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THE RATLIFF-RUSH CLOSURE OF INITIAL IDEALS OF CERTAIN PRIME IDEALS

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ABSTRACT. Let K be a field, and let $m_0, \dots, m_n$ be an almost arithmetic sequence of positive integers. Let C be a monomial curve in the affine $(n+1)$-space, defined parametrically by $x_0 = t^{m_0}, \dots, x_n = t^{m_n}$. In this article we prove that the initial ideal of the defining ideal of C is Ratliff-Rush closed.

The Ratliff-Rush closure. Let R be a commutative Noetherian ring with unity and I a regular ideal in R, that is, an ideal that contains a nonzero divisor. Then the ideals of the form $I^{n+1} : I^n = \{x \in R \mid xI^n \subseteq I^{n+1}\}$ give the ascending chain $I : I^0 \subseteq I^2 : I^1 \subseteq \cdots \subseteq I^n : I^{n+1} \subseteq \cdots$. Let us denote

$$\widetilde{I} = \bigcup_{n \geq 1} (I^{n+1} : I^n).$$

As R is Noetherian, $\widetilde{I} = I^{n+1} : I^n$ for all sufficiently large n. Ratliff and Rush [**8**, Theorem 2.1] proved that $\widetilde{I}$ is the unique largest ideal for which $(\widetilde{I})^n = I^n$ for sufficiently large n. The ideal $\widetilde{I}$ is called the Ratliff-Rush *closure* of I, and I is called *Ratliff-Rush closed* if $I = \widetilde{I}$. It is easy to see that $I \subseteq \widetilde{I}$ and that an element of $(I^n : I^{n+1})$ is an integral over I. Hence, for all regular ideals I,

$$I \subseteq \widetilde{I} \subseteq \bar{I} \subseteq \sqrt{I},$$

where $\bar{I}$ is the integral closure of I. Thus, all radical and integrally closed regular ideals are Ratliff-Rush closed. But there are many ideals which are Ratliff-Rush closed but not integrally closed. For example, the ideal $I = (x^2, y^2) \subset k[x, y]$ is clearly not integrally closed as $xy \in \bar{I}$. Note that if $(xy)\, I^n \subseteq I^{n+1}$ for some n, then by the x-degree count we must have $(xy)\left(y^2\right)^n \in \left(y^2\right)^{n+1}$ which contradicts the y-degree count. Hence $xy \notin \widetilde{I}$. As $\widetilde{I} \subseteq \bar{I} = (x^2, xy, y^2)$, then I is Ratliff-Rush closed.

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Rossi and Swanson [**9**] examine the behavior of the Ratliff-Rush closure with respect to some properties such as the Ratliff-Rush closure of powers of ideals. They established new classes of ideals for which all the powers are Ratliff-Rush closed. They also show that the Ratliff-Rush closure does not behave well under certain operations, such as, taking powers of ideals, leading terms ideals, and the minimal number of generators. They present many examples illustrating the different behaviors of the Ratliff-Rush closure.

As yet, there is no algorithm to compute the Ratliff-Rush closure for regular ideals in general. To compute $\cup_n(I^{n+1} : I^n)$ we need to find a positive integer N such that $\cup_n(I^{n+1} : I^n) = I^{N+1} : I^N$. However, $I^{n+1} : I^n = I^{n+2} : I^{n+1}$ does not imply that $I^{n+1} : I^n = I^{n+3} : I^{n+2}$ (see Example 1.8 in [**9**]). Several different approaches have been used to decide the Ratliff-Rush closure; Heinzer et al. [**4**] established that a regular ideal I (and also every power of I) is Ratliff-Rush closed if and only if the associated graded ring, $\mathrm{gr}_I(R) = \oplus_{n\geq 0} I^n/I^{n+1}$, has a nonzero divisor (has positive depth). Elias [**3**] established a procedure for computing the Ratliff-Rush closure of $\mathbf{m}$-primary ideals of a Cohen-Macaulay local ring with maximal ideal $\mathbf{m}$. Al-Ayyoub [**2**] produced an algorithm for computing the Ratliff-Rush closure of (x, y)-primary ideals in the polynomial ring $K[x, y]$ with K a field.

