ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 40, Number 5, 2010

THE SHERRINGTON KIRKPATRICK MODEL WITH FERROMAGNETIC INTERACTION

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ABSTRACT. We consider a spin model with both ferromagnetic interaction and Sherrington-Kirkpatrick couplings in a high temperature region, with the presence of an external field. We generalize some results obtained in the standard SK model, studying the overlap and the magnetization and limit for the free energy. These results show how ferromagnetic interaction affects the behavior of the model.

1. Introduction. We consider the Sherrington Kirkpatrick model with ferromagnetic interaction. The configuration space is $\Sigma_N = \{-1,1\}^N$, and the energy of each configuration $\sigma \in \Sigma_N$ is represented by the Hamiltonian

$$-H_N(\sigma) = \frac{\beta_1}{2N}\bigg(\sum_{i\le N}\sigma_i\bigg)^2 + \frac{\beta_2}{\sqrt{N}}\bigg(\sum_{i<j\le N} g_{i,j}\sigma_i\sigma_j\bigg) + h\sum_{i\le N}\sigma_i,$$

where $\{g_{i,j}; 1 \le i < j \le N\}$ is a family of independent standard Gaussian random variables and $h > 0$ is the intensity of an external electromagnetic field. The two parameters β_1 and β_2 play the role of two inverse temperatures. If $\beta_1 = 0$, the Hamiltonian is equivalent to the one of the Sherrington Kirkpatrick model. On the other hand, if $\beta_2 = 0$, the model reduces to Curie Weiss model, that is, the canonical model for mean field (deterministic) ferromagnetic interaction. For this type of interaction, in which spins tend to align with the ones in their vicinity, we need a term proportional to $\sigma_i\sigma_j$ in the Hamiltonian, or, equivalently, we can consider the square $(\sum_{i\le N}\sigma_i)^2$ in order to write the Hamiltonian as a function of the *magnetization*

$$m_l = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^l. \tag{1}$$

2010 AMS *Mathematics subject classification.* Primary 60G15, 82B44, 82D30.
Keywords and phrases. Spin glasses, Sherrington-Kirkpatrick model, ferromagnetic interaction.
The first author was partially supported by Fondazione Ing. A. Gini. The second author was partially supported by DGES grant BFM2003-01345.
Received by the editors on December 13, 2007, and in revised form on April 16, 2008.
DOI:10.1216/RMJ-2010-40-5-1441

The Gibbs' measure is given by

$$G_N(\sigma) = \frac{1}{Z_N}\exp(-H_N(\sigma)) \quad \text{with}$$
$$Z_N = \sum_{\sigma\in\Sigma_N}\exp(-H_N(\sigma)),$$

where Z_N denotes the partition function. We will denote by $\langle f\rangle$ the average with respect to the Gibbs' measure of a function $f : \Sigma_N \to \mathbf{R}$ as well as for a function $f : \Sigma_N^n \to \mathbf{R}$. So

$$\langle f\rangle = \frac{1}{Z_N^n}\sum_{\sigma\in\Sigma_N^n} f(\sigma^1,\dots,\sigma^n)\exp\biggl(-\sum_{l\le n}H_N(\sigma^l)\biggr).$$

We write $\nu(f) = \mathbf{E}\langle f\rangle$ where $\mathbf{E}$ denotes the expectation with respect to randomness in the Hamiltonian.

The high temperature regime of the SK model with external field has been widely studied, see e.g., [**4, 5**], but the results on models with ferromagnetic interactions are scarce (see, e.g., [**1, 2, 3**]). The SK model with ferromagnetic interaction is a system with difficulties due to the ferromagnetic interaction but with a familiar disorder. It is interesting since it appears as a first step in the study of models with this kind of interaction.

Our aim is to extend the well-known results obtained in the SK model (see, e.g., [**5**]) to this model, trying to describe the behavior of the model in the high temperature region. In our model, anyway, we have to consider two order parameters and not just one as in the SK model. One of them is the same one considered in the SK model, that is, the *overlap*

$$R_{l,l'} = \frac{1}{N}\sum_{i\le N}\sigma_i^l\sigma_i^{l'}$$

where σ^l, $\sigma^{l'}$ are understood as two independent configurations under G_N, and the other is *magnetization*, defined in (1). Our first result regards the behavior of these two quantities: we will prove that they converge in L^2 to two constants, q and μ, respectively, that are the unique solutions of the replica-symmetric equations of this model:

$$\begin{cases} q = \mathbf{E}\tanh^2(\beta_2 z\sqrt{q}+\beta_1\mu+h) \\ \mu = \mathbf{E}\tanh(\beta_2 z\sqrt{q}+\beta_1\mu+h), \end{cases} \tag{2}$$

where z is a standard Gaussian random variable. It will then be natural to obtain some extra information on the exponential moments of the quantities $R_{1,2}-q$ and $m_1-\mu$. We will also obtain precise estimations for the second order moments of $R_{1,2}-q$ and $m_1-\mu$. Then we will study a quantity that is closely related to the free energy considered by physicists, $p_N(\beta_1,\beta_2)=(1/N)\mathbf{E}\log Z_N$, and we will prove that, when the size of the system tends to infinity, $p_N(\beta_1,\beta_2)$ converges almost surely to the function

$$F(\beta_1,\beta_2)=\frac{\beta_2^2}{4}(1-q)^2+\log 2+\mathbf{E}\left(\log\cosh(\beta_2 z\sqrt{q}+\beta_1\mu+h)\right)-\frac{\beta_1\mu^2}{2}.$$

Moreover, we will talk about the regularity of the system and, in order to understand the behavior of the Gibbs' measure, we will study the family of random variables $\{\langle\sigma_i\rangle\}_{i=1,\dots,n}$.

Our methods of proof follow closely [**5**]. However, the presence of the ferromagnetic interaction requires a careful study at each step of our computations.

Our paper is organized as follows. In Section 2 we introduce the cavity method for our model. In Section 3 we will prove that system (2) admits a unique solution (q,μ), and we will prove the convergence of $(R_{1,2},m_1)$ to it. Section 4 is devoted to the study of $p_N(\beta_1,\beta_2)$. In Section 5 we will compute the moments of order $2k$, $k>0$, of the quantities $R_{1,2}-q$ and $m_1-\mu$ that allow us to study the behavior of the Gibbs' measure in Section 6. Finally, in Section 7, we will give a more precise value to $\nu((R_{1,2}-q)^2)$, $\nu((m_1-\mu)^2)$ and $\nu((R_{1,2}-q)(m_1-\mu))$ in order to obtain central limit results.

We will denote by K almost all constants, although their value may change from line to line.

2. The cavity method. With this method we reduce a system with N spins into one with $N-1$ spins, creating a cavity, so we can think the last spin σ_N independent from the others. The main idea of the cavity method is to reorder in the Hamiltonian all the terms that depend on the last spin.

Let $\rho=(\sigma_1,\dots,\sigma_{N-1})\in\Sigma_{N-1}$, and let

$$\beta_1^-=\frac{N-1}{N}\beta_1,\qquad \beta_2^-=\sqrt{\frac{N-1}{N}}\beta_2, \tag{3}$$

that will play the role of β_1 and β_2 in our reduced system. The Hamiltonian becomes

$$-H_N(\sigma) = -H_{N-1,\beta_1^-,\beta_2^-}(\rho) + \frac{\beta_1}{2N} + \sigma_N\left(g(\rho)+h\right),$$

where $-H_{N-1,\beta_1^-,\beta_2^-}(\rho)$ is the Hamiltonian of the reduced system with $N-1$ spins and $g(\rho)$ is defined as

$$g(\rho) = \frac{\beta_1}{N}\sum_{i\leq N-1}\sigma_i + \frac{\beta_2}{\sqrt{N}}\sum_{i\leq N-1} g_{i,N}\sigma_i.$$

We will denote by $\langle\cdot\rangle_-$ the average with respect to the Gibbs' measure in Σ_{N-1} with reference to the Hamiltonian $-H_{N-1}(\rho)$.

For a function $f:\Sigma_N \to \mathbf{R}$, the following equality holds

$$\langle f\rangle = \frac{\langle Avf \exp\sigma_N(g(\rho)+h)\rangle_-}{Z},$$

where Av means average on the values $\sigma_N = \pm 1$ and

$$Z = \langle Av \exp\sigma_N(g(\rho)+h)\rangle_- = \langle\cosh(g(\rho)+h)\rangle_- .$$

For functions in Σ_N^n, we also have

$$\langle f\rangle = \frac{\left\langle Avf \exp\sum_{l\leq n}\sigma_N^l\left(g(\rho^l)+h\right)\right\rangle_-}{\left(\langle\cosh(g(\rho^l)+h)\rangle_-\right)^n}.$$

To simplify notation we will write $\varepsilon_l = \sigma_N^l$.

In order to construct a continuous path between the original configuration and a configuration where the last spin is independent of the others, let us define for a function $f:\Sigma_N^n \to \mathbf{R}$

$$\langle f\rangle_t = \frac{\left\langle Avf \exp\sum_{l\leq n}\varepsilon_l(g_t(\rho^l)+h)\right\rangle_-}{Z_t^n},$$

where

$$\begin{aligned} g_t(\rho^l) &= \frac{\sqrt{t}\beta_2}{\sqrt{N}}\sum_{i\leq N-1} g_{i,N}\sigma_i^l + \sqrt{1-t}(\beta_2 z\sqrt{q}) \\ &\quad + \frac{t\beta_1}{N}\sum_{i\leq N-1}\sigma_i^l + (1-t)\beta_1\mu \end{aligned} \tag{4}$$

and

$$Z_t = \left\langle Av \exp \varepsilon(g_t(\rho^l) + h)\right\rangle_- = \left\langle \cosh(g_t(\rho^l) + h)\right\rangle_- .$$

Moreover, let us write

$$\xi_{n,t} = \exp \sum_{l \leq n} \varepsilon_l (g_t(\rho^l) + h)$$

and

$$\nu_t(f) = \mathbf{E}\langle f \rangle_t.$$

Notice that it will be simpler to compute $\nu_0(f)$ than $\nu_1(f) = \nu(f)$ and that these two quantities are obviously related by

$$\nu(f) - \nu_0(f) = \int_0^1 \nu_t'(f)\, dt.$$

In the following lemma we show how to compute $\nu_0(f)$. The proof is an obvious extension of Lemma 2.4.4 in [**5**].

