+1)^3 = 0$, $y = -(x+1)^3+2x^2-1 = x^4+2x^2-1$; then

$$t = \frac{-2x^2}{x^2 y} = 2\Big(-\frac{1}{y}\Big).$$

Observe that $-\tau/(1+\tau)$ is a root of $x^4 + (x+1)^3 = 0$; we have

$$y = \frac{\tau^3 - \tau^2 - 3\tau - 2}{(\tau+1)^3} = -\frac{1}{\tau}\frac{-\tau^4 + \tau^3 + 3\tau^2 + 2\tau}{(\tau+1)^3} = -\frac{1}{\tau};$$

hence $t = 2\tau$, $x = -\tau/(1+\tau) = -\tau/(-\tau^4) = 1/\tau^3$ and $y = -1/\tau$.

There are 5 non-singular surfaces X_t with a non-empty associated set $(B^4_{01}, B^3_{02})_t$, for $t = 2$ and for $t = 2\tau$. This proves the first part of Theorem 1.2.

For X_2, the two lines in $(B^4_{01}, B^3_{02})_2$ satisfy the following

$$\begin{pmatrix} 1 & -1 & \omega & -\omega & 0 \\ 1 & \omega^2 & -1 & 0 & -\omega^2 \end{pmatrix} = (1,2)(3,4)\begin{pmatrix} 1 & -1 & \omega^2 & -\omega^2 & 0 \\ 1 & \omega & -1 & 0 & -\omega \end{pmatrix}.$$

For $X_{2\tau}$, the three lines in $(B^4_{01}, B^3_{02})_{2\tau}$ are the following:

$$\begin{aligned} L_0 &= \begin{pmatrix} 1 & -1 & \tau & -\tau & 0 \\ 1 & \tau & -1 & 0 & -\tau \end{pmatrix} \\ L_1 &= \begin{pmatrix} 1 & -1 & -1/\tau & 1/\tau & 0 \\ 1 & 1/\tau^3 & -1 & 0 & -1/\tau^3 \end{pmatrix} \\ L_2 &= \begin{pmatrix} 1 & -1 & 1/\tau^3 & -1/\tau^3 & 0 \\ 1 & -1/\tau & -1 & 0 & 1/\tau \end{pmatrix}. \end{aligned}$$

We check $(1,0,3,4)L_0$ and $(1,0,4,2)L_0$ to see if they are equal to L_1 and L_2; first observe

$$\begin{aligned} (1,0,3,4)\cdot L_0 &= \begin{pmatrix} -1 & 0 & \tau & 1 & -\tau \\ \tau & -\tau & -1 & 1 & 0 \end{pmatrix} \\ (1,0,4,2)\cdot L_0 &= \begin{pmatrix} -1 & \tau & 0 & -\tau & 1 \\ \tau & -1 & -\tau & 0 & 1 \end{pmatrix}; \end{aligned}$$

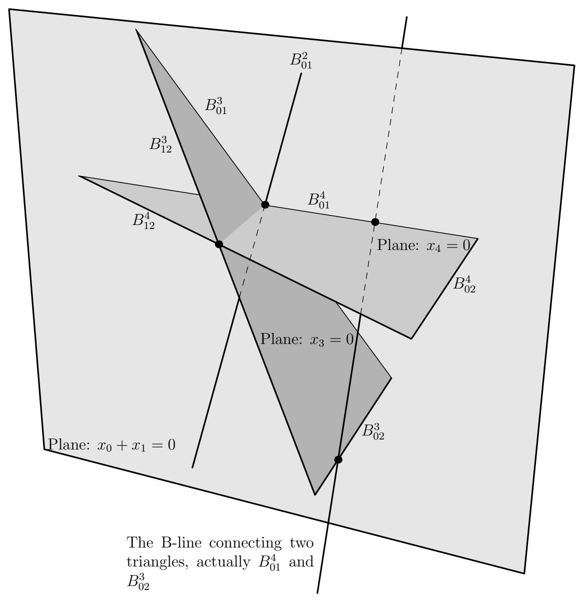

FIGURE 2. B-line connects two base line triangles.

then it is easy to verify

$$(1,0,3,4) \cdot L_0 = \begin{pmatrix} \tau & \tau^4 \\ \tau & 0 \end{pmatrix} L_1, \qquad (1,0,4,2) \cdot L_0 = \begin{pmatrix} \tau^4 & \tau \\ 0 & \tau \end{pmatrix} L_2,$$

which means L_0, L_1 and L_2 differ by a permutation in S_5. For the image of L_0, see Figure 2. From Theorem 4.11 and the above statements, Theorem 1.1 is proved.

