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BANACH-STEINHAUS TYPE THEOREMS FOR STATISTICAL AND $\mathcal{I}$-CONVERGENCE WITH APPLICATIONS TO MATRIX MAPS

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ABSTRACT. Let (A_n) be a sequence of bounded linear operators from a Banach space X into a Banach space Y. It is proved that if X has a countable fundamental set Φ and the ideal $\mathcal{I}$ of subsets of $\mathbf{N}$ has property (APO), then $(A_n x)$ is boundedly $\mathcal{I}$-convergent for each $x \in X$ if and only if $\sup_n \|A_n\| < \infty$ and $(A_n \phi)$ is $\mathcal{I}$-convergent for any $\phi \in \Phi$. This result is applied to characterize some sequence-to-sequence transformations defined by infinite matrices of bounded linear operators.

1. Introduction and auxiliary results. Let $\mathbf{N} = \{1, 2, \dots\}$, and let X, Y be two Banach spaces over the field $\mathbf{K}$ of real or complex numbers. A subset Φ of X is called *fundamental* if the linear span of Φ is dense in X. By $B(X, Y)$ we denote the space of all bounded linear operators from X into Y. We write $\sup_n$, $\lim_n$, $\sum_n$, $\cup_n$ and $\cap_n$ instead of $\sup_{n\in\mathbf{N}}$, $\lim_{n\to\infty}$, $\sum_{n=1}^{\infty}$, $\cup_{n=1}^{\infty}$ and $\cap_{n=1}^{\infty}$, respectively.

Let $A_n \in B(X, Y)$, $n \in \mathbf{N}$. A well-known *principle of uniform boundedness* asserts that if $\sup_n \|A_n x\| < \infty$ for every $x \in X$, then there exists a constant $M > 0$ such that

$$\|A_n\| \le M, \quad n \in \mathbf{N}. \tag{1.1}$$

By investigation of the convergence of various linear processes the following corollary from this principle is useful (see, for example, [**4**, page 248] or [**9**, page 173]).

Theorem 1 (Banach-Steinhaus). *Let $\Phi \subset X$ be a fundamental set. The limit $\lim_n A_n x$ exists for any $x \in X$ if and only if* (1.1) *holds and $\lim_n A_n \phi$ exists for every $\phi \in \Phi$. Moreover, the limit operator*

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A, $Ax = \lim_n A_n x$, is bounded and linear, i.e., $A \in B(X,Y)$, and $\|A\| \le M$.

Let $\omega(X)$ be the linear space of all X-valued sequences $\mathfrak{x} = (x_k)$. It is well known that the sets $\ell_\infty(X)$, $c(X)$ and $c_0(X)$ of bounded, convergent and convergent to zero element θ X-valued sequences are Banach spaces with the norm $\|\mathfrak{x}\|_\infty = \sup_k \|x_k\|$, and $\ell_p(X) = \{\mathfrak{x} \in \omega(X) : \sum_k \|x_k\|^p < \infty\}$ $(1 \le p < \infty)$ is a Banach space with the norm $\|\mathfrak{x}\|_p = (\sum_k \|x_k\|^p)^{1/p}$. In the case $X = \mathbf{K}$ we write ω, c, c_0, ℓ and ℓ_p instead of $\omega(\mathbf{K})$, $c(\mathbf{K})$, $c_0(\mathbf{K})$, $\ell_1(\mathbf{K})$ and $\ell_p(\mathbf{K})$, respectively.

Now let $\lambda(X)$ be a subspace of $\omega(X)$, $\mu(Y)$ a subspace of $\omega(Y)$ and $\mathfrak{A} = (A_{nk})$ an infinite matrix of operators $A_{nk} \in B(X,Y)$ $(n,k \in \mathbf{N})$. We say that $\mathfrak{A}$ maps $\lambda(X)$ into $\mu(Y)$, and write $\mathfrak{A} \in (\lambda(X), \mu(Y))$, if for all $\mathfrak{x} = (x_k) \in \lambda(X)$ the series $\mathfrak{A}_n\mathfrak{x} = \sum_k A_{nk}x_k$ $(n \in \mathbf{N})$ converge and the sequence $\mathfrak{A}\mathfrak{x} = (\mathfrak{A}_n\mathfrak{x})$ belongs to $\mu(Y)$. If $Y = X$ and the subspaces $\lambda(X), \mu(X) \subset \omega(X)$ are equipped with the limits λ-lim and μ-lim, respectively, then we write $\mathfrak{A} \in (\lambda(X), \mu(X))_{\mathrm{reg}}$ (and read: $\mathfrak{A}$ maps $\lambda(X)$ into $\mu(X)$ regularly) if $\mathfrak{A} \in (\lambda(X), \mu(X))$ and $\mu\text{-}\lim_n \mathfrak{A}_n\mathfrak{x} = \lambda\text{-}\lim_k x_k$ for all $\mathfrak{x} = (x_k) \in \lambda(X)$.

