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INVARIANT IDEALS FOR UNIFORM JOINT LOCALLY QUASINILPOTENT OPERATORS

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ABSTRACT. In this article we will see sufficient conditions for which an N-tuple of operators (not necessarily commuting) have a common nontrivial invariant closed ideal. The results of this work extend some results of Abramovich-Aliprantis-Burkinsaw, see [**1, 3**] to the case of several variables. The concepts of joint local quasinilpotence and joint compact-friendly will be basics.

1. Introduction. Let $S, B : E \to E$ be two operators on a Banach lattice E, with B positive. We say that the operator S is dominated by the operator B provided

$$|S(x)| \leq B(|x|)$$

for all $x \in E$.

We say that S is compact-friendly if there exists a positive operator R in the commutant of S such that it is dominated by a nonzero operator which, in turn, is dominated by a positive compact operator.

That is, S is compact-friendly if and only if there exist three nonzero operators $R, K, A : E \to E$, with R, K positive and K compact such that

$$RS = SR; \quad |A(x)| \leq R(|x|) \quad \text{and} \quad |A(x)| \leq K(|x|)$$

for each $x \in E$.

On the other hand, let S be a positive linear operator defined on a Banach lattice E. We say that S is local quasinilpotent at $x \in X$ if

$$\lim_{n\to\infty} \|S^n x\|^{1/n} = 0.$$

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The notions of compact-friendly and local quasinilpotence were introduced by Abramovich-Aliprantis-Burkinsaw in [**3**] to obtain nontrivial invariant subspaces for positive operators.

The study of these properties gave rise to significant and important results about the invariant subspace problem for positive operators, see [**2, 3, 4**]. These studies have been continued later in [**6**]. Our interest in this article is to extend these notions to the case of several variables and to obtain invariant subspaces for N tuples of operators. Contributions to the invariant subspace problem for N-tuples of operators can be found in [**6, 7, 8, 9**], see [**10**] for a recent survey.

During this article we will denote $T = (T_1, \dots, T_N)$ an N-tuple of not-necessary commuting operators. In Section 2 we will remember the notion of an N tuple joint local quasinilpotence introduced in [**6**] and several of its properties. We will introduce the notion of joint compact-friendly, and we will study some of its properties.

Section 3 is dedicated to extend some results of [**1, 3**] for N tuples of operators. Finally, the article ends with a section of concluding remarks and open problems.

2. Joint compact-friendly. Let $T = (T_1, \dots, T_N)$ be an $N-$tuple of positive operators defined on a Banach lattice E. Let us denote by T^n the collection of all possible products of n elements in T.

Let us recall the following notion introduced in [**6**].

Definition 2.1. We will say that $T = (T_1, \dots, T_N)$ is uniform joint locally quasinilpotent at $x_0 \in E$ if

$$\lim_{n\to\infty} \max_{S\in T^n} \|Sx_0\|^{1/n} = 0.$$

The above notion allows us to obtain local quasinilpotence for operators $p(T_{i_1}, \cdots, T_{i_m})$, for some polynomials p.

Proposition 2.2. *Let $T = (T_1, \dots, T_N)$ be an N-tuple of continuous linear operators on a Banach lattice E (not necessarily commuting), and let us suppose that T is uniform joint locally quasinilpotent at*

$x_0 \in X \setminus \{0\}$. Then, for all polynomials p of m variables, such that $p(0, \cdots, 0) = 0$, we have

$$\lim_{n\to\infty} \|p(T_{i_1}, \dots, T_{i_m})^n x_0\|^{1/n} = 0,$$

where $i_j \in \{1, \dots, N\}$; $j \in \{1, \dots, m\}$, that is, the operator $p(T_{i_1}, \dots, T_{i_m})$ is locally quasinilpotent at x_0.