5. Number of B-lines and configuration. From now on, let us always assume τ to be a root of $\tau^4 + \tau + 1 = 0$. We will complete the proof of Theorem 1.2 in this section. By Section 4, the only non-singular surfaces containing B-lines are X_0, X_2 and $X_{2\tau}$. For the surface X_0, its associated set of a skew pair is always empty; hence, it contains only B-lines on the plane $z_i + z_j = 0$, by Theorem 4.11. Then we know $\#E_0 = 20$. Next we will count the number of lines on X_2 and $X_{2\tau}$. The trick is to use an injective mapping π from a B-line to a set of fundamental planes to determine the stabilizer under the S_5 action.

To prove there is no isomorphism between any of our surfaces and the quintic Fermat surface, we only need to show the line configurations on our surfaces are different from that on the quintic Fermat surface, since the isomorphism between two quintic surfaces in $\mathbf{P}^3$ must be a linear one which preserves the line configurations.

5.1. Counting the number of lines. We construct the mapping π in subsection 5.1.1 and prove it commutes with a permutation (Theorem 5.6). Then the stabilizers of the set of additional lines are determined respectively in subsections 5.1.2 and 5.1.3. Consequently, the numbers of lines are determined (Proposition 5.10 and Proposition 5.13).

5.1.1. Fundamental planes and the injective mapping. For the surface X_t where $t = 2$ or $t = 2\tau$, we call a plane D a fundamental plane for X_t if D contains at least one base line and one B-line of X_t. Denote by G_t the set of fundamental planes for surface X_t.

Denote by M_{ij}^k the pencil of planes containing the base line $\langle B_{ij}^k \rangle$, denote by $M_{ij}^k(\eta)$ the polynomial $z_i+z_j+\eta z_k$; then $\langle M_{ij}^k(\eta)\rangle$ is a plane in the pencil. For $\{i', j'\} = \{0, 1, 2, 3, 4\}-\{i, j, k\}$, we have $\langle B_{ij}^k\rangle = \langle B_{i'j'}^k\rangle$; hence, M_{ij}^k and $M_{i'j'}^k$ are the same pencil. By $z_{i'} + z_{j'} = -(z_i + z_j + z_k)$, we have $\langle M_{ij}^k(\eta)\rangle = \langle M_{i'j'}^k(1 - \eta)\rangle$ being the same ideal.

Proposition 5.1. *Let D be a fundamental plane in the set G_t; then D belongs to a unique pencil M_{ij}^k.*

Proof. If D belongs to two different pencils M_{ij}^k and $M_{i'j'}^{k'}$, then D contains a B-line L, which intersects both $\langle B_{ij}^k\rangle$ and $\langle B_{i'j'}^{k'}\rangle$. By Lemma 4.5, $(B_{ij}^k, B_{i'j'}^{k'})$ is skew, contradicting the fact that they are on the same plane D. □

By this lemma, we know that a fundamental plane of G_t is equal to $\langle M_{ij}^k(\eta)\rangle$ for a suitable η.

Proposition 5.2. *Let L be a* B*-line. We define a map π which maps a* B*-line to a subset of G_t as the following*:

$$\pi : L \longmapsto \{D \mid D \text{ is spanned by a base line and } L\}.$$

Then π is injective and $G_t = \cup_{L\in E_t}\pi(L)$ where E_t is the B-line set of X_t.