Using Theorem 1, Zeller [**13**] (see also [**11**]) and Kangro [**5**] described the matrix classes $(c(X), c(Y))$ and $(\ell_1(X), c(Y))$ as follows.

Theorem 2. *Let $\mathfrak{A} = (A_{nk})$ be an infinite matrix with $A_{nk} \in B(X,Y)$. Then*:

(1) *$\mathfrak{A} \in (c(X), c(Y))$ if and only if there exists a constant $M > 0$ such that*

$$\sup_{\|x_k\| \le 1} \left\| \sum_{k=1}^{r} A_{nk}x_k \right\| \le M \quad (n, r \in \mathbf{N}), \tag{1.2}$$

$$\exists \lim_n A_{nk}x \quad (k \in \mathbf{N},\ x \in X), \tag{1.3}$$

$$\sum_k A_{nk}x \text{ converge for each } n \in N \text{ and } x \in X, \tag{1.4}$$

$$\exists \lim_n \sum_k A_{nk}x \quad (x \in X);$$

(2) $\mathfrak{A} \in (\ell_1(X), c(Y))$ *if and only if* (1.3) *is satisfied and there exists a constant* $M > 0$ *such that*

$$\|A_{nk}\| \leq M \quad (n, k \in \mathbf{N}). \tag{1.5}$$

Statistical convergence of number sequences was introduced by Fast [**2**] and investigated in a number of papers (for references see [**1**]). This notion has been extended in different ways. For instance, Maddox [**12**] and Kolk [**6**] considered the statistical convergence of sequences taking values in a locally convex space or a Banach space, respectively. Another extension of statistical convergence is related to generalized densities.

Let $T = (t_{nk})$ be a regular scalar matrix (i.e., $T \in (c, c)_{\mathrm{reg}}$) with the elements $t_{nk} \geq 0$ $(n, k \in \mathbf{N})$. A set $K \subset \mathbf{N}$ is said to have *T-density* $\delta_T(K)$ if the limit

$$\delta_T(K) = \lim_n \sum_{k \in K} t_{nk}$$

exists (cf. [**3**]). A sequence $\mathfrak{x} = (x_k) \in \omega(X)$ is called *T-statistically convergent* to a point $l \in X$, briefly $st_T\text{-}\lim x_k = l$, if

$$\delta_T(\{k : \|x_k - l\| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$ (see [**6**]). If T is the identity matrix, then T-statistical convergence is just the ordinary convergence in X and if T is the Cesàro matrix C_1, then T-statistical convergence is statistical convergence as defined by Fast.

A further extension of statistical convergence is given in [**8**]. Recall that a subfamily $\mathcal{I}$ of the family $2^{\mathbf{N}}$ of all subsets of $\mathbf{N}$ is called an *ideal* if for each $K, L \in \mathcal{I}$ we have $K \cup L \in \mathcal{I}$ and for each $K \in \mathcal{I}$ and each $L \subset K$ we have $L \in \mathcal{I}$. An ideal $\mathcal{I}$ is called *non-trivial* if $\mathcal{I} \neq \varnothing$ and $\mathbf{N} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called *admissible* if $\mathcal{I}$ contains all finite subsets of $\mathbf{N}$.

Now let $\mathcal{I} \subset 2^{\mathbf{N}}$ be a non-trivial ideal. A sequence $\mathfrak{x} = (x_k) \in \omega(X)$ is said to be *$\mathcal{I}$-convergent* to $l \in X$, briefly $\mathcal{I}\text{-}\lim x_k = l$, if for each $\varepsilon > 0$ the set $\{k \in \mathbf{N} : \|x_k - l\| \geq \varepsilon\}$ belongs to $\mathcal{I}$.