Lemma 2.1. *Let Y be the random variable defined as*

$$Y = \beta_2 z \sqrt{q} + \beta_1 \mu + h. \tag{5}$$

For any function $f^- : \Sigma_{N-1}^n \to \mathbf{R}$ and any subset I of $\{1, \dots, n\}$, we have

$$\nu_0\left(f^- \prod_{i \in I} \varepsilon_i\right) = \mathbf{E}(\tanh Y)^{\operatorname{card} I} \nu_0(f^-) = \nu_0\left(\prod_{i \in I} \varepsilon_i\right) \nu_0(f^-).$$

We now compute the derivative of $\nu_t(f)$ with respect to t.

Proposition 2.2.

$$\begin{aligned} \nu_t'(f) = {} & \beta_2^2 \left(\sum_{1 \leq l < l' \leq n} \nu_t\left(f \varepsilon_l \varepsilon_{l'} (R_{l,l'} - q)\right)\right) \\ & - n\beta_2^2 \sum_{l \leq n} \nu_t\left(f \varepsilon_l \varepsilon_{n+1} (R_{l,n+1} - q)\right) \\ & + \beta_2^2 \frac{n(n+1)}{2} \nu_t\left(f \varepsilon_{n+1} \varepsilon_{n+2} (R_{n+1,n+2} - q)\right) \\ & + \beta_1 \left(\sum_{l \leq n} \nu_t(f \varepsilon_l (m_l - \mu)) - n \nu_t(f \varepsilon_{n+1} (m_{n+1} - \mu))\right). \end{aligned} \tag{6}$$

Remark 2.3. Define

$$R^-_{l,l'} = \frac{1}{N}\sum_{i=1}^{N-1} \sigma_i^l \sigma_i^{l'}, \qquad m_l^- = \frac{1}{N}\sum_{i=1}^{N-1} \sigma_i^l;$$

thus, the following relations hold

$$\varepsilon_l \varepsilon_{l'} R_{l,l'} = \frac{1}{N} + \varepsilon_l \varepsilon_{l'} R^-_{l,l'}, \qquad \varepsilon_l m_l = \frac{1}{N} + \varepsilon_l m_l^-. \tag{7}$$

Proof of Proposition 2.2. It suffices to prove

$$\begin{aligned}
\nu_t'(f) = \beta_2^2 \Bigg(& \sum_{1 \le l < l' \le n} \nu_t\left(f \varepsilon_l \varepsilon_{l'} (R^-_{l,l'} - q)\right)\Bigg) \\
& - n\beta_2^2 \sum_{l \le n} \nu_t \left(f \varepsilon_l \varepsilon_{n+1} (R^-_{l,n+1} - q)\right) \\
& + \beta_2^2 \frac{n(n+1)}{2} \nu_t \left(f \varepsilon_{n+1} \varepsilon_{n+2} (R^-_{n+1,n+2} - q)\right) \\
& + \beta_1 \Bigg(\sum_{l \le n} \nu_t (f \varepsilon_l (m_l^- - \mu)) - n \nu_t (f \varepsilon_{n+1} (m^-_{n+1} - \mu))\Bigg)
\end{aligned} \tag{8}$$

and then use relations (7). This proof is an extension of the proof of Proposition 2.4.5 in [**5**], checking that

$$\begin{aligned}
& \mathbf{E}\Bigg(\frac{\langle Avf \sum_{l \le n} \varepsilon_l (f_1(\rho^l) - \beta_1 \mu) \xi_{n,t}\rangle_-}{Z_t^n}\Bigg) \\
& \qquad - n\mathbf{E}\Bigg(\frac{\langle Avf \xi_{n,t}\rangle_- \, \langle Av \varepsilon (f_1(\rho) - \beta_1 \mu)\xi_{1,t}\rangle_-}{Z_t^{n+1}}\Bigg) \\
& \quad = \mathbf{E}\Bigg(\frac{\langle Avf \sum_{l \le n} \varepsilon_l f_1(\rho^l) \xi_{n,t}\rangle_-}{Z_t^n}\Bigg) \\
& \qquad - n\mathbf{E}\Bigg(\frac{\big\langle Avf (\varepsilon_{n+1} f_1(\rho^{n+1}))\xi_{n+1,t}\big\rangle_-}{Z_t^{n+1}}\Bigg) \\
& \qquad - \beta_1 \mu \mathbf{E}\Bigg(\frac{\langle Avf \sum_{l \le n} \varepsilon_l \xi_{n,t}\rangle_-}{Z_t^n} - n \frac{\langle Avf \varepsilon_{n+1} \xi_{n+1,t}\rangle_-}{Z_t^{n+1}}\Bigg) \\
& \quad = \beta_1 \Bigg(\nu_t \Bigg(f \sum_{l \le n} \varepsilon_l (m_l^- - \mu)\Bigg) - n\nu_t (f \varepsilon_{n+1} (m^-_{n+1} - \mu))\Bigg). \qquad \square
\end{aligned}$$

As a consequence of Proposition 2.2 we can bound $\nu_t(f)$ by $\nu(f)$.

Proposition 2.4. *If f is a nonnegative function on Σ_N^n, we have*

$$\nu_t(f) \le \nu(f) \exp\left(4n^2\beta_2^2 + 4n\beta_1\right).$$

Proof. Since we can assume that $|q| \le 1$ and $|\mu| \le 1$, the proof is an extension of Lemma 2.4.6 in [**5**]. □

Proposition 2.5. *Consider a function $f : \Sigma_N^n \to \mathbf{R}$ and numbers $\alpha_1,\ \alpha_2,\ \tau_1,\ \tau_2 > 1$ such that $1/\alpha_1 + 1/\alpha_2 = 1$ and $1/\tau_1 + 1/\tau_2 = 1$. Then*

$$\begin{aligned}\nu(f) &\le \nu_0(f)\\ &\quad + 2n^2\beta_2^2 \exp(4n^2\beta_2^2 + 4n\beta_1)\left(\nu|f|^{\tau_1}\right)^{1/\tau_1}\left(\nu|R_{1,2} - q|^{\tau_2}\right)^{1/\tau_2}\\ &\quad + 2n\beta_1 \exp(4n^2\beta_2^2 + 4n\beta_1)\left(\nu|f|^{\alpha_1}\right)^{1/\alpha_1}\left(\nu|m_1 - \mu|^{\alpha_2}\right)^{1/\alpha_2}.\end{aligned}$$

Proof. Notice that

$$\nu(f) = \nu_0(f) + \int_0^1 \nu_t'(f)\,dt \le \nu_0(f) + \sup_{0\le t\le 1} |\nu_t'(f)|.$$

Applying Hölder's inequality in (6), we get that

$$\begin{aligned}|\nu_t'(f)| &\le 2n^2\beta_2^2\left(\nu_t|f|^{\tau_1}\right)^{1/\tau_1}\left(\nu_t|R_{1,2} - q|^{\tau_2}\right)^{1/\tau_2}\\ &\quad + 2n\beta_1\left(\nu_t|f|^{\alpha_1}\right)^{1/\alpha_1}\left(\nu_t|m_l - \mu|^{\alpha_2}\right)^{1/\alpha_2}.\end{aligned}$$

Then, we use Proposition 2.4. □

3. L^2 convergence to the parameters. Before proving that $R_{l,l'}$ and m_l converge in the L^2 sense to q and μ, respectively, we will check that system (2) admits a unique solution (q, μ). Actually, it is enough to use a Banach fixed point theorem for the function

$$\begin{aligned}T : [-1,1]\times[0,1] &\longrightarrow [-1,1]\times[0,1]\\ (\mu, q) &\longrightarrow (\mathbf{E}\tanh(Y), \mathbf{E}\tanh^2(Y)),\end{aligned}$$

where Y is the random variable defined in (5). Indeed, T is a contraction since it can be checked that

$$d^2(T(\mu, q), T(\mu', q')) \leq \alpha^2 d^2((\mu, q), (\mu', q')),$$

where $\alpha^2 = 10(\beta_1^2 + \beta_2^4)$ and $\alpha < 1$ when β_1 and β_2 are small enough.

Now we can focus our attention on the main theorem of this section. Let us assume that from now on our high temperature region will be determined by the following relations

$$\begin{cases} 16\beta_2^2 \exp(16\beta_2^2 + 8\beta_1) \leq 1/4, \\ 8\beta_1 \exp(16\beta_2^2 + 8\beta_1) \leq 1/4. \end{cases} \tag{9}$$

Notice that, under (9), $\alpha < 1$ and we have a unique solution (μ, q).