From the definition, it is clear that the Ratliff-Rush closure of a monomial ideal is a monomial ideal, and this makes some computations easier. The following two theorems and proposition serve us as a technique to compute the Ratliff-Rush closure of the monomial ideals of interest in this article.

Lemma 1.1 [**4**, Property 1.7]. *Let R, S be Noetherian rings. Assume R is a faithfully flat S-algebra and $I \subset S$ an ideal. Then* $\mathbf{R}$ *is Ratliff-Rush closed in R if and only if I is Ratliff-Rush closed in S.*

Proof. The proof follows directly from Theorems 7.4 and 7.5 of [**5**]. □

In the proof of the main theorem of the paper we need the following proposition which is a special case of the above lemma.

Proposition 1.2. *Let $R = K[x_0, \dots, x_n]$ and $S = K[x_0, \dots, x_m]$ with $m \le n$ where K is a field. Let $I \subset S$ be an ideal. Then* $\mathbf{R}$ *is Ratliff-Rush closed in R if and only if I is Ratliff-Rush closed in S.*

Theorem 1.3. *Let I be an ideal in the polynomial ring $R = K[x_0, \dots, x_n]$ with K a field. Let $r \ge 1$. If I is primary to $(x_r, \dots, x_n)$ and $\widetilde{I} \cap (I : (x_r, \dots, x_n)) \subseteq I$, then I is Ratliff-Rush closed.*

Proof. Assume I is not Ratliff-Rush closed. Let m be an element such that $m \in \widetilde{I} \backslash I$. As I is primary to $(x_r, \dots, x_n)$, then there exists an integer k such that $(x_r, \dots, x_n)^k \subseteq I$. In particular, $(x_r, \dots, x_n)^l m \subseteq I$ for some l. Choose $l \ge 1$ the smallest possible such integer. Then $(x_r, \dots, x_n)^{l-1} m \nsubseteq I$. Let $m' \in (x_r, \dots, x_n)^{l-1}$ be a monomial such that $m'm \notin I$. Then $(x_r, \dots, x_n) m'm \subseteq (x_r, \dots, x_n)^l m \subseteq I$. Thus, $m'm \in I : (x_r, \dots, x_n)$ and $m'm \in \widetilde{I}$ as $m \in \widetilde{I}$. Therefore, $m'm \in \widetilde{I} \cap (I : (x_r, \dots, x_n)) \backslash I$. □

The defining ideals of certain monomial curves. Let $n \ge 2$, K a field, and let $x_0, \dots, x_n, t$ be indeterminates. Let $m_0, \dots, m_n$ be an almost arithmetic sequence of positive integers, that is, some $n-1$ of these form an arithmetic sequence, and assume $\gcd(m_0, \dots, m_n) = 1$. Let P be the kernel of the K-algebra homomorphism $\eta : K[x_0, \dots, x_n] \to K[t]$, defined by $\eta(x_i) = t^{m_i}$. A set of generators for the ideal P was explicitly constructed in [**7**]. We call these generators the *"Patil-Singh generators"*. Al-Ayyoub [**1**] proved that Patil-Singh generators form a Groebner basis for the prime ideal P with respect to the grevlex monomial order using the grading $\mathrm{wt}\,(x_i) = m_i$ with $x_0 < x_1 < \dots < x_n$ (in this case

$$\prod_{i=0}^{n} x_i^{a_i} >_{\text{grevlex}} \prod_{i=0}^{n} x_i^{b_i}$$

if in the ordered tuple $(a_0 - b_0, a_1 - b_1, \dots, a_n - b_n)$ the left-most nonzero entry is negative).

We first introduce some notation and terminology that [**7**] used in their construction of the generating set for the ideal P. Let $n \ge 2$ be an integer, and let $p = n-1$. Let $m_0, \dots, m_p, m_n$ be an almost arithmetic sequence of positive integers and $\gcd(m_0, \dots, m_n) = 1$, $0 < m_0 < \dots < m_p$, and m_n arbitrary. Let Γ denote the numerical semi-group that is minimally generated by $m_0, \ldots, m_p, m_n$, i.e., $\Gamma = \sum_{i=0}^{n} \mathbf{N_0} m_i$ where $\mathbf{N_0} = \mathbf{N} \cup \{\mathbf{0}\}$. Put $\Gamma' = \sum_{i=0}^{p} \mathbf{N_0} m_i$ and $\Gamma = \Gamma' + \mathbf{N_0} m_n$.

