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CROSSED PRODUCTS OF NONCOMMUTATIVE CW-COMPLEXES BY FINITE GROUPS

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ABSTRACT. In this paper we will construct a new class of examples of the so-called noncommutative CW-complexes ($NCCW$-complexes). We show that if G is a finite group acting on a $NCCW$-complex $\mathbf{A}_n$ by a natural class of automorphisms, then the crossed product $\mathbf{A}_n \rtimes G$ is an $NCCW$-complex. As a result, we find that whenever G is a finite group of diffeomorphisms acting on a smooth manifold M, then the resulting crossed product $C(M) \rtimes G$ has the structure of an $NCCW$-complex. Partial results are given in the case of twisted crossed products.

1. Introduction. The goal of this paper is to give a systematic study of crossed products of the form $\mathbf{A} \rtimes G$, where G is a finite group and $\mathbf{A}$ can be decomposed in a way analogous to the cellular decomposition of a topological CW-complex as first defined in [**1**]. The results given provide the general framework for the computations of certain K-theories carried out in [**7**].

Motivations for this study come variously from [**9, 11, 12**]. First, crossed products resulting from the action of finite groups on simplicial complexes were studied at some length by Yang in [**12**]. The primary goal of this author was to compute the K-theory of group C^*-algebras of planar crystallographic groups. These algebras were realized as subhomogeneous algebras over simplicial complexes where dimension drops occur only on lower dimensional skeleta. While this realization did allow for computation of $K_*(C^*(G))$ for all 17 planar crystallographic groups, nontrivial analysis was required in the computations.

The motivation for studying these crossed products from the point of view of noncommutative CW-complexes can be summarized by the characterization of noncommutative CW-complexes as "algebras of matrix-valued functions over topological spaces homeomorphic to

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CW-complexes..." [**9**]. The pullback constructions of [**9**] allow us to decompose the crossed products of [**12**] in a way that more fully utilizes the simplicial decomposition of the underlying space. Then it is possible to use the algebraic-topological approach to computing the K-theory of a pullback given in [**11**], thus giving a very algorithmic way of computing K_* groups.

2. Preliminaries. In this section we will recall some definitions from [**1, 2, 9**], as well as some important facts. We will begin by setting some notation, borrowed directly from Pedersen's paper [**9**]. If $\mathbf{A}$ is a C^*-algebra, then

$$\mathbf{I}^n\mathbf{A} = C([0,1]^n, \mathbf{A}), \quad \mathbf{I}_0^n\mathbf{A} = C_0((0,1)^n, \mathbf{A}), \quad \mathbf{S}^n\mathbf{A} = C(S^n, \mathbf{A})$$

where we identify the n-sphere S^n with the boundary of $[0,1]^{n+1}$.

Definition 2.1. A *zero dimensional noncommutative CW-complex* ($NCCW$-complex) is any finite dimensional C^*-algebra $\mathbf{A}_0$. In general we recursively define an *n-dimensional $NCCW$-complex* to be any C^*-algebra $\mathbf{A}_n$ appearing in a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{I}_0^n\mathbf{F}_n \arrow[r] \arrow[d, equal] & \mathbf{A}_n \arrow[r, "\pi_n"] \arrow[d, "f_n"] & \mathbf{A}_{n-1} \arrow[r] \arrow[d, "\varphi_n"] & 0 \\
0 \arrow[r] & \mathbf{I}_0^n\mathbf{F}_n \arrow[r] & \mathbf{I}^n\mathbf{F}_n \arrow[r, "\delta"] & \mathbf{S}^{n-1}\mathbf{F}_n \arrow[r] & 0
\end{tikzcd}$$

where the rows are extensions and the righthand square is a pullback. $\mathbf{A}_{n-1}$ denotes an $(n-1)$-dimensional $NCCW$-complex, $\mathbf{F}_n$ is some finite dimensional C^*-algebra, δ is the boundary restriction map, and φ_n is an arbitrary morphism.

The map φ_n is will henceforth be referred to as the *connecting morphism*. The maps f_n and π_n are the projections onto the first and second coordinates, respectively, in the realization of $\mathbf{A}_n$ as a restricted direct sum

$$\mathbf{A}_n = \mathbf{I}^n\mathbf{F}_n \bigoplus_{\mathbf{S}^{n-1}\mathbf{F}_n} \mathbf{A}_{n-1}.$$

Any $NCCW$-complex $\mathbf{A}_n$ of dimension $n \geq 1$ will be assumed to have lower dimensional complexes $\mathbf{A}_k$ with $k < n$ such that $\mathbf{A}_{k-1}$ is the image under the projection π_k appearing in the diagram making $\mathbf{A}_k$ a k-dimensional $NCCW$-complex.

Example 2.2. Recall from [**3**] that a space X is a finite, n-dimensional CW-complex if there is a filtration $X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n = X$ such that X^0 is a finite discrete space and, for $k = 1, \ldots, n$, X^k arises in the pushout diagram

$$\begin{tikzcd} X^k & X^{k-1} \arrow[l] \\ \coprod_{\lambda_k} \mathbf{I}^k \arrow[u] & \coprod_{\lambda_k} S^{k-1} \arrow[l, "\iota"'] \arrow[u, "\gamma_k"'] \end{tikzcd}$$

Here λ_k denotes some finite index set, $\mathbf{I}^k = [0,1]^k$, the horizontal maps are the obvious inclusions, and γ_k is an arbitrary continuous map. By dualizing this diagram we obtain a pullback

$$\begin{tikzcd} C(X^k) \arrow[r, "\pi_k"] \arrow[d] & C(X^{k-1}) \arrow[d, "\gamma_k^* = \varphi_k"] \\ \mathbf{I}^k\mathbf{C}^{\lambda_k} \arrow[r, "\iota^* = \delta"] & \mathbf{S}^k\mathbf{C}^{\lambda_k} \end{tikzcd}$$

This makes $C(X)$ into a $NCCW$-complex that happens to be commutative.