Theorem 3.1. *For β_1 and β_2 satisfying* (9) *and for q and μ solutions of* (2)*, the following inequalities hold*

$$Q_N := \nu((R_{1,2} - q)^2) \leq \frac{K}{N},$$
$$M_N := \nu((m_1 - \mu)^2) \leq \frac{K}{N}.$$

Proof. Let $f = (m_1 - \mu)^2$. By symmetry, $\nu(f) = \nu((\varepsilon_1 - \mu)(m_1 - \mu))$ and, using Lemma 2.1 with $f^- = m_1^- - (N - 1/N)\mu$, we can write

$$\nu_0((\varepsilon_1 - \mu)(m_1 - \mu)) = \frac{1}{N}\nu_0((\varepsilon_1 - \mu)^2) + \nu_0\left((\varepsilon_1 - \mu)\, f^-\right) = \frac{1 - \mu^2}{N}.$$

Applying Proposition 2.5 with $\alpha_1 = \alpha_2 = \tau_1 = \tau_2 = n = 2$, we get that

$$\begin{aligned} \nu(f) \leq {} & \nu_0((\varepsilon_1 - \mu)(m_1 - \mu)) \\ & + 16\beta_2^2 \exp(16\beta_2^2 + 8\beta_1)(\nu(R_{1,2} - q)^2)^{1/2}(\nu(m_1 - \mu)^2)^{1/2} \\ & + 8\beta_1 \exp(16\beta_2^2 + 8\beta_1)\nu(m_1 - \mu)^2, \end{aligned}$$

and so

$$\begin{aligned} M_N \leq {} & \frac{1 - \mu^2}{N} + 16\beta_2^2 \exp(16\beta_2^2 + 8\beta_1) M_N^{1/2} Q_N^{1/2} \\ & + 8\beta_1 \exp(16\beta_2^2 + 8\beta_1) M_N. \end{aligned} \tag{10}$$

Using the same arguments for Q_N, we obtain

$$Q_N \le \frac{1-q^2}{N} + 16\beta_2^2 \exp(16\beta_2^2 + 8\beta_1) Q_N + 8\beta_1 \exp(16\beta_2^2 + 8\beta_1) Q_N^{1/2} M_N^{1/2}. \tag{11}$$

Then the problem reduces to study the system (10)–(11). Observe that hypothesis (9) and relation $(Q_N M_N)^{1/2} \le (Q_N + M_N)/2$ yield that system (10)–(11) implies that

$$\begin{cases} M_N \le (K/N) + (Q_N + M_N)/8 + (1/4)M_N \\ Q_N \le (K/N) + (Q_N + M_N)/8 + (1/4)Q_N, \end{cases}$$

and the result follows easily. □

4. Study of the free energy.

Set

$$p_N(\beta_1, \beta_2) = \frac{1}{N}\mathbf{E} \log Z_N = \frac{1}{N}\mathbf{E}\left[\log\left(\sum_{\sigma\in\Sigma_N} \exp(-H_N(\sigma))\right)\right].$$

This quantity is closely related to the *free energy* considered by physicists, up to a scaling factor, and we call it the free energy of our system. In this section we will prove that the limit of $p_N(\beta_1, \beta_2)$, when $N \to \infty$, is the function

$$F(\beta_1, \beta_2) = \frac{\beta_2^2}{4}(1-q)^2 + \log 2 + \mathbf{E}\left(\log\cosh(\beta_2 z\sqrt{q} + \beta_1\mu + h)\right) - \frac{\beta_1\mu^2}{2}.$$

Theorem 4.1. *If β_1 and β_2 satisfy hypothesis* (9)*, we have*

$$\lim_{N\to\infty} p_N(\beta_1, \beta_2) = F(\beta_1, \beta_2).$$

Proof. Let us recall first that, under (9), (q, μ) are unique and well defined. Moreover, since they satisfy (2) it can be checked (see, for instance, [**5**, Lemma 2.4.16]) that

$$\left|\frac{\partial F}{\partial \mu}(\beta_1, \beta_2, h, q, \mu)\right| + \left|\frac{\partial F}{\partial q}(\beta_1, \beta_2, h, q, \mu)\right| = 0. \tag{12}$$

Now, if we fix β_2, we have

$$|F(\beta_1,\beta_2) - p_N(\beta_1,\beta_2)| \le |F(0,\beta_2) - p_N(0,\beta_2)| + \int_0^{\beta_1} \left| \frac{dF(x,\beta_2)}{dx} - \frac{\partial p_N(x,\beta_2)}{\partial x} \right| dx.$$

Thanks to Theorem 2.4.18 in [**5**], we know that $|F(0,\beta_2) - p_N(0,\beta_2)| \le K/N$. On the other hand, using (12), we get

$$\frac{dF}{d\beta_1} = -\frac{\mu^2}{2} + \mu \mathbf{E}(\tanh Y) = \frac{\mu^2}{2}$$

and

$$\begin{aligned}\frac{\partial p_N}{\partial \beta_1} &= \frac{1}{2N^2}\mathbf{E}\left(\frac{1}{Z_N} \sum_\sigma \left(\sum_{i=1}^N \sigma_i \right)^2 \exp(-H_N(\sigma)) \right) \\ &= \frac{1}{2N^2}\mathbf{E}\left\langle \left(\sum_{i=1}^N \sigma_i \right)^2 \right\rangle.\end{aligned}$$

The proof finishes easily using Theorem 3.1. □

Remark 4.2. The value of the free energy can also be obtained for more values of β_1, β_2 and h without the computations of the previous sections from the well-known result for the classical SK model ($\beta_1 = 0$, β_2 small and $h > 0$). Doing a Gaussian integration and using a large deviations argument, it can be obtained that

$$F(\beta_1,\beta_2) = \sup_\mu F^{SK(h+\beta_1\mu)}(\beta_2) - \frac{\beta_1\mu^2}{2},$$

where $F^{SK(h+\beta_1\mu)}(\beta_2)$ denotes the free energy for the SK model with external field $h + \beta_1\mu$. We would like to thank the referees for this observation.

5. Exponential moments. The aim of this section is to control the exponential moments of our model, that is, to obtain that:

$$\nu\left(\exp \frac{N}{L}(R_{1,2} - q)^2 \right) \le L \text{ and } \nu\left(\exp \frac{N}{L}(m_1 - \mu)^2 \right) \le L.$$

The main tool is the control the moments of order $2k$, $k > 0$, given in the following theorem.

Theorem 5.1. *For all $k \geq 0$ and for β_1 and β_2 satisfying (9) the following inequalities hold*

$$\nu\left((R_{1,2} - q)^{2k}\right) \leq \left(\frac{Lk}{N}\right)^k \quad \textit{and} \quad \nu\left((m_1 - \mu)^{2k}\right) \leq \left(\frac{Lk}{N}\right)^k.$$

Notice that these estimates will permit us to prove, at the end of this section, the following bounds

$$|\nu(R_{1,2} - q)| \leq \frac{K}{N} \quad \text{and} \quad |\nu(m_1 - \mu)| \leq \frac{K}{N}.$$

We will prove Theorem 5.1 by induction, considering that we have already proved the induction step $k = 1$ in Theorem 3.1. The induction hypothesis is, for all $l \leq k$,

$$\nu(R_{1,2} - q)^{2l} \leq \left(\frac{Ll}{N}\right)^l \quad \text{and} \quad \nu(m_1 - \mu)^{2l} \leq \left(\frac{Ll}{N}\right)^l. \tag{13}$$

To prove this theorem, anyway, we will need the following lemma, whose proof is an obvious extension of Lemma 2.5.2 in [**5**].

Lemma 5.2. *We assume (14) and $k \leq N - 1$. Then, if $L_0 \geq 4$, we have for all $j \leq 2k$,*

$$\nu(|R^-_{1,2} - q|^j) \leq \left(\frac{L_0(j+1)}{N}\right)^{j/2},$$

$$\nu(|R^-_{1,2} - q|^{2k}) \leq 3\left(\frac{L_0(k+1)}{N}\right)^k,$$

and for all $j \leq 2k$,

$$\nu(|m^-_1 - \mu|^j) \leq \left(\frac{L_0(j+1)}{N}\right)^{j/2},$$

$$\nu(|m^-_1 - \mu|^{2k}) \leq 3\left(\frac{L_0(k+1)}{N}\right)^k.$$

We can now prove Theorem 5.1.

Proof of Theorem 5.1. Let $f = (m_1 - \mu)^{2k+2}$. Symmetry implies $\nu(f) = \nu(\hat{f})$ with $\hat{f} = (\varepsilon_1 - \mu)(m_1 - \mu)^{2k+1}$. Using Proposition 2.5 with $n = 2$, $\tau_1 = \alpha_1 = (2k+2)/(2k+1)$, $\tau_2 = \alpha_2 = 2k+2$ and hypothesis (9), we have

(14)

$$\begin{aligned}
\nu(f) \leq \nu_0(\hat{f}) &+ 8\beta_2^2 \exp(16\beta_2^2 + 8\beta_1)\Big(\nu\Big((\hat{f})^{(2k+2)/(2k+1)}\Big)\Big)^{(2k+1)/(2k+2)} \\
&\times \left(\nu\left((R_{1,2} - q)^{2k+2}\right)\right)^{1/(2k+2)} \\
&+ 4\beta_1 \exp(16\beta_2^2 + 8\beta_1)\left(\nu\left((\hat{f})^{(2k+2)/(2k+1)}\right)\right)^{(2k+1)/(2k+2)} \\
&\times \nu\left(((m_1 - \mu)^{2k+2}\right))^{1/(2k+2)} \\
\leq \nu_0(\hat{f}) &+ \frac{1}{4}\left[\nu\left((m_1 - \mu)^{2k+2}\right)\right]^{(2k+1)/(2k+2)} \\
&\times \left[\nu\left((R_{1,2} - q)^{2k+2}\right)\right]^{1/(2k+2)} + \frac{1}{4}\nu\left((m_1 - \mu)^{2k+2}\right).
\end{aligned}$$

So, using that for any numbers a, $b < 1$ such that $a + b = 1$ and x, $y > 0$, we have $x^a y^b \leq x + y$, and (14) becomes

$$\nu(f) \leq 2\nu_0(\hat{f}) + \frac{1}{2}\nu\left((R_{1,2} - q)^{2k+2}\right).$$