Notation 1.4. For $c, d \in \mathbf{Z}$ let $[c, d] = \{t \in \mathbf{Z} \mid c \leq t \leq d\}$. For $t \geq 0$, let $q_t \in \mathbf{Z}$, $r_t \in [1, p]$ and $g_t \in \Gamma'$be defined by $t = q_t p + r_t$ and $g_t = q_t m_p + m_{r_t}$.

Let $S = \{\gamma \in \Gamma \mid \gamma - m_0 \notin \Gamma\}$. The following is a part of Lemma (1.6) given in [**6**] that gives an explicit description of S.

Lemma 1.5 [**6**, Lemma 1.6]. *Let* $u = \min\{t \geq 0 \mid g_t \notin S\}$ *and* $v = \min\{b \geq 1 \mid b m_n \in \Gamma'\}$. *Then there exist unique integers* $w \in [0, v-1]$, $z \in [0, u-1]$, $\lambda \geq 1$, $\mu \geq 0$, *and* $\nu \geq 2$ *such that*

(i) $g_u = \lambda m_0 + w m_n$

(ii) $v m_n = \mu m_0 + g_z$

(iii)

$$g_{u-z} + (v - w) m_n = \begin{cases} (\lambda + \mu + 1)\, m_0 & \text{if } r_{u-z} < r_u; \\ (\lambda + \mu)\, m_0 & \text{if } r_{u-z} \geq r_u. \end{cases}$$

Notation 1.6. Let $q = q_u$, $r = r_u$. For the rest of this article the symbols $q, r, u, v, w, z, \lambda$ and μ will have the meaning assigned to them by the lemma and the notations above.

Let

$$\varepsilon = \begin{cases} 0 & \text{if } r > r_z; \\ 1 & \text{if } r \leq r_z, \end{cases}$$

We state Patil-Singh generators as follows:

$$\varphi_i = x_{i+r} x_p^{q} - x_0^{\lambda-1} x_i x_n^{w}, \qquad \text{for} \quad 0 \leq i \leq p - r;$$

$$\psi_j = x_{\varepsilon p + r - r_z + j} x_p^{q - q_z - \varepsilon} x_n^{v-w} - x_0^{\lambda+\mu-\varepsilon} x_j, \quad \text{for} \quad j \in [0\ , (1-\varepsilon)p + r_z - r]\,;$$

$$\theta = x_n^{v} - x_0^{\mu} x_{r_z} x_p^{q_z}$$

$$\alpha_{i,j} = x_i x_j - x_{i-1} x_{j+1} \qquad \text{for} \quad 1 \leq i \leq j \leq p - 1.$$

Theorem 1.7 [**1**, Theorem 2.11]. *The set* $\{\varphi_i \mid 0 \le i \le p-r\} \cup \{\theta\} \cup \{\alpha_{i,j} \mid 1 \le i \le j \le p-1\} \cup \{\psi_j \mid 0 \le j \le (1-\varepsilon)p + r_z - r\}$ *forms a Groebner basis for the ideal* P *with respect to the grevlex monomial order with* $x_0 < x_1 < \cdots < x_n$ *and with the grading* $wt(x_i) = m_i$.

The main result. In this section we prove that the initial ideal $\operatorname{in} P$, of the defining ideal of the monomial curves introduced above, is Ratliff-Rush closed. The previous section states a Groebner basis for the defining ideal P with respect to the grevlex monomial order with the grading $\operatorname{wt}(x_i) = m_i$ with $x_0 < x_1 < \cdots < x_n$. Therefore, $\operatorname{in} P$ is generated by the following monomials

$$\begin{aligned}
&x_i x_p^q, && \text{for} \quad i \in [r, p]\,;\\
&x_j x_p^{q-q_z-\varepsilon} x_n^{\upsilon-w}, && \text{for} \quad j \in [\varepsilon p + r - r_z, p]\,;\\
&x_n^{\upsilon},\\
&x_i x_j, && \text{for} \quad 1 \le i \le j \le p-1.
\end{aligned}$$

Now we state the main result of the article.