The fact that, in the definition of an $NCCW$-complex, there is no restriction made on the connecting morphism is worthy of some remarks. In the previous example, it is possible that the k-skeleton is equal to the $(k-1)$-skeleton. This occurs when the index set λ_k is empty. In the general situation this corresponds to the finite dimensional C^*-algebra $\mathbf{F}_k$ appearing in the diagram with $\mathbf{I}^k\mathbf{F}_k$ being equal to zero. In this case, we could also write $\mathbf{A}_k$ as a $(k-1)$-dimensional $NCCW$-complex. Occasionally, we will need to assume this does not happen, so we will make the following definition:

Definition 2.3. Suppose $n \geq 1$ and $\mathbf{A}_n$ is an n-dimensional $NCCW$-complex with lower dimensional complexes $\mathbf{A}_0, \cdots \mathbf{A}_{n-1}$. $\mathbf{A}_n$ is called *strongly n-dimensional* if all the connecting morphisms $\varphi_k : \mathbf{A}_{k-1} \to \mathbf{S}^{k-1}\mathbf{F}_k$ are nonzero. A CW-complex X is strongly n-dimensional if $C(X)$ is a strongly n-dimensional $NCCW$-complex when it is decomposed as in the previous example.

If $\mathbf{A}_1$ is a strongly one-dimensional $NCCW$-complex, then the finite dimensional algebras $\mathbf{A}_0$ and $\mathbf{F}_1$ must be unital. Then $\mathbf{A}_1$ is unital if and only if φ_1 is a unital morphism. In general, if $\mathbf{A}_n$ is a strongly n-dimensional $NCCW$-complex, $\mathbf{A}_n$ is unital if and only if $\mathbf{A}_{n-1}$ is unital and the connecting morphism φ_n is unital. This is due to the fact that the only unit in $\mathbf{I}^n\mathbf{F}_n$ for the ideal $\mathbf{I}_0^n\mathbf{F}_n \subset \mathbf{I}^n\mathbf{F}_n$ is the unit $1 \in \mathbf{I}^n\mathbf{F}_n$. It follows that all the lower dimensional complexes $\mathbf{A}_k$ and their corresponding connecting morphisms φ_k are unital. In practice, all our $NCCW$-complexes will be unital, so we will assume they are from now on. This is really a very small restriction to make (see [**9**, subsection 11.2]). Moreover, we will assume that nonzero morphisms between nonzero finite dimensional C^*-algebras are unital.

Another possibility in a commutative CW-complex is that some parts of a given CW-complex appear to have lower dimension than the rest of the complex. For instance, take the disjoint union of a closed disk and a closed interval. In the noncommutative case this corresponds to a connecting morphism not being injective. So from [**9**] we recall the following definition.

Definition 2.4. An n-dimensional $NCCW$-complex is called *proper* if all the connecting morphisms appearing in its construction are injective. This occurs if and only if the ideals $\mathbf{I}_0^k\mathbf{F}_k$ are essential in $\mathbf{A}_k$.

Definition 2.5. Suppose $\mathbf{A}_n$ is an n-dimensional $NCCW$-complex. Define the *canonical ideals* in $\mathbf{A}_n$ to be the decreasing family of closed ideals

$$\mathbf{A}_n = I_0 \supset I_1 \supset \cdots \supset I_n \neq 0$$

by setting I_k equal to the kernel of the composition

$$\mathbf{A}_n \xrightarrow{\pi_n} \mathbf{A}_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_k} \mathbf{A}_{k-1}.$$

So we have $I_n = \mathbf{I}_0^n\mathbf{F}_n$, $\mathbf{A}_n/I_{k+1} \cong \mathbf{A}_k$, and $I_k/I_{k+1} \cong \mathbf{I}_0^k\mathbf{F}_k$. For the purpose of consistency, we will let $I_k = 0$ for $k > n$.

Example 2.6. To make the previous definition more concrete, let X be a finite, n-dimensional CW-complex containing cells of each dimension $\leq n$. Then, as in the previous example, we have a decomposition of $C(X)$ as an n-dimensional $NCCW$-complex, with $\mathbf{A}_k = C(X^k)$ where X^k denotes the k-skeleton of X. Then $I_k = C_0(X \setminus X^{k-1})$ and $I_k/I_{k+1} \cong \mathbf{I}_0^n\mathbf{C}^{\lambda_k}$, where λ_k denotes the number of open k-cells.

Remark 2.7. Suppose $\mathbf{A}_n$ is strongly n-dimensional. Then there is a particularly nice way of writing elements of the algebra and of the canonical ideals. Suppose the lower dimensional complexes are of the form $\mathbf{A}_k = \mathbf{I}^k\mathbf{F}_k \oplus_{\mathbf{S}^{k-1}\mathbf{F}_k} \mathbf{A}_{k-1}$, then we may write $\mathbf{A}_n$ as an iterated restricted direct sum

$$\mathbf{A}_n = \mathbf{I}^n\mathbf{F}_n \bigoplus_{\mathbf{S}^{n-1}\mathbf{F}_n} \mathbf{I}^{n-1}\mathbf{F}_{n-1} \bigoplus_{\mathbf{S}^{n-2}\mathbf{F}_{n-1}} \cdots \bigoplus_{\mathbf{F}_1^2} \mathbf{A}_0.$$

Then we may regard elements of $\mathbf{A}_n$ as $(n+1)$-tuples $(a_n, \ldots, a_0)$, where $a_k \in \mathbf{I}^k\mathbf{F}_k$ for $k > 0$ and $a_0 \in \mathbf{A}_0$, such that $\delta(a_k) = \varphi_k((a_{k-1}, \ldots, a_0))$ in $\mathbf{S}^{k-1}\mathbf{F}_k$ for $1 \leq k \leq n$. Then the canonical ideals have the form

$$I_k = \{(a_n, a_{n-1}, \ldots, a_k, 0, \ldots, 0) \in \mathbf{A}_n : a_k \in \mathbf{I}_0^k\mathbf{F}_k\}.$$

Then the quotients become very transparent.