Letting $g = (R_{1,2} - q)^{2k+2}$, $\hat{g} = (\varepsilon_1\varepsilon_2 - q)(R_{1,2} - q)^{2k+1}$, and by similar arguments we have that

$$\nu(g) \leq 2\nu_0(\widehat{g}) + \frac{1}{2}\nu(f).$$

So we have to study the inequalities

$$\begin{cases} \nu(f) \leq 2\nu_0\left((\varepsilon_1 - \mu)(m_1 - \mu)^{2k+1}\right) + \frac{1}{2}\nu(g) \\ \nu(g) \leq 2\nu_0\left((\varepsilon_1\varepsilon_2 - q)(R_{1,2} - q)^{2k+1}\right) + \frac{1}{2}\nu(f). \end{cases}$$

If we prove that

$$\begin{cases} \nu_0\left((\varepsilon_1 - \mu)(m_1 - \mu)^{2k+1}\right) \leq 32L_0^k\left((k+1)/N\right)^{k+1} \\ \nu_0\left((\varepsilon_1\varepsilon_2 - q)(R_{1,2} - q)^{2k+1}\right) \leq 32L_0^k\left((k+1)/N\right)^{k+1}, \end{cases} \tag{15}$$

the system becomes

$$\begin{cases} \nu(f) \le 64L_0^k\left((k+1)/N\right)^{k+1} + \frac{1}{2}\nu(g) \\ \nu(g) \le 64L_0^k\left((k+1)/N\right)^{k+1} + \frac{1}{2}\nu(f), \end{cases}$$

and we conclude easily choosing $L_0 = 128$.

To prove (15) we use Lemma 2.1. It implies that $\nu_0((\varepsilon_1-\mu)(m_1^- - \mu)^{2k+1}) = 0$, and hence

$$\nu_0\left((\varepsilon_1-\mu)(m_1-\mu)^{2k+1}\right) = \nu_0\left((\varepsilon_1-\mu)((m_1-\mu)^{2k+1} \right.$$
$$\left. -(m_1^- - \mu)^{2k+1})\right).$$

Using the inequality $|x^{2k+1} - y^{2k+1}| \le (2k+1)|x-y|(x^{2k}+y^{2k})$, Proposition 2.4 and relations (7), we have

$$\nu_0\left((\varepsilon_1-\mu)(m_1-\mu)^{2k+1}\right) \le 4\frac{k+1}{N}\exp(16\beta_2^2+8\beta_1)$$
$$\times\left[\nu(m_1-\mu)^{2k} + \nu(m_1^- - \mu)^{2k}\right].$$

Since (9) holds, we can assume $\exp(16\beta_2^2+8\beta_1) \le 2$. So, using Lemma 5.2 and the induction hypothesis, we have

$$\begin{aligned} \nu_0&\left((\varepsilon_1-\mu)(m_1-\mu)^{2k+1}\right) \\ &\le 8\frac{k+1}{N}\left(\left(\frac{L_0k}{N}\right)^k + 3\left(\frac{L_0(k+1)}{N}\right)^k\right) \\ &\le 32\frac{k+1}{N}\left(\frac{L_0(k+1)}{N}\right)^k. \end{aligned}$$

Similarly, we have

$$\nu_0\left((\varepsilon_1\varepsilon_2-q)(R_{1,2}-q)^{2k+1}\right) \le 32\frac{k+1}{N}\left(\frac{L_0(k+1)}{N}\right)^k. \qquad \square$$

The last goal of this section is to prove the following theorem.

Theorem 5.3. *For β_1 and β_2 satisfying (9), the following inequalities hold*

$$|\nu(R_{1,2}-q)| \le \frac{K}{N} \quad \textit{and} \quad |\nu(m_1-\mu)| \le \frac{K}{N}.$$

First, we need the following lemma, that is an extension of Corollary 2.3.4 in [**5**]:

Lemma 5.4. *For a function* $f : \Sigma_N^n \to \mathbf{R}$, *we have*

$$|\nu(f) - \nu_0(f)| \leq \frac{K(n)}{\sqrt{N}} \left(\nu(f^2)\right)^{1/2}, \tag{16}$$

$$|\nu(f) - \nu_0(f) - \nu_0'(f)| \leq \frac{K(n)}{N} \left(\nu(f^2)\right)^{1/2}. \tag{17}$$

From now on, set

$$\begin{aligned} \widehat{q} &= \mathbf{E} \tanh^4(\beta_2 z\sqrt{q} + \beta_1\mu + h) \\ \widehat{\mu} &= \mathbf{E} \tanh^3(\beta_2 z\sqrt{q} + \beta_1\mu + h). \end{aligned}$$

Proof of Theorem 5.3. Let $f = m_1 - \mu$. By symmetry, we have $\nu(f) = \nu(\varepsilon_1 - \mu)$. Thanks to Lemma 5.4, we have

$$|\nu(f) - \nu_0(\varepsilon_1 - \mu) - \nu_0'(\varepsilon_1 - \mu)| \leq \frac{K(n)}{N}, \tag{18}$$

where $\nu_0(\varepsilon_1 - \mu) = 0$ because of Lemma 2.1. To compute $\nu_0'(\varepsilon_1 - \mu)$ we use (8) with $n = 1$ and Lemma 2.1, and we get

$$\nu_0'(\varepsilon_1 - \mu) = \beta_2^2(\widehat{\mu} - \mu)\nu_0(R_{1,2}^- - q) + \beta_1(1 - q)\nu_0(m_1^- - \mu).$$

Since (16) implies that

$$\begin{cases} |\nu_0(R_{1,2}^- - q) - \nu(R_{1,2} - q)| \leq K/N \\ |\nu_0(m_1^- - \mu) - \nu(m_1 - \mu)| \leq K/N, \end{cases} \tag{19}$$

(18) becomes

$$|\nu(f) - \beta_2^2(\widehat{\mu} - \mu)\nu(R_{1,2} - q) - \beta_1(1 - q)\nu(m_1 - \mu)| \leq \frac{K(n)}{N}.$$

Reasoning analogously with $g = R_{1,2} - q$, we have

$$|\nu(R_{1,2} - q) - \beta_2^2(1 - 4q + 3\widehat{q})\nu(R_{1,2} - q) - 2\beta_1(\mu - \widehat{\mu})\nu(m_1 - \mu)| \leq \frac{K}{N}.$$

So we have to study the system

$$\begin{cases} |(1 - \beta_2^2(1 - 4q + 3\widehat{q}))\nu(R_{1,2} - q) \\ \quad -2\beta_1(\mu - \widehat{\mu})\nu(m_1 - \mu)| \leq K/N \\ |(1 - \beta_1(1 - q))\nu(m_1 - \mu) \\ \quad -\beta_2^2(\widehat{\mu} - \mu)\nu(R_{1,2} - q)| \leq K/N. \end{cases}$$

Clearly, there exist two constants L and L' such that $|L| + |L'| \leq K/N$ and

$$\begin{cases} (1 - \beta_2^2(1 - 4q + 3\widehat{q}))\nu(R_{1,2} - q) \\ \quad = 2\beta_1(\mu - \widehat{\mu})\nu(m_1 - \mu) + L \\ (1 - \beta_1(1 - q))\nu(m_1 - \mu) \\ \quad = \beta_2^2(\widehat{\mu} - \mu)\nu(R_{1,2} - q) + L'. \end{cases}$$

Thus,

$$\left[1 - \beta_1(1 - q) + \frac{2\beta_1\beta_2^2(\widehat{\mu} - \mu)^2}{1 - \beta_2^2(1 - 4q + 3\widehat{q})}\right]\nu(m_1 - \mu) = L' + \frac{L}{1 - \beta_2^2(1 - 4q + 3\widehat{q})}.$$

Since $0 \leq \widehat{q} \leq q \leq 1$, we have that $1 - 4q + 3\widehat{q} < 1$, and for $\beta_2^2 < 1/2$, we get

$$\frac{1}{2} \leq 1 - \beta_2^2(1 - 4q + 3\widehat{q}).$$

Moreover, for $\beta_1 < 1/2$, we get that

$$1 - \beta_1(1 - q) + \frac{2\beta_1\beta_2^2(\widehat{\mu} - \mu)^2}{1 - \beta_2^2(1 - 4q + 3\widehat{q})} \geq 1 - \beta_1(1 - q) > \frac{1}{2}.$$

Thus,

$$|\nu(m_1 - \mu)| \leq 4|L| + 2|L'|.$$

Using similar arguments we study $|\nu(R_{1,2} - q)|$. □

6. Regularity of the system. One way of looking at the regularity of the system when $N \to \infty$ is to investigate the limit of the laws of the random variables $(\langle \sigma_1 \rangle, \ldots, \langle \sigma_n \rangle)$. In fact, one way to study the self averaging phenomenon for the model is to show that those quantities converge to some independent and identically distributed centered random variables that can be clearly identified, by analogy with the fact that the magnetization vanishes for the Ising model at high temperature.

It turns out that the above sequence is formed by asymptotically i.i.d. random variables, and the limit law of each one of them is the law of the random variable $Y = \tanh(\beta_2 z \sqrt{q} + \beta_1 \mu + h)$, where z is as usual a standard Gaussian random variable.

The central theorem of this section reads as follows.

Theorem 6.1. *If β_1 and β_2 satisfy* (9)*, we can find independent standard Gaussian random variables $\{z_i\}_{i \leq n}$ such that*

$$\mathbf{E}\left[\sum_{i \leq n} \left(\langle \sigma_i \rangle - \tanh(\beta_2 z_i \sqrt{q} + \beta_1 \mu + h) \right)^2 \right] \leq \frac{K}{N}.$$

To prove it we need some preliminary results.