The following subsequence characterization of $\mathcal{I}$-convergence is important for us. An admissible ideal $\mathcal{I} \subset 2^{\mathbf{N}}$ is said to have *property* (APO) if for every countable family of mutually disjoint sets $K_1, K_2, \dots$ from $\mathcal{I}$ there exist sets $L_1, L_2, \dots$ from $2^{\mathbf{N}}$ such that the symmetric difference $K_i \Delta L_i$ is a finite set for every $i \in \mathbf{N}$ and $L = \cup_i L_i \in \mathcal{I}$. By an *index set* we mean any infinite set $\{k_i\} \subset \mathbf{N}$ with $k_i < k_{i+1}$ for each $i \in \mathbf{N}$.

Proposition 1 [**8**, Theorem 3.2]. *Let $\mathcal{I} \subset 2^{\mathbf{N}}$ be an admissible ideal. If the ideal $\mathcal{I}$ has property* (APO)*, then $\mathcal{I}$-$\lim x_k = l$ in a Banach space X if and only if there exists an index set $M = \{m_i\}$ such that $\mathbf{N} \backslash M \in \mathcal{I}$ and $\lim_i x_{m_i} = l$ in X.*

The fact that $\mathcal{I}$-convergent sequences may be unbounded justifies the following definition.

Definition. An X-valued sequence $\mathfrak{x} = (x_k)$ is called *boundedly $\mathcal{I}$-convergent* (*boundedly T-statistically convergent*) to $l \in X$ if $\mathfrak{x}$ is bounded and $\mathcal{I}$-$\lim x_k = l$ (st_T-$\lim x_k = l$).

By $bc_{\mathcal{I}}(X)$ ($bst_T(X)$) we denote the set of all boundedly $\mathcal{I}$-convergent (boundedly T-statistically convergent) X-valued sequences. For $X = \mathbf{K}$ we write $bc_{\mathcal{I}}$ and bst_T instead of $bc_{\mathcal{I}}(\mathbf{K})$ and $bst_T(\mathbf{K})$, respectively.

Based on Proposition 1 and Theorem 1, we prove Banach-Steinhaus type theorems for bounded $\mathcal{I}$-convergence and bounded T-statistical convergence. As applications of these results, we characterize matrix classes $(\lambda(X), \mu(Y))$, where $\lambda \in \{c,\ c_0,\ \ell_1\}$ and $\mu \in \{bc_{\mathcal{I}},\ bst_T\}$. Some special matrix transformations are also considered.

2. Banach-Steinhaus type theorems and matrix maps. In the following let $X,\ Y$ be two Banach spaces, $A_n \in B(X,Y)$ $(n \in \mathbf{N})$, $\mathcal{I} \subset 2^{\mathbf{N}}$ a non-trivial admissible ideal and $T = (t_{nk})$ a regular matrix of non-negative scalars. We start with a Banach-Steinhaus type theorem for $\mathcal{I}$-convergence.

Theorem 3. *Suppose that X has a countable fundamental set Φ. If the ideal $\mathcal{I}$ has property* (APO)*, then the sequence $(A_n x)$ is boundedly $\mathcal{I}$-convergent for any $x \in X$ if and only if* (1.1) *holds and $(A_n \phi)$ is $\mathcal{I}$-convergent for every $\phi \in \Phi$. Thereby, the limit operator A, $Ax = \mathcal{I}$-$\lim A_n x$, is bounded and linear, and $\|A\| \le M$.*

Proof. If $(A_n x) \in bc_{\mathcal{I}}(X)$ for any $x \in X$, then $\mathcal{I}$-$\lim A_n \phi$ exists for every $\phi \in \Phi$. Moreover, since $(A_n x) \in \ell_\infty(X)$ for any $x \in X$, (1.1) is satisfied by the principle of uniform boundedness.