Theorem 1.8. *Let* P *be the defining ideal of the monomial curves as defined before. Then the ideal* $\operatorname{in} P$ *is Ratliff-Rush closed.*

Here is an outline for the proof of Theorem 1.8: from the generators above, it is clear that the monomial ideal $\operatorname{in} P$ is primary to $(x_1, \ldots, x_n)$. Therefore, we can use Theorem 1.3 to prove that $(\operatorname{in} P)\, R$ is Ratliff-Rush closed in the polynomial ring $R = K[x_1, \ldots, x_n]$, and hence by Proposition 1.2, Ratliff-Rush closed in the polynomial ring $K[x_0, \ldots, x_n]$. In order to establish the details of this outline, we need to compute $(\operatorname{in} P : (x_1, \ldots, x_n))/\operatorname{in} P$. The following proposition is the first step in doing so.

Proposition 1.9. *With notation as before, then*

$$(\operatorname{in} P : (x_1, \ldots, x_{p-1}))/\operatorname{in} P = (\overline{x}_1, \ldots, \overline{x}_{p-1}),$$

where $\overline{x}_i$ *is the image of* x_i *in the ring* $R/\operatorname{in} P$.

Proof. Let $\lambda = \min\{r, \varepsilon p + r - r_z\}$, and let $\sigma = \max\{r, \varepsilon p + r - r_z\}$. Note that $(\operatorname{in} P : (x_i))/\operatorname{in} P = (\overline{x}_1, \dots, \overline{x}_{p-1})$ for $1 \le i < \lambda$, and $(\operatorname{in} P : (x_i))/\operatorname{in} P = (\overline{x}_1, \dots, \overline{x}_{p-1}, \varepsilon x_p^q, (1-\varepsilon)\overline{x}_p^{q-q_z-\varepsilon}\overline{x}_n^{\upsilon-w})$ for $\lambda \le i < \sigma$. Also note that $(\operatorname{in} P : (x_i))/\operatorname{in} P = (\overline{x}_1, \dots, \overline{x}_{p-1}, \overline{x}_p^q, \overline{x}_p^{q-q_z-\varepsilon}\overline{x}_n^{\upsilon-w})$ for $\sigma < i \le p-1$. Hence, it follows that $(\operatorname{in} P : (x_1, \dots, x_{p-1}))/\operatorname{in} P = \cap_{i=1}^{p-1}(\operatorname{in} P : (x_i))/\operatorname{in} P = (\overline{x}_1, \dots, \overline{x}_{p-1})$. □

Notation 1.10. To simplify notation, in the sequel if a monomial happens to have an indeterminate with a negative exponent, then that monomial is treated as 0. For example, $x_1^{-2}x_3 + x_2^2 - x_1x_3$ is $x_2^2 - x_1x_3$.

Proposition 1.11. *Let* $p = n-1$ *as before, then* $(\operatorname{in} P : (x_1, \dots, x_p))/\operatorname{in} P$ *is minimally generated in* $K[x_1, \dots, x_n]/\operatorname{in} P$ *by* $\{\overline{x}_i\overline{x}_p^q \mid 1 \le i \le r-1\} \cup \{\overline{x}_i\overline{x}_p^{q-1} \mid r \le i \le p-1\} \cup \{\overline{x}_i\overline{x}_p^{q-q_z-\varepsilon}\overline{x}_n^{\upsilon-w} \mid 1 \le i \le \varepsilon p + r - r_z - 1\} \cup \{\overline{x}_i\overline{x}_p^{q-q_z-\varepsilon-1}\overline{x}_n^{\upsilon-w} \mid \varepsilon p + r - r_z \le i \le p-1\}$.