Lemma 6.2. *Denote by q_- and μ_- the solutions of* (2) *when β_1 and β_2 are replaced by β_1^- and β_2^- defined in* (3)*. Then, for β_1 and β_2 satisfying* (9)*, we have*

$$|q - q_-| \leq \frac{K}{N} \quad \textit{and} \quad |\mu - \mu_-| \leq \frac{K}{N}.$$

Proof. Clearly,

$$\begin{aligned} |q(\beta_1, \beta_2) - q(\beta_1^-, \beta_2^-)| \leq & \sup_{\beta_1, \beta_2} \left| \frac{\partial q}{\partial \beta_1} \right| |\beta_1 - \beta_1^-| \\ & + \sup_{\beta_1, \beta_2} \left| \frac{\partial q}{\partial \beta_2} \right| |\beta_2 - \beta_2^-|. \end{aligned}$$

Since system (2) can be seen as

$$\begin{cases} q = F_1(\beta_1, \beta_2, q(\beta_1, \beta_2), \mu(\beta_1, \beta_2)) \\ \mu = F_2(\beta_1, \beta_2, q(\beta_1, \beta_2), \mu(\beta_1, \beta_2)), \end{cases}$$

for $i = 1, 2$ we have

$$\frac{\partial q}{\partial \beta_i} = \frac{(\partial F_1/\partial \beta_i) + (\partial F_1/\partial \mu)(\partial \mu/\partial \beta_i)}{1 - (\partial F_1/\partial q)}$$

and

$$\frac{\partial \mu}{\partial \beta_i} = \frac{(\partial F_2/\partial \beta_i) + (\partial F_2/\partial q)(\partial q/\partial \beta_i)}{1 - (\partial F_2/\partial \mu)}.$$

Computing these derivatives, we can conclude that

$$\begin{aligned} |q(\beta_1, \beta_2) - q(\beta_1^-, \beta_2^-)| &= K(\beta_1, \beta_2)|\beta_2 - \beta_2^-| \\ &\leq \frac{K(\beta_1, \beta_2)}{N}, \\ |\mu(\beta_1, \beta_2) - \mu(\beta_1^-, \beta_2^-)| &\leq \frac{K(\beta_1, \beta_2)}{N}. \qquad \square \end{aligned}$$

Now set $e = \sum_{i \leq N} \langle \sigma_i \rangle$, $\|c\|^2 = \sum_{i \leq N-1} \langle \sigma_i \rangle_-^2$ and

$$g(c) = \frac{\beta_2}{\sqrt{N}} \sum_{i \leq N-1} g_{i,N} \langle \sigma_i \rangle_- + \frac{\beta_1}{N} \sum_{i \leq N-1} \langle \sigma_i \rangle_- . \tag{20}$$

Lemma 6.3. *We can find a standard Gaussian random variable z that depends only upon $(g_{i,j})_{i<j \leq N}$, but it is probabilistically independent of the $(g_{i,j})_{i<j \leq N-1}$, such that*

$$\mathbf{E}\left((\langle \sigma_N \rangle - \tanh(\beta_2 z \sqrt{q} + \beta_1 \mu + h))^2 \right) \leq \frac{K}{N}.$$

Proof. Let

$$z = \frac{1}{\|c\|} \sum_{i \leq N-1} g_{i,N} \langle \sigma_i \rangle_- ,$$

and let Y be the random variable defined in (5). Using the inequalities $|\tanh x - \tanh y| \le |x-y|$ and $(x+y)^2 \le 2x^2 + 2y^2$, we have
(21)
$$\begin{aligned}\mathbf{E}\Big((\langle\sigma_N\rangle - \tanh(Y))^2\Big) \le\ & 2\mathbf{E}\left((\langle\sigma_N\rangle - \tanh(g(c)+h))^2\right)\\ & + 4\mathbf{E}\Bigg(\bigg(\frac{\beta_2}{\sqrt{N}}\sum_{i\le N-1} g_{i,N}\langle\sigma_i\rangle_- - \beta_2 z\sqrt{q}\bigg)^2\Bigg)\\ & + 4\mathbf{E}\Bigg(\bigg(\frac{\beta_1}{N}\sum_{i\le N-1}\langle\sigma_i\rangle_- - \beta_1\mu\bigg)^2\Bigg).\end{aligned}$$

We will prove in Lemma 6.4 that

$$\mathbf{E}\left((\langle\sigma_N\rangle - \tanh(g(c)+h))^2\right) \le \frac{K}{N}. \tag{22}$$

Following the proof of Lemma 2.4.14 in [**5**], we can obtain

$$\mathbf{E}\Bigg(\bigg(\frac{\beta_2}{\sqrt{N}}\sum_{i\le N-1} g_{i,N}\langle\sigma_i\rangle_- - \beta_2 z\sqrt{q}\bigg)^2\Bigg) \le \frac{K}{N}. \tag{23}$$

Finally, using a similar argument we have

$$\begin{aligned}\mathbf{E}\Bigg(\bigg(\frac{\beta_1}{N}\sum_{i\le N-1}\langle\sigma_i\rangle_- - \beta_1\mu\bigg)^2\Bigg) & \\ \le\ & \beta_1^2\mathbf{E}\Bigg(\bigg(\frac{1}{N}\sum_{i\le N-1}\langle\sigma_i\rangle_- - \mu_-\bigg)^2\Bigg) + \frac{K}{N}\\ \le\ & \beta_1^2\mathbf{E}\left\langle\left(m_1^- - \mu_-\right)^2\right\rangle_- + \frac{K}{N}\\ \le\ & \frac{K}{N}.\end{aligned} \tag{24}$$

The proof finishes putting together (21), (22), (23) and (24). □

Lemma 6.4. *For β_1 and β_2 satisfying* (9) *and $g(c)$ defined in* (20), *we have*
$$\mathbf{E}\Big((\langle\sigma_N\rangle - \tanh(g(c)+h))^2\Big) \le \frac{K}{N}.$$

Remark 6.5. In the proof of Lemma 6.4 we will use Gronwall's inequality in the following way: let $g(t)$ be a function such that $g(0) \leq L/N$ and $g'(t) \leq Lg(t)$. Then $g(t) \leq L/N$.

Proof. Consider $g_t(\rho)$ defined in (4), and set $g_t(c)$ in a similar way

$$\begin{aligned} g_t(c) = {} & \frac{\sqrt{t}\beta_2}{\sqrt{N}} \sum_{i \leq N-1} \langle \sigma_i \rangle_- + \sqrt{1-t}(\beta_2 z \sqrt{q}) \\ & + \frac{t\beta_1}{N} \sum_{i \leq N-1} \langle \sigma_i \rangle_- + (1-t)\beta_1 \mu. \end{aligned}$$

We consider the function

$$\varphi(t) = \mathbf{E}\left((U(t) - V(t))^2 \right),$$

where

$$\begin{aligned} U(t) &= \langle \sigma_N \rangle_t, \\ V(t) &= \tanh(g_t(c) + h) \\ &= \frac{\langle Av\varepsilon \exp(\varepsilon(g_t(c) + h)) \rangle_-}{\langle \cosh(g_t(c) + h) \rangle_-}, \end{aligned}$$

where $U(t)$ and $V(t)$ are defined similarly, putting $g_t(c)$ instead of $g_t(\rho)$. Obviously, our aim is to check that

$$\varphi(1) \leq \frac{K}{N}.$$

Since $g_0(\rho) = g_0(c)$, we have $\varphi(0) = \mathbf{E}((U(0) - V(0))^2) = 0$ and $\varphi(1) = \varphi(1) - \varphi(0)$. Set $\varphi(t) = \varphi_1(t) - 2\varphi_2(t) + \varphi_3(t)$, where

$$\varphi_1(t) = \mathbf{E}(U^2(t)), \quad \varphi_2(t) = \mathbf{E}(U(t)V(t)), \quad \varphi_3(t) = \mathbf{E}(V^2(t)). \tag{25}$$

Then, it is enough to prove that

$$|\varphi(1) - \varphi(0)| \leq |\varphi_1(1) - \varphi_1(0)| + 2|\varphi_2(1) - \varphi_2(0)| + |\varphi_3(1) - \varphi_3(0)| \leq \frac{K}{N}.$$

We begin with the study of $\varphi_1(t)$. The same kind of computations will be useful to study $\varphi_2(t)$ and $\varphi_3(t)$, provided that we will prove that relation (17) also holds for $\varphi_2(t)$ and $\varphi_3(t)$.

Step 1. Using symmetry, we have $\varphi_1(t) = \nu_t(\varepsilon_1\varepsilon_2)$, and thanks to (17) we have that

$$|\varphi_1(1) - \varphi_1(0) - \varphi_1'(0)| \le \frac{K}{N}.$$

So it is sufficient to prove that $\varphi_1'(0) \le K/N$. Using Proposition 2.2 with $n = 2$ and $f = \varepsilon_1\varepsilon_2$ and Lemma 2.1, we have

$$\varphi_1'(0) = \beta_2^2(1 - 4q + 3\widehat{q})\nu_0(R_{1,2}^- - q) + 2\beta_1(\mu - \widehat{\mu})\nu_0(m_1^- - \mu),$$

and we can conclude using (19) and Theorem 5.3.