The following, from [**9**], generalizes the notion of a cellular map of CW-complexes. They will be used decisively in the next section to make the action of a finite group on an $NCCW$-complex compatible with the cellular decomposition.

Definition 2.8. Suppose

$$\mathbf{A} = \mathbf{I}^n\mathbf{F}_n \bigoplus_{\mathbf{S}^{n-1}\mathbf{F}_n} \mathbf{I}^{n-1}\mathbf{F}_{n-1} \bigoplus_{\mathbf{S}^{n-2}\mathbf{F}_{n-1}} \cdots \bigoplus_{\mathbf{F}_1^2} \mathbf{A}_0$$

and

$$\mathbf{B} = \mathbf{I}^m\mathbf{G}_m \bigoplus_{\mathbf{S}^{m-1}\mathbf{G}_m} \mathbf{I}^{m-1}\mathbf{G}_{m-1} \bigoplus_{\mathbf{S}^{m-2}\mathbf{G}_{m-1}} \cdots \bigoplus_{\mathbf{G}_1^2} \mathcal{B}_0$$

are $NCCW$-complexes of dimensions n and m, with canonical ideals $\{I_k\}_{k=0}^n$ and $\{J_k\}_{k=0}^m$, respectively. For $0 \le k \le n$, let $I_k/I_{k+1} = \mathbf{I}_0^k\mathbf{F}_k$ and $J_k/J_{k+1} = \mathbf{I}_0^k\mathbf{G}_k$. A morphism $\alpha : \mathbf{A} \to \mathbf{B}$ is called *simplicial* if

1. $\alpha(I_k) \subseteq J_k$. This guarantees the existence of an induced morphism between the quotient algebras $\widetilde{\alpha}_k : \mathbf{I}_0^k\mathbf{F}_k \to \mathbf{I}_0^k\mathbf{G}_k$.

2. There exists a morphism $\psi_k : \mathbf{F}_k \to \mathbf{G}_k$ and a homeomorphism ι_k of $[0,1]^k$ such that $\widetilde{\alpha}_k = \iota_k^* \otimes \psi_k$, where ι_k^* denotes the dualized version of ι_k.

When $\mathbf{A}$ is an $NCCW$-complex and α is a simplicial automorphism of $\mathbf{A}$, it is assumed that α is simplicial for a fixed decomposition of $\mathbf{A}$. That is, the decomposition is the same when viewing $\mathbf{A}$ as the domain or the codomain of α. If one considers the $\mathbf{A} = C[0,1]$ decomposed as an $NCCW$-complex with one 1-cell and two 0-cells for the domain of the identity morphism, and with two one-cells and three 0-cells for the codomain, then the identity is not simplicial. This is obviously corrected by giving $\mathbf{A}$ the same, finer, decomposition of both the domain and codomain.

3. Crossed products of $NCCW$-complexes.

Proposition 3.1. *Suppose $\mathbf{A}_n$ and $\mathbf{B}_n$ are strongly n-dimensional ($n \ge 1$) $NCCW$-complexes, with decompositions as in Definition* 2.8*, and $\alpha : \mathbf{A}_n \to \mathbf{B}_n$ is a simplicial morphism. Then $\alpha_{n-1}(a + I_n) = \alpha(a) + J_n$ defines a simplicial morphism $\alpha_{n-1} : \mathbf{A}_{n-1} \to \mathbf{B}_{n-1}$ and α is the unique $*$-homomorphism making the following diagram commute*:

$$\begin{tikzcd}
 & \mathbf{I}_0^n\mathbf{G}_n \arrow[rr] \arrow[dd, equal] & & \mathbf{B}_n \arrow[rr, "\pi_n"] \arrow[dd, "g_n"] & & \mathbf{B}_{n-1} \arrow[dd, "\sigma_n"] \\
\mathbf{I}_0^n\mathbf{F}_n \arrow[ur, "\iota_n^*\otimes\psi_n"'] \arrow[rr] \arrow[dd, equal] & & \mathbf{A}_n \arrow[ur, "\alpha"] \arrow[rr, "\pi_n"] \arrow[dd, "f_n"] & & \mathbf{A}_{n-1} \arrow[ur, "\alpha_{n-1}"'] \arrow[dd, "\gamma_n"] & \\
 & \mathbf{I}_0^n\mathbf{G}_n \arrow[rr] & & \mathbf{I}^n\mathbf{G}_n \arrow[rr, "\delta"] & & \mathbf{S}^{n-1}\mathbf{G}_n \\
\mathbf{I}_0^n\mathbf{F}_n \arrow[ur, "\iota_n^*\otimes\psi_n"'] \arrow[rr] & & \mathbf{I}^n\mathbf{F}_n \arrow[ur, "\iota_n^*\otimes\psi_n"'] \arrow[rr, "\delta"'] & & \mathbf{S}^{n-1}\mathbf{F}_n \arrow[ur, "\iota_n^*\otimes\psi_n"'] &
\end{tikzcd}$$

where γ_n and σ_n are the respective connecting morphisms.