Proof. We need to compute $\left(\cap_{i=1}^{p-1}(\operatorname{in} P : (x_i))/\operatorname{in} P\right) \cap (\operatorname{in} P : (x_p))/\operatorname{in} P$. Note that $(\operatorname{in} P : (x_p))/\operatorname{in} P$ is minimally generated by the following set of monomials

$$\left\{\overline{x}_{\varepsilon p + r - r_z}\overline{x}_p^{q-q_z-\varepsilon-1}\overline{x}_n^{\upsilon-w}, \dots, \overline{x}_{p-1}\overline{x}_p^{q-q_z-\varepsilon-1}\overline{x}_n^{\upsilon-w}, \overline{x}_p^{q-q_z-\varepsilon}\overline{x}_n^{\upsilon-w}\right\} \cup \left\{\overline{x}_r\overline{x}_p^{q-1}, \dots, \overline{x}_{p-1}\overline{x}_p^{q-1}, \overline{x}_p^q\right\}.$$

As the intersection of two monomial ideals is generated by the least common multiple of the monomial generators of each of the two ideals, then the proposition follows by Proposition 1.9. □

We next compute $(\operatorname{in} P : (x_n))/\operatorname{in} P$. For the sake of notation we do so in two cases. Also, at the same time we will prove Theorem 1.8 for each of these cases separately. With the notation from the previous section, consider the following two cases. Case 1: $\varepsilon > 0$ or $q_z > 0$, and Case 2: $\varepsilon = q_z = 0$.

Case 1: $\varepsilon > 0$ *or* $q_z > 0$. In this case $\operatorname{in} P$ is generated by the following set of monomials

$$\begin{array}{lll} x_i x_p^q, & \text{for} & r \le i \le p; \\ x_j x_p^{q-q_z-\varepsilon} x_n^{\upsilon-w}, & \text{for} & \varepsilon p + r - r_z \le j \le p; \\ x_n^{\upsilon}, & & \\ x_i x_j, & \text{for} & 1 \le i \le j \le p-1. \end{array}$$

Therefore, $(\operatorname{in} P : (x_n))/\operatorname{in} P$ is minimally generated by

$$\{\overline{x}_{p+r-r_z}\overline{x}_p^{q-q_z-\varepsilon}\overline{x}_n^{\upsilon-w-1}, \dots, \overline{x}_{p-1}\overline{x}_p^{q-q_z-\varepsilon}\overline{x}_n^{\upsilon-w-1}\} \\ \cup \left\{\overline{x}_p^{q-q_z-\varepsilon+1}\overline{x}_n^{\upsilon-w-1}\right\} \\ \cup \left\{\overline{x}_n^{\upsilon-1}\right\}.$$

As the intersection of two monomial ideals is generated by the least common multiple of the monomial generators of each of the two ideals, then by Proposition 1.11 it is straightforward to compute that $\operatorname{in} P : ((x_1, \dots, x_n))/\operatorname{in} P = (\cap_{i=1}^n (\operatorname{in} P : (x_i))/\operatorname{in} P = (\cap_{i=1}^p (\operatorname{in} P : (x_i))/\operatorname{in} P \cap (\operatorname{in} P : (x_n))/\operatorname{in} P$ is generated by the monomials in the set $\varrho \cup \chi$, where $\varrho = \{\overline{x}_i \overline{x}_p^{q-q_z-\varepsilon-1}\overline{x}_n^{\upsilon-1} \mid \varepsilon p + r - r_z \le i \le p-1\}$ and χ consists of the following monomials

$$\begin{array}{lll} \overline{x}_i\overline{x}_p^q\overline{x}_n^{\upsilon-w-1}, & \text{for} & 1 \le i \le r-1; \\ \delta_{q_z 0}\overline{x}_i\overline{x}_p^{q-1}\overline{x}_n^{\upsilon-w-1}, & \text{for} & r \le i \le \varepsilon p + r - r_z - 1; \\ \overline{x}_i\overline{x}_p^{q-1}\overline{x}_n^{\upsilon-w-1}, & \text{for} & \varepsilon p + r - \varepsilon r_z \le i \le p-1; \\ \overline{x}_i\overline{x}_p^{q-q_z-\varepsilon}\overline{x}_n^{\upsilon-1}, & \text{for} & 1 \le i \le \varepsilon p + r - r_z - 1. \end{array}$$