Step 2. Study of $\varphi_3(t)$. For a function $f : \Sigma_N^n \to \mathbf{R}$, set

$$\langle f \rangle_t' = \frac{\left\langle Avf \exp\left(\sum_{l\le n} \varepsilon_l (g_t(c) + h)\right)\right\rangle_-}{\langle \cosh^n (g_t(c) + h)\rangle_-}.$$

Using symmetry,

$$\varphi_3(t) = \mathbf{E}\,\langle \varepsilon_1\varepsilon_2\rangle_t'.$$

The only difference between $\langle\cdot\rangle_t$ and $\langle\cdot\rangle_t'$ is that, instead of $g_t(\rho^l)$, we will have $g_t(c)$. So, (8) remains valid, provided one replaces $\nu_t(\cdot)$ by $\mathbf{E}\langle\cdot\rangle_t'$, $R_{l,l'}^-$ by $\|c\|^2/N$ and m_l^- by $e_- = (1/N)\sum_{i\le N-1}\langle\sigma_i\rangle_-$. Thus,

$$\begin{aligned}
\frac{d}{dt}\mathbf{E}(\langle f\rangle_t') = {} & \beta_2^2 \sum_{1\le l<l'\le n} \mathbf{E}\left\langle f\varepsilon_l\varepsilon_{l'}\left(\frac{\|c\|^2}{N} - q\right)\right\rangle_t' \\
& - n\beta_2^2 \sum_{l\le n} \mathbf{E}\left\langle f\varepsilon_l\varepsilon_{n+1}\left(\frac{\|c\|^2}{N} - q\right)\right\rangle_t' \\
& + \beta_2^2 \frac{n(n+1)}{2}\mathbf{E}\left\langle f\varepsilon_{n+1}\varepsilon_{n+2}\left(\frac{\|c\|^2}{N} - q\right)\right\rangle_t' \\
& + \beta_1 \sum_{l\le n}\mathbf{E}\,\langle f\varepsilon_l(e_- - \mu)\rangle_t' \\
& - n\beta_1 \mathbf{E}\,\langle f\varepsilon_{n+1}(e_- - \mu)\rangle_t'.
\end{aligned} \tag{26}$$

To prove that (17) holds, we have to verify first that (16) also holds. We will use Remark 6.5 to prove that

$$\mathbf{E}\left\langle\left(\frac{\|c\|^2}{N}-q\right)^2\right\rangle_t' \le \frac{K}{N} \quad \text{and} \quad \mathbf{E}\left\langle(e_- - \mu)^2\right\rangle_t' \le \frac{K}{N}.$$

Some easy computations give that the functions

$$\psi(t)=\mathbf{E}\left\langle\left(\frac{\|c\|^2}{N}-q\right)^2\right\rangle_t' \quad \text{and} \quad \eta(t)=\mathbf{E}\left\langle(e_- - \mu)^2\right\rangle_t'$$

satisfy the hypothesis to apply the Gronwall's inequality (see Remark 6.5). Indeed, for $t=0$, $\mathbf{E}\langle\cdot\rangle_0' = \mathbf{E}\langle\cdot\rangle_0 = \nu_0(\cdot)$. Thus, thanks to Proposition 2.4 and Theorem 3.1, we have

$$\begin{aligned}\psi(0) &= \nu_0\left(\frac{1}{N}\sum_{i\le N-1}\langle\sigma_i\rangle_-^2 - q\right)^2\\ &= \nu_0\left(\left\langle R_{1,2}^- - q\right\rangle_-^2\right)\\ &\le \nu_0(R_{1,2}^- - q)^2 \le \frac{K}{N}.\end{aligned}$$

To show that $\psi'(t)\le L\psi(t)$, it is enough to use (26).

We have to prove now that $\varphi_3'(0)\le K/N$. Using (26) with $n=2$ and $f=\varepsilon_1\varepsilon_2$, we have

$$\begin{aligned}\varphi_3'(0) &= \beta_2^2(1-4q+3\hat{q})\nu_0\left(\frac{\|c\|^2}{N}-q\right)\\ &\quad + 2\beta_1(\mu-\widehat{\mu})\nu_0(e_- - \mu),\end{aligned}$$

and so $\varphi_3'(0)\le K/N$.

Step 3. To study $\varphi_2(t)$, we will do the same things as Step 2.

Set

$$\langle f\rangle_t'' = \frac{\left\langle Avf\exp\left(\sum_{1\le l\le n}\varepsilon_l(g_t(\rho^l)+h)\right)\exp(\sum_{n+1\le l'\le 2n}\varepsilon_{l'}(g_t(c)+h))\right\rangle_-}{\langle\cosh^n(g_t(\rho)+h)\cosh^n(g_t(c)+h)\rangle_-}.$$

Then

$$\varphi_2(t) = \mathbf{E}\,\langle \varepsilon_1 \varepsilon_2 \rangle_t'' .$$

Now, in the adapted version of (8) some of the terms $R^-_{l,l'}$ are replaced by $N^{-1}\sum_{i\le N-1}\sigma_i^l\langle\sigma_i\rangle_-$ and others by $\|c\|^2/N$, while the terms m_l^- are again replaced by e_-. The analysis of the problem proceeds similarly. So we have to prove that the functions $\psi(t)$, $\eta(t)$ and

$$\mathbf{E}\left\langle \left(\frac{1}{N}\sum_{i\le N-1}\sigma_i\,\langle\sigma_i\rangle_- - q\right)^2\right\rangle_t''$$

verify the hypothesis of Remark 6.5. We will verify this just for the last function, considering that, for $\psi(t)$ and $\eta(t)$, the previous reasoning holds. In this case, we have that for $t=0$, $\mathbf{E}\langle\cdot\rangle_0'' = \mathbf{E}\langle\cdot\rangle_0 = \nu_0(\cdot)$, and we obtain

$$\begin{aligned}
\mathbf{E}&\left\langle \left(\frac{1}{N}\sum_{i\le N-1}\sigma_i\,\langle\sigma_i\rangle_- - q\right)^2\right\rangle_0'' \\
&= \nu_0\left(\left(\frac{1}{N}\sum_{i\le N-1}\sigma_i\,\langle\sigma_i\rangle_- - q\right)^2\right) \\
&= \frac{1}{N^2}\nu_0\left(\left(\sum_{i\le N-1}\sigma_i\,\langle\sigma_i\rangle_-\right)^2\right) \\
&\quad - 2q\nu_0\left(\frac{1}{N}\sum_{i\le N-1}\sigma_i\,\langle\sigma_i\rangle_-\right) + q^2.
\end{aligned}$$

Notice that

$$\nu_0\left(\frac{1}{N}\sum_{i\le N-1}\sigma_i\,\langle\sigma_i\rangle_-\right) = \nu_0(R^-_{1,2}) = \nu(R_{1,2}) + \frac{K}{N} = q + \frac{K}{N},$$

thanks to (19) and Theorem 5.3. Besides, using Hôlder's inequality, we have

$$\frac{1}{N^2}\nu_0\left(\left(\sum_{i\le N-1}\sigma_1\,\langle\sigma_i\rangle_-\right)^2\right) = \nu_0(R^-_{1,2}R^-_{1,3}) \le q^2 + \frac{K}{N},$$

and so

$$\mathbf{E}\left\langle\left(\frac{1}{N}\sum_{i\le N-1}\sigma_i\left\langle\sigma_i\right\rangle_- - q\right)^2\right\rangle_0'' \le \frac{K}{N}.$$

On the other hand, it suffices to proceed as we did in Step 2 for $\psi(t)$ and $\eta(t)$, in order to obtain

$$\begin{aligned}
|\varphi_2'(0)| &\le (1+4q+3\widehat{q})\beta_2^2|\nu_0(R_{1,2}^- - q)| \\
&\quad + 8\widehat{q}\beta_2^2\left|\nu_0\left(\frac{\|c\|^2}{N} - q\right)\right| \\
&\quad + (8q+12\widehat{q})\beta_2^2\left|\nu_0\left(\frac{1}{N}\sum_{i\le N-1}\sigma_i\left\langle\sigma_i\right\rangle_- - q\right)\right| \\
&\quad + 2(\mu+\widehat{\mu})\beta_1|\nu_0(m_1^- - \mu)| \\
&\quad + 2(\mu+\widehat{\mu})\beta_1|\nu_0(e_- - \mu)| \\
&\le \frac{K}{N}. \qquad \square
\end{aligned}$$

Corollary 6.6. *For β_1 and β_2 satisfying* (9), *we have*

$$\mathbf{E}\left(\langle\sigma_1\rangle - \langle\sigma_1\rangle_-\right)^2 \le \frac{K}{N}.$$

Proof. We will proceed as in the previous proof. Set

$$\begin{aligned}
\widehat{U}(t) &= \langle\sigma_1\rangle_t \\
\widehat{V}(t) &= \langle\sigma_1\rangle_- \\
&= \frac{\langle Av\sigma_1\exp(\varepsilon(g_t(c)+h))\rangle_-}{\langle\cosh(g_t(c)+h)\rangle_-}
\end{aligned}$$

and

$$\widehat{\varphi}(t) = \mathbf{E}\left(\widehat{U}(t) - \widehat{V}(t)\right)^2.$$

Clearly, $\widehat{\varphi}(0)=0$, and we only need to prove that

$$|\widehat{\varphi}(1)-\widehat{\varphi}(0)| \leq |\widehat{\varphi_1}(1)-\widehat{\varphi_1}(0)|+2|\widehat{\varphi_2}(1)-\widehat{\varphi_2}(0)| + |\widehat{\varphi_3}(1)-\widehat{\varphi_3}(0)| \leq \frac{K}{N},$$

where we define $\widehat{\varphi_1}(t)$, $\widehat{\varphi_2}(t)$ and $\widehat{\varphi_3}(t)$ as in (25). Using symmetry we have

$$\widehat{\varphi_1}(t)=\nu_t(\sigma_1^1\sigma_1^2), \quad \widehat{\varphi_2}(t)=\mathbf{E}\left\langle \sigma_1^1\sigma_1^2\right\rangle_t'', \quad \widehat{\varphi_3}(t)=\mathbf{E}\left\langle \sigma_1^1\sigma_1^2\right\rangle_t'.$$