Proof. First we note that α_{n-1} is a well-defined morphism because $\alpha(I_n) \subseteq J_n$. To show that α_{n-1} is simplicial, put $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{A}_{n-1}$, and $\mathbf{B}_{n-1}$ into their standard forms as in Remark 2.7 and let $\{I'_k\}_{k=0}^{n-1}$ and $\{J'_k\}_{k=0}^{n-1}$ be the canonical ideals for $\mathbf{A}_{n-1}$ and $\mathcal{B}_{n-1}$, respectively. Then we have isomorphisms

$$\mathbf{I}_0^k\mathbf{F}_k \cong I'_k/I'_{k+1} \cong (I_k/I_n)/(I_{k+1}/I_n) \cong I_k/I_{k+1} \cong \mathbf{I}_0^k\mathbf{F}_k$$

and

$$\mathbf{I}_0^k\mathbf{G}_k \cong J'_k/J'_{k+1} \cong (J_k/J_n)/(J_{k+1}/J_n) \cong J_k/J_{k+1} \cong \mathbf{I}_0^k\mathbf{G}_k.$$

When written down explicitly using the decomposition in Remark 2.7 the isomorphisms given above are both the identity $\mathrm{id}^* \otimes \mathrm{id}$. This implies that the induced morphism $\widetilde{\alpha}_{n-1,k} = (\mathrm{id}^* \otimes \mathrm{id}) \circ \widetilde{\alpha}_k \circ (\mathrm{id}^* \otimes \mathrm{id})$, and thus α_{n-1} is simplicial.

To show that this diagram commutes, we first note that the front and back commute by assumption. Obviously the leftmost square commutes, and the bottom commutes because any homeomorphism of $\mathbf{I}^n$ must preserve the boundary. The top right square commutes by the definition of α_{n-1}. The top left square commutes because α is simplicial. To show that the middle square commutes, consider the approximate identity for $\{a_k\} = \{h_k \otimes \mathrm{id}\} \in \mathbf{I}_0^n\mathbf{G}_n$, where $\{h_k\}$ is an approximate identity for $C_0(0,1)^n$ such that $h_k = 1$ on sets of the form $[(1/k), 1-(1/k)]^n$. Since ι_n is a homeomorphism, $h'_k = (\iota_n^*)^{-1}(h_k) \in \mathbf{I}_0^n\mathbf{F}_n$. Let $\{a'_k\} = \{h'_k \otimes \mathrm{id}\}$. Now suppose $(f,a) \in \mathbf{A}_n$. We know that the middle square commutes on the ideals $\mathbf{I}_0^n\mathbf{F}_n$ and $\mathbf{I}_0^n\mathbf{G}_n$. So we have that $\iota^* \otimes \psi_n \circ f_n((a'_k, 0)(f,a)) = g_n \circ \alpha((a'_k,0)(f,a))$ for all k. Now, for each $\mathbf{x} \in (0,1)^n$ we can find a k so that $a_k(\mathbf{x})f(\mathbf{x}) = f(\mathbf{x})$. Then we have that

$$\begin{aligned}\iota^* \otimes \psi_n \circ f_n((f,a))(\mathbf{x}) &= \iota^* \otimes \psi_n \circ f_n((a'_k,0)(f,a))(\mathbf{x})\\ &= g_n \circ \alpha((a'_k,0)(f,a))(\mathbf{x})\\ &= g_n \circ \alpha((f,a))(\mathbf{x}).\end{aligned}$$

This holds true for all $\mathbf{x}$ in a dense subset of $[0,1]^n$, so the middle square commutes.

We show that the rightmost square commutes by a diagram chase. Indeed, for $a \in \mathbf{A}_n$ we have

$$\begin{aligned}\sigma_n(\alpha_{n-1}(a+I_n)) &= \sigma_n(\alpha_{n-1}(\pi_n(a)))\\ &= \sigma_n(\pi_n(\alpha(a)))\\ &= \delta(g_n(\alpha(a)))\\ &= \delta(\iota_n^* \otimes \psi_n(f_n(a)))\\ &= \iota_n^* \otimes \psi_n(\delta(f_n(a)))\\ &= \iota_n^* \otimes \psi_n(\gamma_n(\pi_n(a)))\\ &= \iota_n^* \otimes \psi_n(\gamma_n(a+I_n)).\end{aligned}$$

So the right square commutes. The uniqueness of α follows from the universal property of $\mathbf{B}$ as a pullback. □

Now we are prepared to prove the main results on crossed products of $NCCW$-complexes with the assumption that the action is simplicial.

Proposition 3.2. *Suppose $\mathbf{A}_n$ is a strongly n-dimensional $NCCW$-complex and G is a locally compact group. If $(\mathbf{A}_n, G, \alpha)$ is a C^*-dynamical system with α_g simplicial for all $g \in G$, then there are C^*-dynamical systems $(\mathbf{A}_{n-1}, G, \alpha_{n-1})$, $(\mathbf{I}^n\mathbf{F}_n, G, \widetilde{\alpha})$, and $(\mathbf{S}^{n-1}\mathbf{F}_n, G, \overline{\alpha})$*

such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{I}_0^n\mathbf{F}_n \arrow[r] \arrow[d, Rightarrow, no head] & \mathbf{A}_n \arrow[r, "\pi_n"] \arrow[d, "f_n"] & \mathbf{A}_{n-1} \arrow[r] \arrow[d, "\varphi_n"] & 0 \\
0 \arrow[r] & \mathbf{I}_0^n\mathbf{F}_n \arrow[r] & \mathbf{I}^n\mathbf{F}_n \arrow[r, "\delta"] & \mathbf{S}^{n-1}\mathbf{F}_n \arrow[r] & 0
\end{tikzcd}$$

is G-equivariant.