Conversely, assume that (1.1) holds and $\mathcal{I}$-$\lim A_n \phi_j$ exists for every $j \in \mathbf{N}$, where $\Phi = \{\phi_j\}$. Since $\mathcal{I}$ has property (APO), by Proposition 1 there exist index sets $M_j = \{m_i(j)\}$ $(j \in \mathbf{N})$ such that

$$\exists \lim_i A_{m_i(j)} \phi_j \quad (j \in \mathbf{N}) \tag{2.1}$$

and $M_j' = \mathbf{N} \setminus M_j \in \mathcal{I}$ for any $j \in \mathbf{N}$. Defining $K_1' = M_1'$ and $K_{j+1}' = M_{j+1}' \setminus \cup_{k=1}^{j} M_k'$ $(j \in \mathbf{N})$, we get a countable family of mutually disjoint sets $K_j' \in \mathcal{I}$. By property (APO) we can find the sets $L_j' \in 2^{\mathbf{N}}$ $(j \in \mathbf{N})$ such that $|K_j' \Delta L_j'| < \infty$ and $\cup_j L_j' \in \mathcal{I}$. Letting $N_j' = \cup_{k=1}^{j} L_k'$, it is easily seen that $|N_j' \Delta M_j'| < \infty$ and $\cup_j N_j' = \cup_j L_j'$.

Thus we are constructing sets $N_j' \in 2^{\mathbf{N}}$ $(j \in \mathbf{N})$ such that the symmetric differences $N_j' \Delta M_j'$ are finite and $\cup_j N_j' \in \mathcal{I}$. Now, defining $N_j = \mathbf{N} \setminus N_j'$ and $N = \cap_j N_j$, we have, in view of $N_j \Delta M_j = N_j' \Delta M_j'$ and $\mathbf{N} \setminus N = \cup_j N_j'$, that $|N_j \Delta M_j| < \infty$ and N is an index set with $\mathbf{N} \setminus N \in \mathcal{I}$. Consequently, denoting $N = \{n_i\}$, from (2.1) it follows

$$\exists \lim_i A_{n_i} \phi_j \quad (j \in \mathbf{N})$$

which together with (1.1) gives

$$\exists \lim_i A_{n_i} x \quad (x \in X)$$

because of Theorem 1. But this is equivalent to

$$(A_n x) \in bc_{\mathcal{I}}(X) \quad (x \in X)$$

by Proposition 1 and (1.1).

Finally, since the limit operator A is determined by $Ax = \lim_i A_{n_i} x$ $(x \in X)$ and $\sup_i \|A_{n_i}\| \leq M$ by (1.1), A must be in $B(X, Y)$ and $\|A\| \leq M$ on the grounds of Theorem 1. $\square$

It is known that $\mathcal{I}_T = \{K \subset \mathbf{N} : \delta_T(K) = 0\}$ is a non-trivial admissible ideal (see [**8**, page 671]) with the property (APO) (see [**3**, Proposition 3.2]). Since $\mathcal{I}_T$-convergence coincides with T-statistical convergence, from Theorem 3 we immediately get the following Banach-Steinhaus type theorem for T-statistical convergence.

Theorem 4. *Suppose that X has a countable fundamental set Φ. Then $(A_n x)$ is boundedly T-statistically convergent for all $x \in X$ if and only if* (1.1) *holds and st_T-$\lim A_n \phi$ exists for any $\phi \in \Phi$. In this case the limit operator A, $Ax = st_T$-$\lim A_n x$ $(x \in X)$, belongs to $B(X, Y)$ and $\|A\| \leq M$.*

Let $\mathfrak{A} = (A_{nk})$, where $A_{nk} \in B(X, Y)$ $(n, k \in \mathbf{N})$. Based on Theorems 3 and 4 we describe matrix transformations $\mathfrak{A}$ from $c(X)$, $c_0(X)$ and $\ell(X)$ into $bc_{\mathcal{I}}(Y)$ and $bst_T(Y)$ under some restrictions on X and $\mathcal{I}$.