Notice that $\widehat{\varphi_3}(t)$ does not depend on t. So $\widehat{\varphi_3}'(0)=0$, and we just have to prove that

$$|\widehat{\varphi_1}'(0)| \leq \frac{K}{N}, \qquad |\widehat{\varphi_2}'(0)| \leq \frac{K}{N}.$$

Using (8) and Lemma 2.1 we have

$$\begin{aligned}\widehat{\varphi_1}'(0) &= \beta_2^2 q\nu_0(\sigma_1^1\sigma_1^2(R_{1,2}^- - q)) - 4\beta_2^2 q\nu_0(\sigma_1^1\sigma_1^2(R_{1,3}^- - q)) \\ &\quad + 3\beta_2^2 q\nu_0(\sigma_1^1\sigma_1^2(R_{3,4}^- - q)) + 2\beta_1\mu\nu_0(\sigma_1^1\sigma_1^2(m_1^- - \mu)) \\ &\quad - 2\beta_1\mu\nu_0(\sigma_1^1\sigma_1^2(m_3^- - \mu)).\end{aligned}$$

Observe that, from (16) we have

$$\begin{aligned}&|\nu_0(\sigma_1^1\sigma_1^2(R_{l,l'}^- - q)) - \nu(\sigma_1^1\sigma_1^2(R_{l,l'} - q))| \\ &\qquad + |\nu_0(\sigma_1^1\sigma_1^2(m_1^- - \mu)) - \nu(\sigma_1^1\sigma_1^2(m_1 - \mu))| \leq \frac{K}{N},\end{aligned}$$

and using symmetry we can write

$$\begin{aligned}\nu(\sigma_1^1\sigma_1^2(R_{l,l'} - q)) &= \nu(R_{1,2}(R_{l,l'} - q)) \\ &= \nu((R_{1,2}-q)(R_{l,l'} - q)) + q\nu(R_{l,l'} - q).\end{aligned}$$

Thus,

$$\begin{aligned}\widehat{\varphi_1}'(0) &= \beta_2^2 q\nu(\sigma_1^1\sigma_1^2(R_{1,2} - q)) - 4\beta_2^2 q\nu(\sigma_1^1\sigma_1^2(R_{1,3} - q)) \\ &\quad + 3\beta_2^2 q\nu(\sigma_1^1\sigma_1^2(R_{3,4} - q)) + 2\beta_1\mu\nu(\sigma_1^1\sigma_1^2(m_1 - \mu)) \\ &\quad - 2\beta_1\mu\nu(\sigma_1^1\sigma_1^2(m_3 - \mu)) + \frac{K}{N} \\ &= \beta_2^2 q\nu\left((R_{1,2}-q)^2\right) - 4\beta_2^2 q\nu\left((R_{1,2}-q)(R_{1,3}-q)\right) \\ &\quad + 3\beta_2^2 q\nu\left((R_{1,2}-q)(R_{3,4}-q)\right) \\ &\quad + 2\beta_1\mu\nu\left((R_{1,2}-q)(m_1-\mu)\right) \\ &\quad - 2\beta_1\mu\nu\left((R_{1,2}-q)(m_3-\mu)\right) + \frac{K}{N}.\end{aligned}$$

Then, using Cauchy-Schwarz's inequality and Theorem 3.1, $|\widehat{\varphi_1}'(0)| \le K/N$.

To prove that $\widehat{\varphi_2}'(0) \le K/N$, we proceed in a similar way. Deriving and using Lemma 2.1, we have

$$\begin{aligned}\widehat{\varphi_2}'(0) &= \beta_2^2 q \nu_0(f(R_{1,2}^- - q)) - 4\beta_2^2 q \nu_0(f(R_{1,5}^- - q)) \\ &\quad + 3\beta_2^2 q \nu_0(f(R_{5,6}^- - q)) \\ &\quad - 3\beta_2^2 q \nu_0\left(f\left(\frac{1}{N}\sum_{i\le N-1} \sigma_i^5 \langle \sigma_i \rangle_- - q\right)\right) \\ &\quad + 3\beta_2^2 q \nu_0\left(f\left(\frac{1}{N}\sum_{i\le N-1} \sigma_i^6 \langle \sigma_i \rangle_- - q\right)\right) \\ &\quad + 2\beta_1 \mu \nu_0(f(m_1^- - \mu)) - 2\beta_1 \mu \nu_0(f(m_5^- - \mu)),\end{aligned}$$

so using symmetry we have $|\widehat{\varphi}_2'(0)| \le K/N$. □

We can now prove Theorem 6.1.

Proof of Theorem 6.1. We will proceed by induction over n considering that, in Lemma 6.3, we proved the case $n = 1$. We suppose that it is true for n, and we will prove it for $n + 1$.

Let $Y_i = \beta_2 z_i \sqrt{q} + \beta_1 \mu + h$ and $Y_i^- = \beta_2^- z_i \sqrt{q_-} + \beta_1^- \mu_- + h$. We have

$$\begin{aligned}\sum_{i\le n} \mathbf{E}\left((\langle \sigma_i \rangle - \tanh(Y_i))^2\right) &\le K \sum_{i\le n} \mathbf{E}\left(\langle \sigma_i \rangle - \langle \sigma_i \rangle_-\right)^2 \\ &\quad + K \sum_{i\le n} \mathbf{E}\left(\langle \sigma_i \rangle_- - \tanh(Y_i^-)\right)^2 \\ &\quad + K \sum_{i\le n} \mathbf{E}\left(\tanh(Y_i^-) - \tanh(Y_i)\right)^2.\end{aligned}$$

From Corollary 6.6, we obtain

$$\sum_{i\le n} \mathbf{E}\left(\langle \sigma_i \rangle - \langle \sigma_i \rangle_-\right)^2 \le \frac{K(n)}{N}.$$

Using Lemma 6.2, we can follow the proof of Theorem 2.4.12 in [**5**] in order to finish the proof. □

7. Second order moments computations. A first step through central limit results is to give a more precise value to $\nu((R_{1,2}-q)^2)$, $\nu((m_1-\mu)^2)$ and $\nu((R_{1,2}-q)(m_1-\mu))$. The estimates are established by our next theorem.

Theorem 7.1.

$$\left|\nu\left((R_{1,2}-q)^2\right)-\frac{1}{N}(A_1+2B_1+E_1)\right|\le\frac{K}{N^{3/2}},$$
$$\left|\nu\left((m_1-\mu)^2\right)-\frac{1}{N}(D_1+G_1)\right|\le\frac{K}{N^{3/2}},$$
$$\left|\nu\left((R_{1,2}-q)(m_1-\mu)-\frac{1}{N}(C_1+F_1)\right)\right|\le\frac{K}{N^{3/2}},$$

where A_1, B_1, C_1, D_1, E_1 and F_1 are constants that we will define later.

We need to introduce some new notation and definitions.

Definition 7.2. Set

$$T_{l,l'}=\frac{(\sigma^l-b)\cdot(\sigma^{l'}-b)}{N};\quad T_l=\frac{(\sigma^l-b)\cdot b}{N};\quad T=\frac{b\cdot b}{N}-q;$$
$$U_l=m_l-\langle m_1\rangle\,;\quad U=\langle m_1\rangle-\mu,$$

where $b=\langle\sigma\rangle=(\langle\sigma_i\rangle)_{i\le N}$. Hence, we have $R_{l,l'}-q=T_{l,l'}+T_l+T_{l'}+T$ and $m_l-\mu=U_l+U$.

Definition 7.3. Set $A=\beta_2^2(1-4q+3\widehat{q})$, $B=\beta_1(\mu-\widehat{\mu})$, $C=\beta_1(1-q)$, $D=-2\beta_2^2(\mu-\widehat{\mu})$, $E=\beta_2^2(1-2q+\widehat{q})$, $F=\beta_1(\mu+\mu q-2\widehat{\mu})$, $G=\beta_2^2(\widehat{q}-q^2)$.

Remark 7.4. Notice that $A<1$ (see the proof of Lemma 2.8.3 in [**5**]), $C<1$ (since $q\in(0,1)$) and $E<1$ (see how to compute (2.255) in [**5**]). On the other hand, from the definition it is clear that $BD<0$. So, under hypothesis (9), the constants A_1, B_1, C_1, D_1, E_1 and F_1 appearing in Propositions 7.6 and 7.7 are well defined.

The following remark and the next two propositions yield the proof of Theorem 7.1.