Proof. We define, for $g \in G$, $(\alpha_{n-1})_g = (\alpha_g)_{n-1}$, $\widetilde{\alpha}_g$ to be the induced morphism $(\iota_n^* \otimes \psi_n)_g$, and $\overline{\alpha}_g \delta(f) = \delta(\widetilde{\alpha}_g f)$. Because α_g is simplicial for all g, it follows from Theorem 3.1 that these define dynamical systems for which the pullback is equivariant. □

Theorem 3.3 ([**9**, 6.3]). *Given a pullback of C^*-algebras*

$$\begin{tikzcd}
\mathbf{D} \arrow[r, "g_2"] \arrow[d, "g_1"] & \mathbf{B} \arrow[d, "\varphi"] \\
\mathbf{A} \arrow[r, "\delta"] & \mathbf{C}
\end{tikzcd}$$

and C^*-dynamical systems $(\mathbf{A}, G, \alpha)$, $(\mathbf{B}, G, \beta)$, and $(\mathbf{C}, G, \gamma)$ such that δ and φ are G-equivariant morphisms then there is a unique C^*-dynamical system $(\mathbf{D}, G, \tau)$ such that $\mathbf{D} \rtimes_\tau G \cong \mathbf{A} \rtimes_\alpha G \oplus_{\mathbf{C} \rtimes_\gamma G} \mathbf{B} \rtimes_\beta G$.

Corollary 3.4. *Suppose $\mathbf{A}_n$ is a strongly n-dimensional $NCCW$-complex, G is a locally compact group, and that $(\mathbf{A}_n, G, \alpha)$ is a C^*-dynamical system, with α_g simplicial for all $g \in G$. Then with the actions defined as in Proposition* 3.2, *we have*

$$\mathbf{A}_n \rtimes_\alpha G \cong (\mathbf{I}^n\mathbf{F}_n \rtimes_{\widetilde{\alpha}} G) \bigoplus_{(\mathbf{S}^{n-1}\mathbf{F}_n \rtimes_{\overline{\alpha}} G)} (\mathbf{A}_{n-1} \rtimes_{\alpha_{n-1}} G).$$

Proof. This is a straightforward application of Proposition 3.2 and Theorem 3.3. □

Before we define a class of C^*-dynamical systems that have $NCCW$-complexes as crossed products we need the following useful lemma.

Recall that, if α and β are actions of a group G on C^*-algebras $\mathbf{A}$ and $\mathbf{B}$, respectively, then $\alpha\otimes\beta$ is the action on $\mathbf{A}\otimes\mathbf{B}$ defined on elementary tensors by $(\alpha\otimes\beta)_g(a\otimes b) = \alpha_g a\otimes\beta_g b$.

Lemma 3.5. *Suppose $\mathbf{A}$ is a unital C^*-algebra, and $(\mathbf{B}, G, \alpha)$ is a C^*-dynamical system with $\mathbf{B}$ unital and G locally compact. Then, with ι denoting the trivial action of G on $\mathbf{A}$, $(\mathbf{A}\otimes\mathbf{B}) \rtimes_{\iota\otimes\alpha} G$ is naturally isomorphic to $\mathbf{A}\otimes(\mathbf{B}\rtimes_\alpha G)$.*

Proof. First we show that both of the above C^*-algebras have the same universal property. Let $(\pi, U, \mathcal{H})$ be a covariant representation of $(\mathbf{A}\otimes\mathbf{B}, G, \iota\otimes\alpha)$. Then as π is a nondegenerate representation of $\mathbf{A}\otimes\mathbf{B}$, there is a unique pair of representations (π_1, π_2) of $\mathbf{A}$ and $\mathbf{B}$, respectively, such that for all $a\in\mathbf{A}$ and $b\in\mathbf{B}$

$$\pi(a\otimes b) = \pi_1(a)\pi_2(b) = \pi_2(b)\pi_1(a).$$

Note that π_2 is unital since it is nondegenerate. Then the covariance condition implies that

$$\pi((\iota\otimes\alpha)_s(a\otimes 1_{\mathbf{B}})) = U_s\pi_1(a)U_s^* = \pi_1(a).$$

So π_1 commutes with both π_2 and U, and thus commutes with the representation $\pi_2\rtimes_\alpha U$ of $\mathbf{B}\rtimes_\alpha G$. So from $(\pi, U, \mathcal{H})$ we obtained a commuting pair $(\pi_1, \pi_2\rtimes_\alpha U)$, which gives a representation of $\mathbf{A}\otimes(\mathbf{B}\rtimes_\alpha G)$. This correspondence is one-to-one since the correspondence between π and (π_1, π_2) was one-to-one.

Conversely, take $(\pi_1', \pi_2'\rtimes_\alpha U')$ to be a commuting pair that gives a nondegenerate representation of $\mathbf{A}\otimes(\mathbf{B}\rtimes_\alpha G)$. If h_λ is an approximate unit for $C_c(G, \mathbf{B})$, then we have for each $a\in\mathbf{A}$ and $b\in\mathbf{B}$

$$\begin{aligned}
\pi_1'(a)\pi_2'(b) &= \lim_\lambda \pi_1'(a)\pi_2'(b)(\pi_2'\rtimes_\alpha U')(h_\lambda)\\
&= \lim_\lambda \pi_1'(a)(\pi_2'\rtimes_\alpha U')(bh_\lambda)\\
&= \lim_\lambda (\pi_2'\rtimes_\alpha U')(bh_\lambda)\pi_1'(a)\\
&= \pi_2'(b)\pi_1'(a).
\end{aligned}$$

Similarly, we can show that π_1' commutes with U' by taking translates of h_λ. So, if we let π' denote the representation of $\mathbf{A}\otimes\mathbf{B}$ coming from (π'_1, π'_2), then we have obtained a unique covariant representation $(\pi', U, \mathcal{H})$ of $(\mathbf{A} \otimes \mathbf{B}, G, \iota \otimes \alpha)$. So the two C^*-algebras have the same universal property and are isomorphic.