Proof. Let us suppose that $k \in \mathbf{N}$ is the number of summands of the polynomial p, and let us denote by $c \in \mathbf{R}_+$ the maximum of the modulus of the coefficients of p. Let r be the degree of p and $M = \max\{\|T_1\|, \|T_2\|, \dots, \|T_N\|\}$.

Since p has k summands, then the expression $p(T_{i_1}, \dots, T_{i_m})^n(x_0)$ has k^n summands like this one

$$aT_{i_1}T_{i_2}\cdots T_{i_s}x_0 \quad i_j \in \{1, \dots, N\}$$

where a is a coefficient of p and $n \le i_s \le r^n$. If we apply the triangular inequality in the expression $\|p(T_{i_1}, \dots, T_{i_m})^n(x_0)\|$, then each summand is less than or equal to

$$cM^{r^n-n}\|T_{i_1}, \dots, T_{i_n}x_0\| \le cM^{r^n-n} \max_{S\in T^n} \|Sx_0\|;$$

therefore,

$$\|p(T_{i_1}, \dots, T_{i_m})^n(x_0)\| \le ck^n M^{r^n-n} \max_{S\in T^n} \|Sx_0\|.$$

Hence,

$$\begin{aligned}\lim_{n\to\infty} \|p(T_{i_1}, \dots, T_{i_m})^n(x_0)\|^{1/n} &\le \lim_{n\to\infty} \left(ck^n M^{r^n-n} \max_{S\in T^n} \|Sx_0\|\right)^{1/n} \\ &= kM^{r-1} \lim_{n\to\infty} \max_{S\in T^n} \|Sx_0\|^{1/n} = 0\end{aligned}$$

which yields the desired result. $\square$

For complementary results and properties, see [**6**].

Definition 2.3. Let $T = (T_1, \ldots, T_N)$ be an N-tuple of positive operators defined on a Banach lattice E. Then we say that it is joint compact-friendly if and only if there exist three nonzero operators $R, K, A : E \to E$, with R, K positive and K compact such that

$$RT_j = T_j R; \quad |A(x)| \leq R(|x|) \quad \text{and} \quad |A(x)| \leq K(|x|)$$

for each $x \in E$ and each $j \in \{1, \ldots, N\}$.

Following, we will see that two compact-friendly operators are not necessarily joint compact-friendly. Let Φ be a function defined on a compact Hausdorff space Ω. Then we define the multiplication operator $M_\Phi : C(\Omega) \to C(\Omega)$ via the formula $M_\Phi(f) = \Phi f$. The following construction (whose proof we can find in [**1**, page 433]) will be necessary.

Theorem 2.4. *Let M_Φ be a positive multiplication operator on a $C(\Omega)$-space, where Ω is a compact Hausdorff space. M_Φ is compact friendly if and only if the multiplier Φ has a flat, i.e., there exists a nonempty open subset $V \subset \Omega$ such that Φ is constant on V.*

Example 2.5. Let us consider the following functions:

$$\Phi_1(x) = \begin{cases} 1/2 & \text{if } 0 \leq x \leq 1/2 \\ x & \text{if } 1/2 < x \leq 1 \end{cases}$$
$$\Phi_2(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1/2 \\ 1/2 & \text{if } 1/2 < x \leq 1. \end{cases}$$

Since Φ_1 and Φ_2 have a flat, then by Theorem 2.4, M_{Φ_1} and M_{Φ_2} are compact-friendly.

Now, if (M_{Φ_1}, M_{Φ_2}) were joint compact-friendly, then $M_{\Phi_1} + M_{\Phi_2}$ would be compact-friendly too. Indeed, since (M_{Φ_1}, M_{Φ_2}) is joint compact-friendly, there exist three nonzero operators $R, K, A : C([0,1]) \to C([0,1])$, with R, K positive and K compact such that

$$RM_{\Phi_j} = M_{\Phi_j} R; \quad |A(x)| \leq R(|x|) \quad \text{and} \quad |A(x)| \leq K(|x|)$$

for each $x \in C([0,1])$ and each $j \in \{1,2\}$. Since $R(M_{\Phi_1} + M_{\Phi_2}) = (M_{\Phi_1} + M_{\Phi_2})R$, we deduce that $M_{\Phi_1} + M_{\Phi_2}$ is compact-friendly.