For $x \in X$ and $n \in \mathbf{N}$ let $\mathfrak{e}(x) = (x, x, \dots)$ be a constant sequence, and let $\mathfrak{e}^k(x) = (e_j^k(x))$ be the sequence with $e_j^k(x) = x$ if $j = k$ and $e_j^k(x) = 0$ otherwise. It is not difficult to see that if Φ is a (countable) fundamental set in X, then $\mathcal{E}_0(\Phi) = \{\mathfrak{e}^k(\phi) : k \in \mathbf{N},\ \phi \in \Phi\}$ is a (countable) fundamental set in Banach spaces $c_0(X)$ and $\ell(X)$, and $\mathcal{E}_0(\Phi) \cup \mathcal{E}_1(\Phi)$ with $\mathcal{E}_1(\Phi) = \{\mathfrak{e}(\phi) : \phi \in \Phi\}$ is a (countable) fundamental set in Banach space $c(X)$.

We begin with a simple lemma.

Lemma. *Let Φ be a fundamental set in X. The following is true*:

(1) *If* (1.2) *holds and*

$$(2.2) \qquad \sum_k A_{nk}\phi \text{ converge for each } n \in N \text{ and } \phi \in \Phi,$$

then, for any $n \in \mathbf{N}$ and $\mathfrak{x} = (x_k) \in c(X)$, the series $\mathfrak{A}_n \mathfrak{x} = \sum_k A_{nk} x_k$ converge, $\mathfrak{A}_n \in B(c(X), Y)$ and there exists a constant $M > 0$ such *that*

$$\|\mathfrak{A}_n\| \leq M \quad (n \in \mathbf{N}); \tag{2.3}$$

(2) *If* (1.2) *holds, then, for any* $n \in \mathbf{N}$ *and* $\mathfrak{x} = (x_k) \in c_0(X)$, *the series* $\mathfrak{A}_n\mathfrak{x} = \sum_k A_{nk}x_k$ *converge,* $\mathfrak{A}_n \in B(c_0(X), Y)$ *and* (2.3) *is satisfied*;

(3) *If* (1.5) *is satisfied, then, for any* $n \in \mathbf{N}$ *and* $\mathfrak{x} = (x_k) \in \ell(X)$, *the series* $\mathfrak{A}_n\mathfrak{x} = \sum_k A_{nk}x_k$ *converge,* $\mathfrak{A}_n \in B(\ell(X), Y)$ *and* (2.3) *is satisfied.*

Proof. To prove the first statement we fix arbitrarily index n and consider the operators $\mathfrak{A}_n^r : c(X) \to Y$, $\mathfrak{A}_n^r\mathfrak{x} = \sum_{k=1}^r A_{nk}x_k$, $r \in \mathbf{N}$. Obviously, $\mathfrak{A}_n^r \in B(c(X), Y)$ for each $r \in \mathbf{N}$ and $\sup_r \|A_n^r\| \leq M$ by (1.2). Moreover, $\lim_r A_n^r\mathfrak{x}$ automatically exists for all $\mathfrak{x} \in \mathcal{E}_0(\Phi)$ and the limits $\lim_r \mathfrak{A}_n^r\mathfrak{x}$, $\mathfrak{x} \in \mathcal{E}_1(\Phi)$, exist by (2.2). So, applying Theorem 1 to the sequence of bounded linear operators $(\mathfrak{A}_n^r)_{r\in\mathbf{N}}$, we have that $\lim_r \sum_{k=1}^r A_{nk}x_k$ exists for each $\mathfrak{x} \in c(X)$, $\mathfrak{A}_n \in B(c(X), Y)$ and $\|\mathfrak{A}_n\| \leq M$ for any $n \in \mathbf{N}$.

The proofs of (2) and (3) are quite similar if we observe that in the case of $\ell(X)$, $\|\mathfrak{A}_n\| = \sup_k \|A_{nk}\|$ $(n \in \mathbf{N})$ (see [**5**, page 113]). □

Proposition 2. *Suppose that* X *has a countable fundamental set* Φ *and the ideal* $\mathcal{I} \subset 2^{\mathbf{N}}$ *has property* (APO). *Then*:

(1) $\mathfrak{A} \in (c(X), bc_{\mathcal{I}}(Y))$ *if and only if* (1.2) *and* (1.4) *hold,*

$$\exists \mathcal{I}\text{-}\lim_n A_{nk}x \quad (k \in \mathbf{N},\ x \in X), \tag{2.4}$$

$$\exists \mathcal{I}\text{-}\lim_n \sum_k A_{nk}x \quad (x \in X); \tag{2.5}$$

(2) $\mathfrak{A} \in (c_0(X), bc_{\mathcal{I}}(Y))$ *if and only if* (1.2) *and* (2.4) *are satisfied*;