Therefore, the preimages of the monomials in $\varrho \cup \chi$ are the only monomials in $(\operatorname{in} P : (x_1, \dots, x_p)) \setminus \operatorname{in} P$ in the ring $K[x_1, \dots, x_n]$. By Theorem 1.3 we prove that $\operatorname{in} P$ is Ratliff-Rush closed by showing that none of these monomials belongs to the Ratliff-Rush closure $\widetilde{\operatorname{in} P}$ of $\operatorname{in} P$. We show this separately for the monomials in ϱ and the monomials in χ. First, assume $\overline{x}_i\overline{x}_p^{q-q_z-\varepsilon-1}\overline{x}_n^{\upsilon-1} \in \varrho$ is in $\widetilde{\operatorname{in} P}$ for $\varepsilon p + r - r_z \le i \le p-1$. Then, by the definition of the Ratliff-Rush closure, we must have $\overline{x}_i\overline{x}_p^{q-q_z-\varepsilon-1}\overline{x}_n^{\upsilon-1}(x_i^2)^m \in (\operatorname{in} P)^{m+1}$ for some $m \ge 1$. By degree count for x_p and x_n, we must have

$$\overline{x}_i\overline{x}_p^{q-q_z-\varepsilon-1}\overline{x}_n^{\upsilon-1}(x_i^2)^m \in (x_i^2)^{m+1},$$

a contradiction by the x_i degree count.

Now assume that $x_i x_p^a x_n^b$ is a monomial in χ ($a \leq q$ and $b < \upsilon$) such that $x_i x_p^a x_n^b \in \widetilde{\operatorname{in} P}$. Then $x_i x_p^a x_n^b\ (x_i^2)^m \in (\operatorname{in} P)^{m+1}$ for some $m \geq 1$. By x_n and x_i-degree count for $1 \leq i \leq p-1$, we must have $x_i^{2m+1} x_p^a x_n^b \in (\delta_{i \geq r} x_i x_p^q, \delta_{i \geq \varepsilon p + r - r_z} x_i x_p^{q-q_z-\varepsilon} x_n^{\upsilon - w})^{m+1}$. Note if $a = q$ then we must have $i < r$; thus, $x_i^{2m+1} x_p^a x_n^b \in (\delta_{i \geq \varepsilon p + r - r_z} x_i x_p^{q-q_z-\varepsilon} x_n^{\upsilon - w})^{m+1}$. Assume $a < q$. Then $x_i^{2m+1} x_p^a x_n^b \notin (\delta_{i \geq r} x_i x_p^q)$; hence, $x_i^{2m+1} x_p^a x_n^b \in (\delta_{i \geq \varepsilon p + r - r_z} x_i x_p^{q-q_z-\varepsilon} x_n^{\upsilon - w})^{m+1}$. In either case it implies that $i \geq \varepsilon p + r - r_z$ and $b \geq \upsilon - w$. But there are no such monomials in χ.

Case 2: $\varepsilon = q_z = 0$. In this case $\operatorname{in} P$ is minimally generated by the following set of monomials

$$\begin{aligned} &x_i x_p^q, && \text{for} \quad r \leq i \leq p; \\ &x_j x_p^q x_n^{\upsilon - w}, && \text{for} \quad r - r_z \leq j \leq r-1; \\ &x_n^{\upsilon}, && \\ &x_i x_j, && \text{for} \quad 1 \leq i \leq j \leq p-1. \end{aligned}$$

Therefore, $(\operatorname{in} P : (x_n))/\operatorname{in} P$ is minimally generated by

$$\left\{\overline{x}_{r-r_z} \overline{x}_p^q \overline{x}_n^{\upsilon - w - 1}, \dots, \overline{x}_{r-1} \overline{x}_p^q \overline{x}_n^{\upsilon - w - 1}\right\} \cup \left\{\overline{x}_n^{\upsilon - 1}\right\}.$$

By Proposition 1.11 it follows that $\operatorname{in} P : ((x_1, \dots, x_n))/\operatorname{in} P = (\cap_{i=1}^n (\operatorname{in} P : (x_i))/\operatorname{in} P = (\cap_{i=1}^p (\operatorname{in} P : (x_i))/\operatorname{in} P \cap (\operatorname{in} P : (x_n))/\operatorname{in} P$ is generated by the monomials in the set $\varrho \cup \chi$, where $\varrho = \{\overline{x}_i \overline{x}_p^{q-1} \overline{x}_n^{\upsilon - 1} \mid r - r_z \leq i \leq p-1\}$ and χ consists of the following monomials

$$\begin{aligned} &\overline{x}_i \overline{x}_p^q \overline{x}_n^{\upsilon - 1}, && \text{for} \quad 1 \leq i \leq r - r_z - 1; \\ &\overline{x}_i \overline{x}_p^q \overline{x}_n^{\upsilon - w - 1}, && \text{for} \quad r - r_z \leq i \leq r-1; \end{aligned}$$