To show that the isomorphism is natural we must make the isomorphism explicit somehow. The isomorphism given by matching the universal properties is nothing more than identifying the universal representations of these two C^*-algebras. Since every $C^*$is naturally isomorphic to its universal representation, we are finished. $\square$

Definition 3.6. A *noncommutative G-CW-complex* ($NCGCW$-complex) is a C^*-dynamical system $(\mathbf{A}_n, G, \alpha)$ where $\mathbf{A}_n$ is a strongly n-dimensional $NCCW$-complex, G is a locally compact group, and for all $g \in G$, α_g is a simplicial morphism so that the induced morphism $(\widetilde{\alpha_g})_k$ has the form $\mathrm{id}^* \otimes (\psi_k)_g$.

Theorem 3.7. *Suppose G is a finite group and $(\mathbf{A}_n, G, \alpha)$ is a $NCGCW$-complex. Then the crossed product $\mathbf{A}_n \rtimes_\alpha G$ is an n-dimensional $NCCW$-complex.*

Proof. We will proceed by induction on n. Clearly, when $n = 0$, we have our result as the crossed product is again finite dimensional. Then, with the notation as in Proposition 3.2, we have that $\mathbf{A}_{n-1} \rtimes_{\alpha_{n-1}} G$ is an $(n-1)$-dimensional $NCCW$-complex, by the induction hypothesis. So, with the notation as in Proposition 3.2 and Corollary 3.4, we must show that the crossed products $\mathbf{I}^n\mathbf{F}_n \rtimes_{\widetilde{\alpha}} G$ and $\mathbf{S}^{n-1}\mathbf{F}_n \rtimes_{\overline{\alpha}} G$ are of the form $\mathbf{I}^n\mathbf{F}'_n$ and $\mathbf{S}^{n-1}\mathbf{F}'_n$, respectively, for some finite dimensional C^*-algebra $\mathbf{F}'_n$. This is true by a simple application of Lemma 3.5. By the naturality of the isomorphism in Lemma 3.5, the induced morphism $\delta' : \mathbf{I}^n\mathbf{F}'_n \to \mathbf{S}^{n-1}\mathbf{F}'_n$ is indeed the boundary restriction morphism. $\square$

Theorem 3.8. *Suppose X is a finite, CW-complex, with skeletal filtration $X^0 \subsetneq X^1 \subsetneq \cdots \subsetneq X^n = X$ and G is a finite group. Moreover, assume G acts on X in such a way that the following conditions hold*:

1. *Whenever e is a k-cell of X, so is $g \cdot e$ for $0 \leq k \leq n$ and all $g \in G$.*

2. *If $g \cdot e = e$, then $g|_e = \mathrm{id}_e$*

(in particular X is a G-CW-complex). Then $C(X) \rtimes G$ is a strongly n-dimensional $NCCW$-complex.

Proof. Let Ω^k denote the collection of all open k-cells in X. Since G acts on Ω^k by permutations, Ω^k is partitioned by the orbits under the action of G. Let $\Omega_G^k = \{e_1, \dots, e_{j_k}\}$ be a cross section of the G action on Ω^k. Then we see that, for each closed k-cell in $e \in X$, there is a unique k-cell $e_j \in \Omega_G^k$ such that $e = g \cdot e_j$ for some $g \in G$. Also, for each $e \in \Omega^k$ there is a continuous surjection $\psi_e : [0,1]^k \to \overline{e}$. Then we have a surjection $g \cdot \psi_{e_j} : [0,1]^k \to \overline{e}$. Also, by condition (2), if $h \in G$ is such that $h \cdot \psi_{e_j}([0,1]^k) = \overline{e}$, then $h \cdot \psi_{e_j} = g \cdot \psi_{e_j}$ since $(g^{-1} \cdot h)|_{e_j} = \mathrm{id}_{e_j}$ implies that $h = gs$ for some s in the stabilizer of e_j under the action of G.

Let l_k denote the number of k-cells for each k, and let $\mathbf{F}_k = \mathbf{C}^{l_k}$. For each $f \in C(X^k)$ and k-cell $e \in \Omega^k$, $f|_{\overline{e}} \in C(\overline{e})$, and there is an injection $(g \cdot \psi_{e_j})^* : C(\overline{e}) \to C([0,1]^k)$ such that $(g \cdot \psi_{e_j})^*(f|_{\overline{e}})(\mathbf{x}) = f(g \cdot \psi_{e_j}(\mathbf{x}))$ for all $f \in C(X^n)$. Now define $\widetilde{\Psi}_k : C(X^k) \to \mathbf{I}^k\mathbf{F}_k$ by $(\widetilde{\Psi}_k(f)(\mathbf{x}))_e = (g \cdot \psi_{e_j})^*(f|_{\overline{e}})(\mathbf{x})$, where $e = g \cdot e_j$ and $e_j \in \Omega_G^k$. Since the maps $g \cdot \psi_{e_j}$ take copies of $\partial[0,1]^k$ to X^{k-1}, there is a well-defined map γ_k such that the diagram

$$\begin{tikzcd} C(X^k) \arrow[r, "\pi_k"] \arrow[d, "\widetilde{\Psi}_k"] & C(X^{k-1}) \arrow[d, "\gamma_k"] \\ \mathbf{I}^k\mathbf{F}_k \arrow[r, "\delta"] & \mathbf{S}^{k-1}\mathbf{F}_k \end{tikzcd}$$

commutes. Since X is (as a set) the disjoint union of X^{k-1} and $X^k \backslash X^{k-1}$ we can see that $\ker \widetilde{\Psi}_k \cap \ker \pi_k = 0$. Also we see that $\widetilde{\Psi}_k(\ker(\pi_k)) = \ker(\delta)$ and $\gamma_k^{-1}(\delta(\mathbf{I}^k\mathbf{F}_k)) = \pi_k(C(X^k))$. Then, by Proposition 3.1 in [**9**], this diagram is a pullback. It now follows from the universal property of pullbacks that $C(X^k)$ is isomorphic to a strongly k-dimensional $NCCW$-complex for each k.