(3) $\mathfrak{A} \in (\ell(X), bc_{\mathcal{I}}(Y))$ *if and only if* (1.5) *and* (2.4) *are satisfied.*

Proof. If $\mathfrak{A} \in (c(X), bc_{\mathcal{I}}(Y))$, then $\mathfrak{A} \in (c(X), \ell_\infty(Y))$ and, by the principle of uniform boundedness, (2.3) must hold. But this yields (1.2) since any sequence $\mathfrak{x}^{[r]} = (x_1, x_2, \dots, x_r, 0, 0, \dots)$ belongs to $c(X)$ and $\|\mathfrak{x}^{[r]}\|_\infty \leq 1$ if $\|x_k\| \leq 1$ $(k = 1, 2, \dots, r)$. Conditions (2.4)–(2.5) are obviously satisfied.

Conversely, if $\mathfrak{A} = (A_{nk})$ satisfies (1.2) and (1.4), then, by statement (1) of the lemma the series $\mathfrak{A}_n\mathfrak{x} = \sum_k A_{nk}x_k$ converge, $\mathfrak{A}_n \in B((c(X), Y)$ and (2.3) holds. Moreover, conditions (2.4) and (2.5) show that $\mathcal{I}$-lim $\mathfrak{A}_n\mathfrak{x}$ exists for any sequence $\mathfrak{x}$ from the countable fundamental set $\mathcal{E}_0(\Phi) \cup \mathcal{E}_1(\Phi)$ of $c(X)$. So, applying Theorem 3 to the sequence of operators $(\mathfrak{A}_n)$, we get $\mathfrak{A} \in (c(X), bc_{\mathcal{I}}(Y))$.

Analogously, using statements (2) and (3) of the lemma, Theorem 3 and the fact that $\mathcal{E}_0(\Phi)$ is a countable fundamental set in $c_0(X)$ and $\ell(X)$, we can prove our statements (2) and (3). □

Similarly to Proposition 2, using only Theorem 4 instead of Theorem 3, we get

Proposition 3. *Suppose that X has a countable fundamental set. Then*:

(1) $\mathfrak{A} \in (c(X), bst_T(Y))$ *if and only if* (1.2) *and* (1.4) *hold*,

$$\text{(2.6)} \qquad \begin{gathered} \exists st_T\text{-}\lim_n A_{nk}x \quad (k \in \mathbf{N},\ x \in X), \\ \exists st_T\text{-}\lim_n \sum_k A_{nk}x \quad (x \in X); \end{gathered}$$

(2) $\mathfrak{A} \in (c_0(X), bst_T(Y))$ *if and only if* (1.2) *and* (2.6) *are satisfied*;

(3) $\mathfrak{A} \in (\ell(X), bst_T(Y))$ *if and only if* (1.5) *and* (2.6) *are satisfied.*

Propositions 2 and 3 lead us to the characterizations of matrix classes $(c(X), bc_{\mathcal{I}}(X))_{\text{reg}}$ and $(c(X), bst_T(X))_{\text{reg}}$ as follows.

Proposition 4. *Suppose that X has a countable fundamental set.*

(1) *If the ideal $\mathcal{I} \subset 2^{\mathbf{N}}$ has property* (APO)*, then* $\mathfrak{A} \in (c(X), bc_{\mathcal{I}}(X))_{\text{reg}}$

if and only if (1.2) *and* (1.4) *hold,*

$$\mathcal{I}\text{-}\lim_n A_{nk}x = \theta \quad (k \in \mathbf{N},\ x \in X),$$
$$\mathcal{I}\text{-}\lim_n \sum_k A_{nk}x = x \quad (x \in X);$$

(2) $\mathfrak{A} \in (c(X), bst_T(X))_{\text{reg}}$ *if and only if* (1.2) *and* (1.4) *hold,*

$$st_T\text{-}\lim_n A_{nk}x = \theta \quad (k \in \mathbf{N},\ x \in X),$$
$$st_T\text{-}\lim_n \sum_k A_{nk}x = x \quad (x \in X).$$