Now, $M_{\Phi_1}+M_{\Phi_2} = M_{\Phi_1+\Phi_2}$, and $\Phi_1+\Phi_2$ does not have a flat; hence, it cannot be compact-friendly, a contradiction. $\square$

In fact, for N-tuples $M = (M_{\Phi_1}, \dots, M_{\Phi_N})$, we have a similar characterization to Theorem 2.4. To see this, it will be necessary to show the following lemma, whose proof we can find in [**1**, page 432].

Lemma 2.6. *Let $\{e_n\}$ be a norm bounded sequence in a Banach lattice E, and let $S : E \to E$ be a bounded operator that is dominated by a compact operator. If E has an order continuous norm and the sequence Se_n is disjoint, then $\|Se_n\| \to 0$.*

The proof of the following theorem is an adaptation of the main result in [**5**].

Theorem 2.7. *Let $M = (M_{\Phi_1}, \dots, M_{\Phi_N})$ be an N-tuple of positive multiplication operators on a $C(\Omega)$-space where Ω is a compact Hausdorff space. Then M is joint compact-friendly if and only if there exists a nonempty open subset $V \subset \Omega$ such that Φ_j is constant on V for each $j \in \{1, \dots, N\}$.*

Proof. In one direction, let us assume that there exists a nonempty open subset $V \subset \Omega$ such that $\Phi_j(w) = c_j$ for each $w \in V$ and each $j \in \{1, \dots, N\}$. Take a nonzero function $\psi \in C(\Omega)$ such that $\psi(w) = 0$ for each $w \in \Omega \setminus V$, and fix some $w_0 \in V$. Now, let us define the rank-one operator $K : C(\Omega) \to C(\Omega)$ via the formula $Kx = x(w_0)\psi$.

We claim that K and M_{Φ_j} commute for each $j \in \{1, \dots, N\}$. Indeed, for $x \in C(\Omega)$, we have

$$(KM_{\Phi_j}x)(w) = K(\Phi_j x)(w) = \Phi_j(w_0)x(w_0)\psi(w) = c_j x(w_0)\psi(w).$$

On the other hand, since $\psi(x) = 0$ for all $x \notin V$ and $\Phi_j = c_j$ on V, we also have

$$(M_{\Phi_j}Kx)(w) = (M_{\Phi_j}x(w_0)\psi)(w) = x(w_0)\psi(w)\Phi_j(w) = c_j x(w_0)\psi(w).$$

Therefore, $KM_{\Phi_j} = M_{\Phi_j}K$ for each $j \in \{1, \dots, N\}$, and K is positive. Hence, $M = (M_{\Phi_1}, \dots, M_{\Phi_N})$ is joint compact-friendly.

For the reverse implication, let us assume that $M = (M_{\Phi_1}, \dots, M_{\Phi_N})$ is joint compact-friendly. Then there exist three nonzero operators $R, K, A : C(\Omega) \to C(\Omega)$ with R, K positive, K compact $M_{\Phi_j} R = R M_{\Phi_j}$ for each $j \in \{1, \dots, N\}$ such that, for each $x \in C(\Omega)$, we have

$$|Ax| \le R|x| \quad \text{and} \quad |Ax| \le K|x|.$$

Taking adjoints, we obtain that $M^*_{\Phi_j} R^* = R^* M^*_{\Phi_j}$, and that

$$|A^* x^*| \le R^* |x^*| \quad \text{and} \quad |A^* x^*| \le K^* |x^*|$$

for each $x^* \in C(\Omega)^*$. Now we have the following three properties:

(a) For each $w \in \Omega$, let us consider the δ-measure δ_w concentrated at the point w. Then, it immediately follows from the identity

$$M^*_{\Phi_j} R^* \delta_w = R^* M^*_{\Phi_j} \delta_w = \Phi_j(w) R^* \delta_w$$

that the support of the measure $R^*\delta_w$ is contained in the sets $W^j_w = \Phi_j^{-1}(\Phi_j(w))$ for each $j \in \{1, \dots, N\}$, and therefore in $W_w = \cap_{j=1}^N W^j_w$.