Proof. $L = \cap\pi(L)$ is unique if $\pi(L)$ contains more than two planes. What remains to be proven is that $\pi(L)$ contains at least two fundamental planes. By definition, the B-line L intersects at least two base lines which are not coplanar. These two planes spanned by L and the two base lines are both contained in $\pi(L)$. □

We have shown in Section 2 that, for σ a permutation of S_5, the induced morphism maps a hypersurface defined by the equation $F(z_0, z_1, z_2, z_3, z_4) = 0$ to the hypersurface defined by $F(z_{\sigma(0)}, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}) = 0$. In particular, it maps the set B^k_{ij} to $B^{\sigma(k)}_{\sigma(i)\sigma(j)}$, and the polynomial $M^k_{ij}(\eta)$ to the polynomial $M^{\sigma(k)}_{\sigma(i)\sigma(j)}(\eta)$. Furthermore, it is also an isomorphism on $\mathbf{C}[z_0, z_1, z_2, z_3, z_4]/\langle s_1\rangle$; hence it maps the base line $\langle B^k_{ij}\rangle$ to the base $\langle B^{\sigma(k)}_{\sigma(i)\sigma(j)}\rangle$, the pencil M^k_{ij} to the pencil $M^{\sigma(k)}_{\sigma(i)\sigma(j)}$, and the plane $\langle M^k_{ij}(\eta)\rangle$ to the plane $\langle M^{\sigma(k)}_{\sigma(i)\sigma(j)}(\eta)\rangle$.

Lemma 5.3. *A permutation of S_5 maps a fundamental plane to a fundamental plane, so the symmetric group S_5 acts on the set G_t.*

Proof. A permutation $\sigma \in S_5$ maps a base line to a base line and a B-line to a B-line; hence it maps a fundamental plane in G_t to another fundamental plane in G_t. So the operation on G_t is closed. □

Lemma 5.4. *π commutes with σ, i.e., $\pi(\sigma(L)) = \sigma(\pi(L))$.*

Proof. Because σ preserves the intersection of two lines, each side is equal to the set of the fundamental planes spanned by $\sigma(L)$ and the base lines which intersect $\sigma(L)$. □

Corollary 5.5. *If S_5 is transitive on E_t, each* B-*line intersects exactly the same number of base lines.*

Proof. For two B-lines L and L_1, choose $\sigma \in S_5$ satisfying $L_1 = \sigma(L)$. Then we have $\#\pi(L_1) = \#\pi(\sigma(L)) = \#\sigma(\pi(L)) = \#\pi(L)$. □

The following is the key to finding the stabilizer of a B-line.

Theorem 5.6. *Let σ be a permutation in S_5. Then $\sigma(L) = L$ if and only if $\sigma(\pi(L)) = \pi(L)$.*

Proof. By Lemma 5.4, we have $\sigma(\pi(L)) = \pi(\sigma(L)) = \pi(L)$, if $\sigma(L) = L$. Conversely, since $\sigma(\pi(L)) = \pi(\sigma(L))$, we have $\sigma(L) = L$ by the injectivity of π. □

5.1.2. Stab$E_{2\tau}$ and $\#E_{2\tau}$. We want to use the injectivity of the mapping π. First we give the representation of the image of $\pi(L)$ for a general B-line.

Proposition 5.7. *For the* B-*line* $L = \begin{pmatrix} 1 & -1 & \tau & -\tau & 0 \\ 1 & \tau & -1 & 0 & -\tau \end{pmatrix}$ *on* $X_{2\tau}$, *we have* $\pi(L) = \{\langle M_{01}^4(1 + (1/\tau))\rangle, \langle M_{02}^3(1 + (1/\tau))\rangle, \langle M_{34}^0(\tau)\rangle\}$.

Proof. We can directly verify $\{\langle M_{01}^4(1 + (1/\tau))\rangle, \langle M_{02}^3(1 + (1/\tau))\rangle, \langle M_{34}^0(\tau)\rangle\} \subset \pi(L)$. For the other inclusion, we only need to prove L doesn't intersect other base lines. The intersection point of L and the coordinate plane H_1 is $(\tau + 1 : 0 : \tau^2 - 1 : -\tau^2 : -\tau)$, which is on the residue conic C_1, not on a base line. Also we have $L \cap H_2 = (\tau + 1 : \tau^2 - 1 : 0 : -\tau : -\tau^2) \in C_2$ not on a base line. □

By Corollary 5.5, each B-line on $X_{2\tau}$ intersects three base lines.