Let α be the action of G on $C(X)$ given by $(\alpha_h f)(y) = f(h^{-1} \cdot y)$. Now, since the action of G on X is cellular, it is easy to see that the canonical ideals of $C(X)$ are preserved by the action of α. Then note that, for $f \in C_0(X^k \backslash X^{k-1})$ and $h \in G$,

$$\begin{aligned}(\widetilde{\Psi}_k(\alpha_h f)(\mathbf{x}))_e &= (g \cdot \psi_{e_j})^*(\alpha_h f|_{\overline{e}})(\mathbf{x}) \\ &= \alpha_h f(g \cdot \psi_{e_j}(\mathbf{x})) \\ &= f(h^{-1} g \cdot \psi_{e_j}(\mathbf{x})) \\ &= (h^{-1} g \cdot \psi_{e_j})^*(f|_{h^{-1}\overline{e}})(\mathbf{x}) \\ &= (\widetilde{\Psi}_k(f)(\mathbf{x}))_{h^{-1}e}.\end{aligned}$$

Thus, the induced action $\widetilde{\alpha}_h$ of G on $\mathbf{I}_0^k \mathbf{F}_k$ is of the form $\mathrm{Id} \otimes P_h$, where $h \mapsto P_h$ is an action of G by permutations on the summands of $\mathbf{F}_k$. So $C(X)$ has an action α of G such that $(C(X), G, \alpha)$ is an n-dimensional $NCGCW$-complex. □

To see that this type of action occurs frequently in practice we display the following:

Theorem 3.9. *Let M be a compact smooth manifold of dimension n, with or without boundary. Let G be a finite group acting on M by diffeomorphisms. Then the crossed product $C(M) \rtimes G$ is an n-dimensional $NCCW$-complex.*

Proof. Part one of the first theorem in [**4**] states that there exists a simplicial complex S, of dimension n, with a simplicial action of G on S, and a G equivariant triangulation $h : S \to M$. Since M is compact, S must have finitely many simplices in each dimension. Then we may realize S as a finite n-dimensional CW-complex whose k-cells are the k-simplices. Then we have $C(S) \rtimes G \cong C(M) \rtimes G$. Since the action of G on S is simplicial, the action satisfies the first hypothesis in Theorem 3.8. So we just need to show that the action can be made to satisfy the second.

Suppose f is a simplicial homeomorphism of S and σ is a zero simplex fixed by f. Then, clearly, $f|_\sigma = \mathrm{Id}_\sigma$. Now suppose τ is a k-simplex of S which is fixed by f. We assume by induction that, for each lower dimensional simplex ρ of τ left fixed by f, $f|_\rho = \mathrm{Id}_\rho$. Let τ' be a fixed k-simplex in the barycentric subdivision of τ. Then τ' is the set of convex combinations of the barycenter of τ and a $(k-1)$-simplex left fixed by f. Since f fixes the barycenter and acts as the identity on the $(k-1)$-simplex, f must act as the identity on τ'.

Now we must apply the preceding paragraph to each element of G. Since S is finite dimensional and G is finite this means we must only take a finite number of barycentric subdivisions of S to satisfy the second hypothesis of Theorem 3.8. Barycentric subdivisions do not change the crossed product, so we are done. □

4. Twisted crossed products. A question related to Morita equivalence is whether or not twisted crossed products of $NCCW$-complexes are again $NCCW$-complexes. The answer turns out to be less satisfying than in the untwisted case: we have examples where the twisted crossed product is an $NCCW$-complex, but the techniques of proof from the untwisted case will not work for a general theorem. However, we do obtain a decomposition of some sort.

Recall from [**8**] that two twisted actions (α, u) and (β, w) of G on $\mathbf{A}$ are called *exterior equivalent* if there exists a Borel map $v : G \to UM(\mathbf{A})$ such that

1. $\beta_s = \operatorname{Ad} v_s \circ \alpha_s$,
2. $w(s,t) = v_s \alpha_s(v_t) u(s,t) v_{st}^*$.

Our point in defining exterior equivalence is that if two dynamical systems are exterior equivalent, then their crossed products are Morita equivalent, which is well characterized by to the following:

Theorem 4.1 [**10**]. *If $\mathbf{A}$ and $\mathbf{B}$ are unital C^*-algebras that are Morita equivalent, then $\mathbf{A}$ is isomorphic to a full corner of the algebra of $n \times n$ matrices over $\mathbf{B}$ for suitable n and $\mathbf{B}$ is isomorphic to a full corner of the algebra of $m \times m$ matrices over $\mathbf{A}$ for suitable m.*

Theorem 4.2 ([**8**, 3.4]). *Let $(\mathbf{A}, G, \alpha, u)$ be a separable twisted dynamical system. There is a strongly continuous action β of G on $\mathbf{A} \otimes \mathcal{K}(L^2(G))$ that is exterior equivalent to the action $(\alpha \otimes \mathrm{Id}, u \otimes \mathrm{Id})$.*

While the explicit formula for the exterior equivalence in the previous theorem is not important for us, the following fact, from [**5**] will prove to be very useful.