Now we consider a matrix transformation $\mathfrak{A}$ in the case $Y = \mathbf{K}$. Then $B(X,Y)$ is the topological dual X' of X and the elements of matrix $\mathfrak{A} = (A_{nk})$ are bounded linear functionals on X, i.e., $A_{nk} \in X'$ $(n,k \in \mathbf{N})$. Let $1 < p < \infty$ and $1/p + 1/q = 1$. It is known that if the series $\mathfrak{A}_n = \sum_k A_{nk}x_k$ converge for every $\mathfrak{x} = (x_k) \in \ell_p(X)$, then $(A_{nk})_{k\in\mathbf{N}} \in \ell_q(X')$ and consequently, $\|\mathfrak{A}_n\| = (\sum_k \|A_{nk}\|^q)^{1/q}$ (see, for example, [**10**, page 247]). Moreover, if X has a countable fundamental set Φ, then $\ell_p(X)$ has countable fundamental set $\mathcal{E}_0(\Phi)$. So, using the same arguments as in the proofs of Propositions 2 and 3, we get

Proposition 5. *Let* $1 < p < \infty$ *and* $1/p + 1/q = 1$. *Suppose that* X *has a countable fundamental set.*

(1) *If* $\mathcal{I}$ *has property* (APO), *then* $\mathfrak{A} \in (\ell_p(X), bc_{\mathcal{I}})$ *if and only if* (2.4) *holds and*

$$\sup_n \sum_k \|A_{nk}\|^q < \infty; \tag{2.7}$$

(2) $\mathfrak{A} \in (\ell_p(X), bst_T)$ *if and only if* (2.6) *and* (2.7) *are satisfied.*

Remark. Kangro [**5**] proved that for $Y = \mathbf{K}$ we have

$$\sup_{\|x_k\|\le 1} \left\| \sum_{k=1}^{r} A_{nk}x_k \right\| = \sum_{k=1}^{r} \|A_{nk}\|.$$

Thus, if $Y = \mathbf{K}$, then condition (1.2) may be replaced with

$$\sup_n \sum_k \|A_{nk}\| < \infty$$

and (2.2) may be omitted in the lemma. Consequently, (1.4) is superfluous in Propositions 2–4 in the case $Y = \mathbf{K}$.

Finally, if $X = Y = \mathbf{K}$, then the matrix map $\mathfrak{A} : \lambda(X) \to \mu(Y)$ reduces to the transformation $A : \lambda \to \mu$ defined by an infinite scalar matrix $A = (a_{nk})$. So, taking into account also the remark, from Propositions 2–5 we deduce

Proposition 6. *Let $A = (a_{nk})$ be an infinite scalar matrix, $1 < p < \infty$ and $1/p + 1/q = 1$. If $\mathcal{I}$ has property* (APO), *then*:

(1) $A \in (c, bc_{\mathcal{I}})$ *if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty, \tag{2.8}$$

$$\exists \mathcal{I}\text{-}\lim_n a_{nk} = a_k \quad (k \in \mathbf{N}), \tag{2.9}$$

$$\exists \mathcal{I}\text{-}\lim_n \sum_k a_{nk} = a; \tag{2.10}$$

(2) $A \in (c, bc_{\mathcal{I}})_{\text{reg}}$ *if and only if* (2.8)–(2.10) *hold with* $a_k = 0$ $(k \in \mathbf{N})$ *and* $a = 1$;

(3) $\mathfrak{A} \in (c_0, bc_{\mathcal{I}})$ *if and only if* (2.8) *and* (2.9) *are satisfied*;

(4) $\mathfrak{A} \in (\ell, bc_{\mathcal{I}})$ *if and only if* (2.9) *holds and*

$$\sup_{n,k} |a_{nk}| < \infty;$$

(5) $\mathfrak{A} \in (\ell_p, bc_{\mathcal{I}})$ *if and only if* (2.9) *holds and*

$$\sup_n \sum_k |a_{nk}|^q < \infty.$$

If $\mathcal{I} = \mathcal{I}_T$, then Proposition 6 gives known characterizations of matrix classes (c, bst_T), $(c, bst_T)_{\text{reg}}$, (ℓ, bst_T) and (ℓ_p, bst_T) (see [**7**, Corollaries 3–6]).

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