Lemma 5.8. *$G_{2\tau}$ can be decomposed into two subsets U, V with $U \cap V = \varnothing$, where*

$$U = \left\{ \left\langle M_{ij}^k \left(1 + \frac{1}{\tau}\right) \right\rangle \mid i, j, k \text{ are distinct indices} \right\}$$
$$V = \left\{ \langle M_{ij}^k(\tau) \rangle \mid i, j, k \text{ are distinct indices} \right\}.$$

Moreover, U and V are both closed and transitive under the operation of S_5.

Proof. From Proposition 5.7, a fundamental plane in $G_{2\tau}$ is $\langle M_{ij}^k(1 + (1/\tau))\rangle$ or $\langle M_{ij}^k(\tau)\rangle$; hence, $G_{2\tau} \subset \cup \pi L \subset U \cup V$. By definition, U and V are transitive and closed under the action of S_5. It remains to prove $U \cap V = \varnothing$. If not, by transitivity, there is a fundamental plane $\langle M^{k_0}_{i_0 j_0}(\tau)\rangle \in V$ equal to $\langle M^4_{01}(1+(1/\tau))\rangle \in U$. The intersection locus of the two fundamental planes is the nullspace of the following matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1+(1/\tau) \\ a_0 & a_1 & a_2 & a_3 & a_4 \end{pmatrix} \quad \text{where} \quad a_i = \begin{cases} 1 & \text{if } i = i_0,\ i = j_0 \\ \tau & \text{if } i = k_0 \\ 0 & \text{if } i \notin \{i_0, j_0, k_0\}, \end{cases}$$

the two fundamental planes coincide if the above matrix drops its rank, which is impossible. So the two planes can't coincide, hence $U \cap V = \varnothing$. □

Theorem 5.9. **Stab**$E_{2\tau}$ *is generated by* $(1,2)(3,4)$.

Proof. By transitivity of the set of $E_{2\tau}$, the stabilizer of $E_{2\tau}$ is the stabilizer of L where L is the B-line in Proposition 5.7. By Theorem 5.6, we look for the permutation σ which maps $\pi(L)$ to $\pi(L)$ itself.

$$\pi(L) = \left\{ \left\langle M^4_{01}\left(1+\frac{1}{\tau}\right)\right\rangle, \left\langle M^3_{02}\left(1+\frac{1}{\tau}\right)\right\rangle, \left\langle M^0_{34}(\tau)\right\rangle \right\}.$$

By Lemma 5.8, we have $\sigma(\langle M^0_{34}(\tau)\rangle) = \langle M^0_{34}(\tau)\rangle$; hence, $\sigma(0) = 0$. Now we have $M^0_{\sigma(3)\sigma(4)}(\tau) = M^0_{34}(\tau)$. If $\sigma(3) = 3$ and $\sigma(4) = 4$, then $\sigma(\langle M^4_{01}(1+(1/\tau))\rangle) = \langle M^4_{0\sigma(1)}(1+(1/\tau))\rangle$; hence $\sigma(1) = 1$ and σ is the identity. If σ interchanges 3 and 4, we have $\sigma(\langle M^4_{01}(1+(1/\tau))\rangle) = \langle M^3_{0\sigma(1)}(1+(1/\tau))\rangle \in \pi(L)$; hence, $\sigma(1) = 2$ and then $\sigma = (1,2)(3,4)$. So the stabilizer of L is generated by $(1,2)(3,4)$. □

Proposition 5.10. $\#E_{2\tau} = 60$, *and the number of all lines on the surface* $X_{2\tau}$ *is* 75.

5.1.3. StabE_2 and $\#E_2$. Choose the B-line $L = \begin{pmatrix} 1 & -1 & \omega & -\omega & 0 \\ 1 & \omega^2 & -1 & 0 & -\omega^2 \end{pmatrix}$ on X_2, where ω is a primitive cubic root of 1. Similar to the case of $X_{2\tau}$, we first compute the image $\pi(L)$ as follows.