(b) Since A^* is dominated by R^*, it immediately follows from (a) that for each $w \in \Omega$ the signed measure $A^*\delta_w$ is also supported by W_w.

(c) Fix any function $h \in C(\Omega)$ such that $\|h\| = 1$ and $Ah \neq 0$. Next, choose a nonempty open set U on which $|Ah(w)| \ge \epsilon$ for some $\epsilon > 0$. Then, for each $w \in U$, we have $\|A^*\delta_\omega\| \ge \epsilon$. To see this, note that

$$\|A^*\delta_w\| \ge |\langle A^*\delta_w, h\rangle| = |\langle \delta_\omega, Ah\rangle| = |Ah(\omega)| \ge \epsilon. \tag{1}$$

From the definition of the sets W_w, we see that $\cup_{w \in \Omega} W_w = \Omega$. To complete the proof, let us assume by way of contradiction that, for each $w \in \Omega$, the set W_w has an empty interior. Then, the nonempty open set U chosen in (c) must meet infinitely many sets W_ω. Pick a sequence $\{w_n\}$ in U satisfying $\Phi(w_n) \neq \Phi(w_m)$ for $m \neq n$, and let us consider $e_n = |A^*\delta_{w_n}|$ for each n. Then $\|e_n\| \ge \epsilon$. Furthermore, since each e_n is supported by the set W_{ω_n} and the sequence $\{W_{\omega_n}\}$ is pairwise disjoint, the sequence $\{e_n\}$ in $C(\Omega)^*$ is also disjoint. Therefore, since the norm in $C(\Omega)^*$ is order-continuous, it follows from Lemma 2.6 that $\|e_n\| \to 0$. However, the last conclusion contradicts (1), and the proof of the theorem is complete. $\square$

3. Main results. Let us see the main results of this paper.

Theorem 3.1. *Let* $(T_1, \dots, T_N)$ *be an* N*-tuple of positive linear operators (not necessarily commuting) defined on a Banach lattice* E*. Assume that there exist positive operators* $S : E \to E$ *such that*

(1) $ST_j \le T_j S$ *for all* $j \in \{1, \dots, N\}$,

(2) S *is quasinilpotent at some* $x_0 > 0$, *and*

(3) S *dominates a nonzero compact operator.*

Then, the operators $\{T_1, \dots, T_N\}$ *have a common nontrivial closed invariant ideal.*

Proof. Let K be the nonzero compact operator where S dominates K, i.e., $|Kx| \le S(|x|)$ for each $x \in E$, and let us assume without loss of generality that $\|K\| = 1$ and $\|T_1 + \cdots + T_N\| < 1$. Let us denote $B = T_1 + \cdots + T_N$. Then the following operator is defined on E

$$A = \sum_{n=0}^{\infty} B^n.$$

In view of the first condition on S, the inequalities $SB^k \le B^k S$ and $SA^k \le A^k S$ hold for each $k \in \mathbf{N}$.

For every $x > 0$, let us denote by $J[x]$ the principal ideal generated by Ax. That is,

$$J[x] = \{y \in E : |y| \le \lambda Ax \text{ for some } \lambda > 0\}.$$

Since $x \in J[x]$, then $J[x] \ne \{0\}$. Furthermore, $J[x]$ is T_j-invariant for each $j \in \{1, \dots, N\}$. In fact, let $y \in J[x]$ and $\lambda > 0$ be such that $|y| \le \lambda Ax$. Then

$$|T_j y| \le T_j |y| \le B(|y|) \le \lambda B(Ax) = \lambda \sum_{n=1}^{\infty} (B^n x) \le \lambda Ax,$$

whence $T_j y \in J[x]$ for each $j \in \{1, \dots, N\}$. So, $\overline{J[x]}$ is a nonzero closed T_j-invariant ideal.