Therefore, the preimages of the monomials in $\varrho \cup \chi$ are the only monomials in $(\operatorname{in} P : (x_1, \dots, x_p))\backslash \operatorname{in} P$ in the ring $K[x_1, \dots, x_n]$. By Theorem 1.3 we prove that $\operatorname{in} P$ is Ratliff-Rush closure by showing that none of these monomials belongs to the Ratliff-Rush closure $\widetilde{\operatorname{in} P}$ of $\operatorname{in} P$. We show this separately for the monomials in ϱ and the monomials in χ. First, assume $\overline{x}_i \overline{x}_p^{q-1} \overline{x}_n^{\upsilon - 1} \in \varrho$ is in $\widetilde{\operatorname{in} P}$ for $r - r_z \leq i \leq p-1$. Then, by the definition of the Ratliff-Rush closure, we must have $\overline{x}_i\overline{x}_p^{q-q_z-\varepsilon-1}\overline{x}_n^{v-1}(x_i^2)^m \in (\operatorname{in} P)^{m+1}$ for some $m \geq 1$. By degree count for x_p and x_n we must have $\overline{x}_i\overline{x}_p^{q-1}\overline{x}_n^{v-1}(x_i^2)^m \in (x_i^2)^{m+1}$, which is a contradiction by the x_i degree count.

Now assume that $x_i x_p^q x_n^b$ is a monomial in χ $(b < v)$ such that $x_i x_p^q x_n^b \in \widetilde{\operatorname{in} P}$. Then $x_i x_p^q x_n^b \ (x_i^2)^m \in (\operatorname{in} P)^{m+1}$ for some $m \geq 1$. By x_n and x_i-degree count for $1 \leq i \leq p-1$, we must have $x_i^{2m+1} x_p^q x_n^b \in (\delta_{i\geq r} x_i x_p^q, \delta_{i \geq r-r_z} x_i \ x_p^q x_n^{v-w})^{m+1}$. Note we must have $i < r$; thus, $x_i^{2m+1} x_p^q x_n^b \in (\delta_{r-r_z \leq i \leq r-1} x_i \ x_p^q x_n^{v-w})^{m+1}$. This implies $r - r_z \leq i \leq r-1$ and $b \geq v - w$. But there are no such monomials in χ.

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REFERENCES

1. I. Al-Ayyoub, *Reduced Gröbner basis of certain toric varieties; A new short proof*, Comm. Algebra **37** No. 9 (2009), 2945–2955.

2. ———, *An algorithm for computing the Ratliff-Rush closure*, J. Algebra Appl. **8** No. 4 (2009), 521–532.

3. J. Elias, *On the computations of the Ratliff-Rush closure*, J. Symbolic Comput. **37** No. 6 (2004), 717–725.

4. W. Heinzer, D. Lantz and K. Shah, *The Ratliff-Rush ideals in a Noetherian ring*, Comm. Algebra **20** (1992), 591–622.

5. H. Matsumura, *Commutative ring theory*, Cambridge Univ. Press, Cambridge, 1986.

6. D.P. Patil, *Minimal sets of generators for the relation ideals of certain monomial curves*, Manuscripta Math. **80** (1993), 239–248.

7. D.P. Patil and B. Singh, *Generators for the derivation modules and the relation ideals of certain curves*, Manuscripta Math. **68** (1990), 327–335.

8. R.J. Ratliff and D.E. Rush, *Two notes on reductions of ideals*, Indiana Univ. Math. J. **27** (1978), 929–934.

9. M.E. Rossi and I. Swanson, *Notes in the behavior of the Ratliff-Rush filtration*, Contemporary Math. **331** (2003), 313–328.

10. I. Sengupta, *A Gröbner basis for certain affine monomial curves*, Comm. Algebra **31** No. 3 (2003), 1113–1129.

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