Proposition 5.11. *$\pi(L)=\{D_1,D_2,D_3\}$, where D_i is the following:*

$$D_1=\langle M_{01}^4(1+\omega)\rangle=\langle M_{23}^4(-\omega)\rangle$$
$$D_2=\langle M_{02}^3(-\omega)\rangle=\langle M_{14}^3(1+\omega)\rangle$$
$$D_3=\langle M_{13}^0(1+\omega)\rangle=\langle M_{24}^0(-\omega)\rangle.$$

Proof. A computation shows L, C_1 and C_2 are concurrent at the point $(\omega:0:0:1:\omega^2)$ which is not on a base line. □

Theorem 5.12. *The stabilizer of L is generated by $(0,3,4)$.*

Proof. Let σ be a permutation such that $\sigma(L)=L$.

Case I. $\sigma(D_3)=D_3$; then $\sigma(0)=0$.

1. If $\sigma(1)=1$ and $\sigma(3)=3$, then $\sigma(D_2)=D_2$; hence $\sigma(2)=2$. So σ is the identity.

2. If σ interchanges 1 and 3, then $\sigma(D_2)=\langle M_{0\sigma(2)}^1(-\omega)\rangle\notin\pi(L)$, a contradiction.

Case II. $\sigma(D_3)=D_2$; then $\sigma(0)=3$.

1. If $\sigma(1)=1$ and $\sigma(3)=4$, then $\sigma(D_2)=D_1$; hence $\sigma(2)=2$. So σ is $(0,3,4)$.

2. If $\sigma(1)=4$ and $\sigma(3)=1$, then $\sigma(D_2)=\langle M_{3\sigma(2)}^1(-\omega)\rangle\notin\pi(L)$, a contradiction.

Case III. $\sigma(D_3)=D_1$; then $\sigma(0)=4$.

1. If $\sigma(1)=1$ and $\sigma(3)=0$, then $\sigma(D_2)=D_3$; hence $\sigma(2)=2$. So σ is $(0,4,3)$.

2. If $\sigma(1)=0$ and $\sigma(3)=1$, then $\sigma(D_2)=\langle M_{4\sigma(2)}^1(-\omega)\rangle\notin\pi(L)$, a contradiction.

Thus the stabilizer of the B-line L is generated by $(0,3,4)$. □

Proposition 5.13. *$\#E_2=40$ and there are 55 lines on X_2.*

5.2. Comparison with the Fermat quintic surface. We recall some algebraic geometry in subsection 5.2.1, which assert the

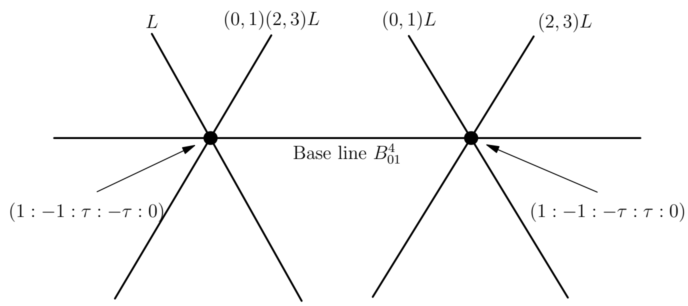

FIGURE 3. Fundamental plane $\langle M^4_{01}(1+(1/\tau))\rangle$, B-line $L = \langle M^4_{01}(1+(1/\tau)) \cup M^3_{02}(1+(1/\tau))\rangle$.

isomorphism between two quintic surfaces in $\mathbf{P}^3$ is linear. In subsection 5.2.2, we find a plane containing five lines on $X_{2\tau}$ which are not concurrent, while in subsection 5.2.3, we show that the coplanar lines on the quintic Fermat surface are always concurrent, hence the quintic Fermat surfaces are not isomorphic to any of our surfaces.

5.2.1. Invariance of line configuration. First note the following important property of the isomorphism between two quintic surfaces in $\mathbf{P}^3$.

Proposition 5.14. *An isomorphism of two quintic surfaces in* $\mathbf{P}^3$ *can be extended to a linear isomorphism of* $\mathbf{P}^3$.