The proof will be finished if we show that $\overline{J[x]} \neq E$ for some $x > 0$. Let us assume by contradiction

$$\overline{J[x]} = E \quad \text{for each} \quad x > 0. \tag{2}$$

Without loss of generality, we can assume $Kx_0 \neq 0$. Indeed, if it were $K(y) = 0$ for each $0 < y \in J[x_0]$, then $K = 0$ on $J[x_0]$, and, consequently, by (2), K would be zero, contrary to the assumptions. Hence, $K(y_0) \neq 0$ for some $0 < y_0 \in J[x_0]$. Moreover, S is quasinilpotent at y_0. In fact,

$$\|S^n(y_0)\|^{1/n} \leq \|S^n(\lambda A(x_0))\|^{1/n} \leq \lambda^{1/n}\|A\|^{1/n}\|S^n(x_0)\|^{1/n} \longrightarrow 0.$$

Therefore, replacing (if necessary) y_0 by x_0, we obtain that there exists an element $x_0 > 0$ at which S is quasinilpotent and $K(x_0) \neq 0$. Also, replacing x_0 by αx_0 for an appropriate scalar $\alpha > 1$, we can suppose that $\|x_0\| > 1$ and $\|Kx_0\| > 1$. Now, let $U = \{z \in E : \|x_0 - z\| \leq 1\}$ be the closed unit ball centered at x_0. By our choice of x_0, we have

$$0 \notin U \quad \text{and} \quad 0 \notin \overline{K(U)}. \tag{3}$$

By (2), we know $\overline{J[x]} = E$ for each $x \neq 0$. Now we claim that for each element $y \geq 0$ the sequence $\{y \wedge nA[|x|]\}$ is norm convergent and $\lim_{n\to\infty} y \wedge nA[|x|] = y$. Indeed, if we consider $y > 0$ and fix $\varepsilon > 0$, since $J(x)$ is dense in E, there exists some $z \in J[x]$ such that $\|z - y\| \leq \varepsilon$. From the lattice inequality $|z^+ \wedge y - y| < |z - y|$, we see (by replacing z by $z^+ \wedge y$) that we can assume $0 \leq z \leq y$. Now, fix some k such that $0 \leq z \leq kA(|x|)$ and so $z \leq y \wedge kA(|x|)$. Hence, if $n \geq k$, then from the inequality $0 \leq y - y \wedge nA(|x|) \leq y - y \wedge kA(|x|) \leq y - z$, we see that $\|y - y \wedge nA(|x|)\| < \varepsilon$. Therefore, $\|y - y \wedge nA(|x|)\| \to 0$. In particular, for each $x \neq 0$ there exists some n such that $\|x_0 - x_0 \wedge nA[|x|]\| < 1$. Since the function $z \to x_0 \wedge nA[|z|]$ is continuous, we see that the set $\{z : \|x_0 - x_0 \wedge nA[|z|]\| < 1\}$ is open for each n. In view of $0 \notin \overline{K(U)}$, the above argument guarantees that

$$\overline{K(U)} \subset \bigcup_{n\in\mathbf{N}} \{z \in E : \|x_0 - x_0 \wedge nA[|z|]\| < 1\}.$$

Now the sequence of sets $\{z : \|x_0 - x_0 \wedge nA[|z|]\| < 1\}$ is increasing with respect to the inclusion, and the compactness of $\overline{K(U)}$ implies

$$\overline{K(U)} \subset \{x \in E : \|x_0 - x_0 \wedge mA[|z|]\| < 1\}$$

for some m. In other words, there exists some fixed m such that if $x \in \overline{K(U)}$ then $x_0 \wedge mA[|x|] \in U$.