Remark 7.5. Using symmetry we can prove that for all l, l', k, k',

$$(l,l') \neq (k,k'), \qquad |\nu(T_{l,l'}T_{k,k'})| = 0;$$

for all l, l', k,

$$|\nu(T_{l,l'}T_k)| + |\nu(T_{l,l'}U_k)| = 0;$$

for all l, l',

$$|\nu(T_{l,l'}T)| + |\nu(T_{l,l'}U)| = 0;$$

for all l, l', $l \neq l'$,

$$|\nu(T_l T_{l'})| = 0;$$

for all l, k, $l \neq k$,

$$|\nu(T_l U_k)| = 0;$$

for all l,

$$|\nu(T_l T)| + |\nu(T_l U)| = 0;$$

for all k, k', $k \neq k'$,

$$|\nu(U_k U_{k'})| = 0;$$

for all k,

$$|\nu(U_k T)| + |\nu(U_k U)| = 0.$$

Proposition 7.6. *If β_1 and β_2 satisfy hypothesis* (9)*, we have*

$$\left|\nu(T_{1,2}^2) - \frac{A_1}{N}\right| \leq \frac{K}{N^{3/2}}, \quad \left|\nu(T_1^2) - \frac{B_1}{N}\right| \leq \frac{K}{N^{3/2}},$$
$$\left|\nu(U_1T_1) - \frac{C_1}{N}\right| \leq \frac{K}{N^{3/2}} \quad \textit{and} \quad \left|\nu(U_1^2) - \frac{D_1}{N}\right| \leq \frac{K}{N^{3/2}},$$

where

$$A_1 = \frac{E}{\beta_2^2(1-E)},$$
$$B_1 = \frac{(1-C)(q-\widehat{q}) + BE(\mu-\widehat{\mu})}{(1-E)\left[(1-A)(1-C)-BD\right]},$$
$$C_1 = \frac{(\mu-\widehat{\mu})(1-A) + D(q-\widehat{q})}{(1-E)\left[(1-A)(1-C)-BD\right]} - \frac{BD(\mu-\widehat{\mu})}{(1-C)\left[(1-A)(1-C)-BD\right]},$$
$$D_1 = \frac{(1-q)(1-A) + D(\mu-\widehat{\mu})}{\left[(1-A)(1-C)-BD\right]} - \frac{BD^2(\mu-\widehat{\mu})}{(1-C)^2\left[(1-A)(1-C)-BD\right]}.$$

Proof. For the sake of completeness, we will give the study of $\nu(T_{1,2}^2)$. Using symmetry we can write

$$\begin{aligned}\nu(T_{1,2}^2) &= \nu\left(\frac{(\sigma^1-\sigma^2)\cdot(\sigma^3-\sigma^4)}{N}\frac{(\sigma^1-\sigma^5)\cdot(\sigma^3-\sigma^6)}{N}\right)\\ &= \frac{1}{N}\nu\left((\varepsilon_1-\varepsilon_2)(\varepsilon_3-\varepsilon_4)(\varepsilon_1-\varepsilon_5)(\varepsilon_3-\varepsilon_6)\right)\\ &\quad+\nu\left((\varepsilon_1-\varepsilon_2)(\varepsilon_3-\varepsilon_4)f^-\right)\end{aligned} \tag{27}$$

where $f^- = R_{1,3}^- - R_{5,3}^- - R_{1,6}^- + R_{5,6}^-$. Moreover, we have

$$\nu_0\left((\varepsilon_1-\varepsilon_2)(\varepsilon_3-\varepsilon_4)(\varepsilon_1-\varepsilon_5)(\varepsilon_3-\varepsilon_6)\right) = 1-2q+\widehat{q}$$

and $(\nu((f^-)^4))^{1/4} \leq K/\sqrt{N}$. Using Lemma 7.8 and (27), we get that

$$\begin{aligned}&\nu\left(f^-(R_{1,3}-R_{5,3}-R_{1,6}+R_{5,6})\right)\\ &\qquad= \frac{1}{E}\nu\left((\varepsilon_1-\varepsilon_5)(\varepsilon_3-\varepsilon_6)f^-\right) + \frac{K}{N^{3/2}}\\ &\qquad= -\frac{1}{NE}\nu\left((\varepsilon_1-\varepsilon_2)(\varepsilon_3-\varepsilon_4)(\varepsilon_1-\varepsilon_5)(\varepsilon_3-\varepsilon_6)\right)\\ &\qquad\quad+\frac{1}{E}\nu(T_{1,2}^2) + \frac{K}{N^{3/2}}\\ &\qquad= -\frac{1-2q+\widehat{q}}{NE} + \frac{1}{E}\nu(T_{1,2}^2) + \frac{K}{N^{3/2}}.\end{aligned}$$

We conclude by noting that

$$\begin{aligned}\Big|\nu\left(f^-(R_{1,3}-R_{5,3}-R_{1,6}+R_{5,6})\right) \\ -\nu\left((R_{1,3}-R_{5,3}-R_{1,6}+R_{5,6})^2\right)\Big| \le \frac{K}{N^{3/2}}\end{aligned}$$

implies

$$\left|\frac{1-E}{E}\nu(T_{1,2}^2)-\frac{1-2q+\widehat{q}}{NE}\right| \le \frac{K}{N^{3/2}}. \qquad \square$$

In a similar way we can prove the next proposition.

Proposition 7.7. *If β_1 and β_2 satisfy hypothesis* (9), *we have*

$$\left|\nu(T^2)-\frac{E_1}{N}\right| \le \frac{K}{N^{3/2}},$$
$$\left|\nu(UT)-\frac{F_1}{N}\right| \le \frac{K}{N^{3/2}}$$

and

$$\left|\nu(U^2)-\frac{G_1}{N}\right| \le \frac{K}{N^{3/2}}$$

where E_1, F_1 and G_1 satisfy

$$\begin{aligned}&[(1-A)(1-C)-FD]\,E_1 \\ &\qquad = (\widehat{q}-q^2)(1-C) \\ &\qquad\quad + \beta_2^2\left((1-C)(\widehat{q}-q^2)+2F(\widehat{\mu}-\mu q)\right)A_1 \\ &\qquad\quad + 2\beta_2^2\left((1-C)(q^2+2q-3\widehat{q}+\frac{2F^2}{\beta_1})\right)B_1 \\ &\qquad\quad + 2\beta_1\left((1-C)(\widehat{\mu}-\mu q)+2F(q-\mu^2)\right)C_1,\end{aligned}$$

$$\begin{aligned}(1-C)F_1 &= \widehat{\mu}-\mu q+\beta_2^2(\widehat{\mu}-\mu q)A_1 \\ &\quad + 2\beta_2^2(\mu+\mu q-2\widehat{\mu})B_1 \\ &\quad + 2\beta_1(q-\mu^2)C_1+\beta_2^2(\widehat{\mu}-\mu)E_1,\end{aligned}$$

$$
\begin{aligned}
(1-C)G_1 = q - \mu^2 &+ \beta_2^2(\mu + \mu q - 2\widehat{\mu})C_1 \\
&+ \beta_1(q-\mu^2)D_1 + \beta_2^2(\widehat{\mu}-\mu)F_1.
\end{aligned}
$$

The proof of the previous propositions are based in the following lemmas that are extensions of the equivalent results for the SK model given in Propositions 2.6.3 and 2.6.8 in [**5**].

Lemma 7.8. *Let* $f^- : \Sigma_{N-1}^n \to \mathbf{R}$. *Then*

$$
\nu_0'\left((\varepsilon_1-\varepsilon_2)(\varepsilon_3-\varepsilon_4)f^-\right) = E\nu_0\left(f^-(R_{1,3}^- - R_{1,4}^- - R_{2,3}^- + R_{2,4}^-)\right)
$$

and

$$
\begin{aligned}
\left|\nu\left((\varepsilon_1-\varepsilon_2)(\varepsilon_3-\varepsilon_4)f^-\right) - E\nu\left(f^-(R_{1,3} - R_{1,4} - R_{2,3} + R_{2,4})\right)\right| \\
\leq \frac{K(n)}{N}\left(\nu\left((f^-)^4\right)\right)^{1/4}.
\end{aligned}
$$

Lemma 7.9. *Let* $f^- : \Sigma_{N-1}^n \to \mathbf{R}$. *Then*

$$
\begin{aligned}
\nu_0'\left((\varepsilon_1-\varepsilon_2)\varepsilon_3 f^-\right) &= \beta_2^2(1-q)\nu_0\left(f^-(R_{1,3}^- - R_{2,3}^-)\right) \\
&+ \beta_2^2(q-\widehat{q})\sum_{4\leq l\leq n}\nu_0\left(f^-(R_{1,l}^- - R_{2,l}^-)\right) \\
&- n\beta_2^2(q-\hat{q})\nu_0\left(f^-(R_{1,n+1}^- - R_{2,n+1}^-)\right) \\
&+ \beta_1(\mu-\widehat{\mu})\nu_0\left(f^-(m_1^- - m_2^-)\right).
\end{aligned}
$$

Besides, if f^- *does not depend on* ρ^3, *we have*

$$
\begin{aligned}
\Big|\nu\left((\varepsilon_1-\varepsilon_2)\varepsilon_3 f^-\right) &- \beta_2^2(1-4q+3\widehat{q})\nu(f^-(R_{1,3}-R_{2,3})) \\
&- \beta_2^2(q-\widehat{q})\sum_{4\leq l\leq n}\nu(f^-(R_{1,l}-R_{2,l}-R_{1,n+1}+R_{2,n+1})) \\
&- \beta_1(\mu-\widehat{\mu})\nu(f^-(m_1-m_2))\Big| \\
&\leq \frac{K}{N}\left(\nu\left((f^-)^4\right)\right)^{1/4}.
\end{aligned}
$$

Lemma 7.10. *Let $f^- : \Sigma_{N-1}^n \to \mathbf{R}$. Then*

$$\begin{aligned}\nu_0'\left((\varepsilon_1-\varepsilon_2)f^-\right) = \beta_2^2(\mu-\widehat{\mu}) &\sum_{3\le l\le n} \nu_0\left(f^-(R^-_{1,l}-R^-_{2,l})\right)\\ &- n\beta_2^2(\mu-\widehat{\mu})\nu_0\left(f^-(R^-_{1,n+1}-R^-_{2,n+1})\right)\\ &+\beta_1(1-q)\nu_0\left(f^-(m_1^- - m_2^-)\right).\end{aligned}$$

Besides, if f^- does not depend on ρ^3, we have

$$\begin{aligned}\Big|\nu\left((\varepsilon_1-\varepsilon_2)f^-\right) &+ 2\beta_2^2(\mu-\widehat{\mu})\nu(f^-(R_{1,3}-R_{2,3}))\\ &-\beta_2^2(\mu-\widehat{\mu})\sum_{4\le l\le n}\nu(f^-(R_{1,l}-R_{2,l}-R_{1,n+1}+R_{2,n+1}))\\ &-\beta_1(1-q)\nu(f^-(m_1-m_2))\Big| \le \frac{K}{N}\left(\nu\left((f^-)^4\right)\right)^{1/4}.\end{aligned}$$

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