Proof. Let $f : X \to Y$ be the isomorphism of two quintic surfaces X and Y. Then f induces a homomorphism on the linear system of canonical divisors $K_X \to K_Y$. Now $K_X = K_Y = O(1)$, hence f is the linear isomorphism. □

Now if the quintic surface X is isomorphic to Y, then the configuration of the lines on X is the same as the configuration of the lines on Y because the isomorphism is linear by Proposition 5.14. Hence X and Y must have the same configuration of the lines. Obviously, the two surfaces X_0 and X_2 are different from the Fermat quintic surface because they contain less than 75 lines.

5.2.2. Split fundamental plane of $X_{2\tau}$. We assert that at least one plane intersects $X_{2\tau}$ in five lines, which are not concurrent. Actually we prove the following stronger proposition.

Proposition 5.15. *Let U be the component of $G_{2\tau}$ defined in Lemma* 5.8. *A fundamental plane in U intersects $X_{2\tau}$ in five lines, which are not concurrent.*

Proof. By Lemma 5.8, U is transitive, we can choose an arbitrary element $\langle M_{01}^4(1+(1/\tau))\rangle$. By Bézout's theorem, it intersects the surface at most five times. Let L be the line in Proposition 5.7. The plane already contains B_{01}^4 and L, and direct computation shows $(0,1)(2,3)L$, $(0,1)L$ and $(2,3)L$ are also on the same plane, and there are only two triple points as follows; $(1:-1:\tau:-\tau:0) = B_{01}^4 \cap L \cap (0,1)(2,3)L$, and $(1:-1:-\tau:\tau:0) = B_{01}^4 \cap (0,1)L \cap (2,3)L$. See Figure 3. □

5.2.3. Split plane on the Fermat quintic surface. In this subsection, let $\mathbf{P}^3$ be the ordinary complex projective space with coordinate ring $\mathbf{C}[z_0, z_1, z_2, z_3]$.

For the Fermat quintic surface, we mean the hypersurface in $\mathbf{P}^3$ defined by $z_0^5 + z_1^5 + z_2^5 + z_3^5 = 0$ (or the surface isomorphic to it). Denote by $L_{ij} := z_i = z_j = 0$ the line in $\mathbf{P}^3$. These lines are not on the Fermat surface, but they set up the skeleton for all the 75 lines on the Fermat surface. On the coordinate plane $z_0 = 0$, there are three L-types lines L_{01}, L_{02}, L_{03} (see Figure 4), which intersect at the following points; $L_{01} \cap L_{02} = (0:0:0:1)$, $L_{01} \cap L_{03} = (0:0:1:0)$ and $L_{02} \cap L_{03} = (0:1:0:0)$. Observe the line L_{03} is the intersection of the planes $z_0 = 0$ and $z_3 = 0$ and for other two L-lines on $z_3 = 0$, we have: $L_{13} \cap L_{03} = (0:0:1:0)$ and $L_{23} \cap L_{03} = (0:1:0:0)$. Moreover, by $L_{12} \cap (z_3 = 0) = (1:0:0:0)$, $L_{12} \cap (z_0 = 0) = (0:0:0:1)$ and $L_{13} \cap L_{23} = (1:0:0:0)$, we get a tetrahedron (see Figure 5). Each line L_{ij} intersects the Fermat surface on 5 points P_{ij}^k for $k = 0, 1, 2, 3, 4$. The line L_{ij} can be organized as 3 skew pairs (whose unions of indices are $\{0, 1, 2, 3\}$):

$$(L_{01}, L_{23}), \qquad (L_{03}, L_{12}), \qquad (L_{02}, L_{13}).$$

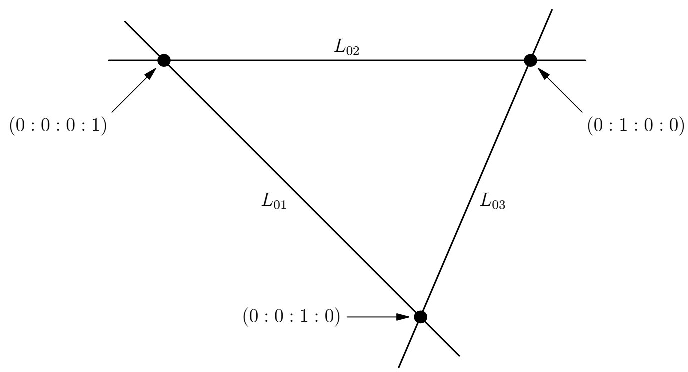

FIGURE 4. The triangle of L-type lines.