Proposition 4.3. *Let $(C(X), G, \alpha, u)$ be a twisted C^*-dynamical system with G finite of order n. If I is an ideal in $C(X)$ such that $\alpha(I) \subseteq I$, then $\beta(M_n(I)) \subseteq M_n(I)$ where β denotes the stabilized action defined above.*

Lemma 4.4. *Suppose $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are C^*-algebras, and let $\mathbf{D} = \mathbf{A} \oplus_{\mathbf{C}} \mathbf{B}$ be the pullback over the morphisms $\varphi_1 : \mathbf{A} \to \mathbf{C}$ and $\varphi_2 : \mathbf{B} \to \mathbf{C}$. If $p \in \mathbf{D}$ is a projection, let p_1 and p_2 denote the projections onto the first and second coordinates, respectively, of p, and let p_0 be their image in $\mathbf{C}$. Then $p\mathbf{D}p \cong p_1\mathbf{A}p_1 \oplus_{p_0\mathbf{C}p_0} p_2\mathbf{B}p_2$.*

Proof. Suppose $(p_1ap_1, p_2bp_2) \in p\mathbf{D}p$. Then $\varphi_1(a) = \varphi_2(b)$ and so $p_0\varphi_1(a)p_0 = p_0\varphi_2(b)p_0$, and we have $(p_1ap_1, p_2bp_2) \in p_1\mathbf{A}p_1 \oplus_{p_0\mathbf{C}p_0} p_2\mathbf{B}p_2$. On the other hand, if $(p_1ap_1, p_2bp_2) \in p_1\mathbf{A}p_1 \oplus_{p_0\mathbf{C}p_0} p_2\mathbf{B}p_2$, then we have $(p_1ap_1, p_2bp_2) = (p_1p_1ap_1p_1, p_2p_2bp_2p_2) \in p\mathbf{D}p$. So $p\mathbf{D}p \cong p_1\mathbf{A}p_1 \oplus_{p_0\mathbf{C}p_0} p_2\mathbf{B}p_2$. $\square$

Now we are ready to give the decomposition for twisted crossed products of $NCCW$-complexes.

Theorem 4.5. *Suppose $\mathbf{A}_k$ is a proper k-dimensional $NCCW$-complex and G is a finite group of order n. Suppose $(\mathbf{A}_k, G, \alpha, u)$ is a twisted C^*-dynamical system such that $\alpha_g(I_j) = I_j$ for all the canonical ideals I_j. Then there exists a sequence of pullbacks*

$$\mathbf{A}_i \rtimes_{\alpha,u} G \cong \mathbf{B}_i \bigoplus_{\mathbf{D}_i} \mathbf{C}_i,$$

where

$$\begin{aligned} &\mathbf{B}_i \sim_m M_n(\mathbf{I}^i\mathbf{F}_i) \rtimes_{\widetilde{\beta}} G, \\ &\mathbf{C}_i \sim_m M_n(\mathbf{A}_{n-1}) \rtimes_{\beta_{i-1}} G, \ \text{and} \\ &\mathbf{D}_i \sim_m M_n(\mathbf{S}^{i-1}\mathbf{F}_i) \rtimes_{\overline{\beta}} G. \end{aligned}$$

Proof. By using Lemma 4.4 we just need to show that we have pullbacks of untwisted crossed products

$$M_n(\mathbf{A}_i) \rtimes_\beta G \cong (M_n(\mathbf{I}^i\mathbf{F}_i) \rtimes_{\widetilde{\beta}} G) \bigoplus_{(M_n(\mathbf{S}^{i-1}\mathbf{F}_i) \rtimes_{\beta_{i-1}} G)} (M_n(\mathbf{A}_{i-1}) \rtimes_{\beta_{i-1}} G)$$

where β denotes the stabilized action given above. Following the proof of Proposition 3.1, existence of dynamical systems $(M_n(\mathbf{A}_{i-1}), G, \beta_{i-1})$ and $(M_n(\mathbf{S}^{i-1}\mathbf{F}_i), G, \overline{\beta})$ such that the connecting morphism is equivariant will follow from the existence of a dynamical system $(M_n(\mathbf{I}^i\mathbf{F}_i), G, \widetilde{\beta})$ such that the projection $f_i : M_n(\mathbf{A}_i) \to M_n(\mathbf{I}^i\mathbf{F}_i)$ is equivariant since $\beta(M_n(\mathbf{I}_0^i\mathbf{F}_i)) = M_n(\mathbf{I}_0^i\mathbf{F}_i)$. Then using Theorem 3.3 we will obtain the pullback of crossed products. The canonical ideals $\{M_n(I_j')\}$ of the lower dimensional complex will also be preserved by β_{i-1} because they are nothing more than quotients of the canonical ideals of the higher dimensional complex. So if we obtain the result when $i = k$, the rest will follow.

Since $\mathbf{A}_n$ is proper so is $M_n(\mathbf{A}_k)$. Then $M_n(\mathbf{I}_0^k\mathbf{F}_k)$ is an essential ideal in $M_n(\mathbf{A}_k)$ and $M_n(\mathbf{I}^k\mathbf{F}_k)$ and the projection f_n is injective. So we have inclusions

$$M_n(\mathbf{I}_0^k\mathbf{F}_k) \subset M_n(\mathbf{A}_k) \subseteq M_n(\mathbf{I}^k\mathbf{F}_k) \subset M(M_n(\mathbf{I}_0^k\mathbf{F}_k)).$$

By [**6**, Proposition 7.1.7], we know that for each $g \in G$, β_g extends to a unique automorphism of $M(M_n(\mathbf{I}_0^k\mathbf{F}_k))$. Then, by [**2**, Lemma 2.1], β_g is the restriction of this extension to $M_n(\mathbf{A}_k)$. Let $\widetilde{\beta}$ denote the restriction of this action on $M_n(\mathbf{I}^k\mathbf{F}_k)$. Then we have our dynamical system such that f_k is equivariant. Since the projection π_k and the boundary restriction δ are equivariant we are done. □

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