In particular, we have $x_1 = x_0 \wedge mA[|Kx_0|] \in U$. Since $K(x_1)$ belongs to $K(U)$, it follows that $x_2 = x_0 \wedge mA[|Kx_1|] \in U$. Proceeding inductively, we obtain a sequence $\{x_n\}$ of positive vectors in U defined by $x_{n+1} = x_0 \wedge mA[|Kx_n|]$. Now, we claim that

$$0 \leq x_n \leq m^n A^n S^n(x_0)$$

holds for each n. The proof is by induction. For $n = 1$, we have the inequality $x_1 = x_0 \wedge mA[|Kx_0|] \leq mA[Sx_0]$. For the induction step, let us recall that $SA^n \leq A^n S$ and that, if $0 \leq x_n \leq m^n A^n S^n(x_0)$ holds for some n, then

$$\begin{aligned} 0 &\leq x_{n+1} = x_0 \wedge mA[|K(x_n)|] \leq mA[|Kx_n|] \\ &\leq mA[Sx_n] \leq m^{n+1} A[SA^n S^n(x_0)] \leq m^{n+1} A^{n+1} S^{n+1}(x_0). \end{aligned}$$

Thus, we have $\|x_n\| \leq m^n \|A\|^n \|S^n x_0\|$, and so

$$\|x_n\|^{1/n} \leq m\|A\| \|S^n x_0\|^{1/n}$$

for each n. Since $\lim_{n\to\infty} \|S^n x_0\|^{1/n} = 0$, it follows that $\lim_{n\to\infty} \|x_n\|^{1/n} = 0$, and consequently $\lim_{n\to\infty} \|x_n\| = 0$. However, since $\{x_n\} \subset U$, this implies that $0 \in \overline{U} = U$, contrary to (3), and the proof of the theorem is finished. $\square$

Remark 3.2. We do not know presently if we can obtain an analogue of the previous result replacing the inequality $ST_j \leq T_j S$ by the reverse inequality $ST_j \geq T_j S$. However, if we assume in addition that the operator S is quasinilpotent, i.e., $\lim \|S^n\|^{1/n} = 0$, then we can obtain an analogue result to Theorem 3.1, whose proof is also similar.

Theorem 3.3. *Let $(T_1, \ldots, T_N)$ be an N-tuple of positive linear operators (not necessarily commuting) defined on a Banach lattice E. Assume that there exists a positive operator $S : E \to E$ such that*

(1) *$ST_j \geq T_j S$ for all $j \in \{1, \ldots, N\}$,*

(2) *S is quasinilpotent, and*

(3) *S dominates a nonzero compact operator K.*

Then, the operators $(T_1, \dots, T_N)$ have a common nontrivial closed invariant ideal.

Sketch of the proof. The operators B and A are constructed as in Theorem 3.1. This time, using the reverse inequalities $ST_j \geq T_j S$ for all j, we obtain that $SA^k \geq A^k S$ for all $k \in \mathbf{N}$. The sequence x_n is constructed as in Theorem 3.1, and the proof finishes if we can show that $\lim_{n\to\infty} \|x_n\| = 0$. This time, we claim that

$$0 \leq x_n \leq m^n S^n A^n(x_0).$$

The proof follows by induction. If $n = 1$, then $x_1 = x_0 \wedge mA[|Kx_0|] \leq mA[Sx_0]$, and since $SA \leq AS$, then $x_1 = x_0 \wedge mA[|Kx_0|] \leq mA[Sx_0] \leq mS[Ax_0]$. Now, if $0 \leq x_n \leq m^n S^n A^n(x_0)$ follows for some n, then

$$\begin{aligned}0 &\leq x_{n+1} = x_0 \wedge mA[|K(x_n)|] \leq mA[|Kx_n|] \\ &\leq mA[Sx_n] \leq m^{n+1} A[SS^n A^n(x_0)] \\ &\leq m^{n+1} S^{n+1} A^{n+1}(x_0).\end{aligned}$$