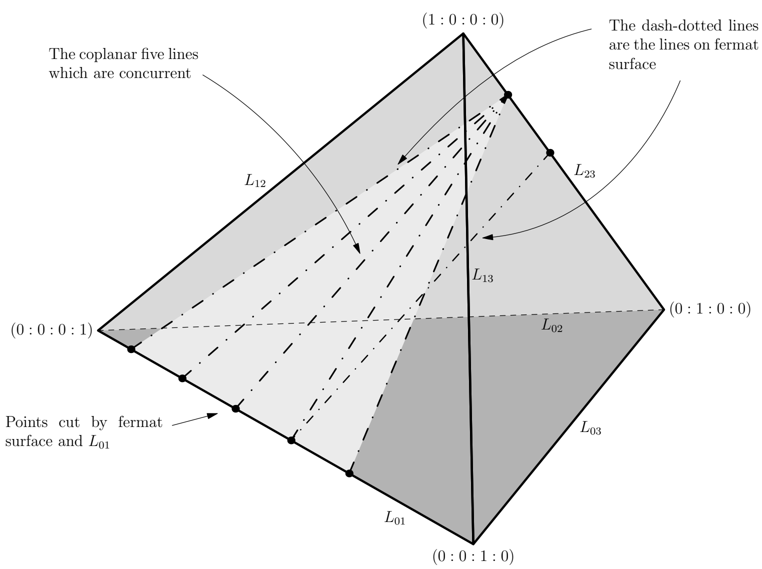

FIGURE 5. Skeleton tetrahedron of the lines on the Fermat surface.

A line is on the Fermat quintic surface if it connects two points P_{ij}^{k} and $P_{i'j'}^{k'}$, which lies on a skew pair of lines $(L_{ij}, L_{i'j'})$ $\{i,j,i',j'\} = \{0,1,2,3\}$.

These lines are $5 \times 5 \times 3 = 75$ in total.

If two lines L_1 and L_2 on the Fermat quintic surface are coplanar at a plane D, we have the following cases:

1. L_1 and L_2 intersect the same skew pairs; hence, L_1 and L_2 intersects at a point P_{ij}^k. Then D is the plane passing $L_{i'j'}$ and P_{ij}^k, which contains five lines through P_{ij}^k.

2. They intersect different skew pairs. But there are only 3 skew pairs; hence, the plane D has at most 3 lines which intersect different skew pairs. If D contains five lines, there are 2 lines intersect the same skew pair and fall back to Case 1.

From the discussion, five coplanar lines on the Fermat quintic surface are always concurrent at one point. But for $X_{2\tau}$, the fundamental planes in U contain five lines which are not concurrent; hence, $X_{2\tau}$ is not isomorphic to the Fermat quintic surface. Combining this with the results of Proposition 5.10 and Proposition 5.13 we have proved Theorem 1.2.

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REFERENCES

1. S. Boissière and A. Sarti, *Counting lines on surfaces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **6**(2007), 39–52.

2. L. Caporaso, J. Harris and B. Mazur, *How many rational points can a curve have?*, *The moduli space of curves* (Texel Island, 1994), Progr. Math. **129**, Birkhäuser Boston, MA, 1995, 13–31.

3. A. Sarti, *Pencils of symmetric surfaces in* $\mathbf{P}^3$, J. Algebra **246** (2001), 429–452.

4. B. Segre, *The maximum number of lines lying on a quartic surface*, Quart. J. Math. **14** (1943), 86–96.

5. ———, *On arithmetical properties of quartic surfaces*, Proc. London Math. Soc. **49** (1947), 353–395.

6. D.V. Straten, *A quintic hypersurface in* $\mathbf{P}^4$ *with* 130 *nodes*, Topology **32** (1993), 857–864.

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