Therefore, $0 \leq x_n \leq m^n S^n A^n(x_0)$, and

$$\|x_n\|^{1/n} \leq m\|S^n\|^{1/n}\|A\|\|x_0\|^{1/n}$$

for each n. However, by hypothesis, $\lim_{n\to\infty} \|S^n\|^{1/n} = 0$; consequently, $\lim \|x_n\| = 0$, which yields the desired result. $\square$

To prove the next result, we need to introduce the concept of the null ideal.

Let $T : E \to E$ be a positive operator defined on a Banach lattice E. Its null ideal is defined via the formula

$$\mathcal{N}_T = \{x \in E : T(|x|) = 0\}.$$

We need the following the following Lemma, too.

Lemma 3.4. *Let J be an ideal in a Banach lattice E. If J is invariant under some positive operator $B : E \to E$, then J is also invariant under every operator S which is dominated by B.*

Proof. If $x \in J$, then since J is solid, (that is, if $y \in J$ and $x \in E$, $|x| \leq |y|$ then $x \in J$) $|x| \in J$. Therefore, $B|x| \in J$.

On the other hand, J solid and $|Sx| \leq B|x|$ implies that $Sx \in J$. Hence, J is S-invariant. □

Now we will extend the following theorem, whose proof can be found in [**2**, page 45].

Theorem 3.5. *Let $B, S : E \to E$ be two commuting nonzero positive operators on a Banach lattice E. If one of them is quasinilpotent at a nonzero positive vector and the other dominates a nonzero compact operator, then B and S have a common nontrivial closed invariant ideal.*

Theorem 3.6. *Let $\{T_1, \ldots, T_N\}$ be N positive operators on a Banach lattice E, and let us suppose that there exists an $S > 0$ such that*

(1) $T_j S = S T_j$ *for each* $j \in \{1, \ldots, N\}$,

(2) $(T_1, \ldots, T_N)$ *is uniform joint local quasinilpotent at* $x_0 > 0$, *and*

(3) $|K(x)| \leq S(|x|)$ *for each* $x \neq 0$.

Then $\{S, T_1, \ldots, T_N\}$ has a common nontrivial closed invariant ideal.

Proof. If T_{j_0} is not strictly positive for some $j_0 \in \{1, \ldots, N\}$, then the null ideal $\mathcal{N}_{T_1 + \cdots + T_N}$ is the desired nontrivial closed invariant subspace. So, we suppose that T_j is strictly positive for each $j \in \{1, \ldots, N\}$. Without loss of generality, we can also assume that $\|T_1 + T_2 + \cdots + T_N + S\| < 1$. Put $A = \sum_{n=0}^{\infty} (T_1 + \cdots + T_N + S)^n$, and let J denote the ideal generated by Ax_0; i.e.,

$$J = \{y \in E : \text{ there exists } \lambda > 0 \text{ such that } |y| \leq \lambda A x_0\}.$$

Clearly, J is a nonzero ideal that is invariant under $T_1 + \cdots + T_N + S$. Since $0 \leq T_1, \ldots, 0 \leq T_N$, $0 \leq S$, we have that $T_1 + \cdots + T_N + S$ dominates S and T_j for each $j \in \{1, \ldots, N\}$. Let us consider $x \in J$. Then $|x| \leq \lambda A x_0$ for some λ; hence, by Lemma 3.4, J is invariant under $T_1, \ldots, T_N, S$.

If $\overline{J} \neq E$, then $\overline{J}$ is a common nontrivial closed invariant ideal for $T_1, \ldots, T_N, S$. So, let us consider the case $\overline{J} = E$. In this case, since $K \neq 0$, there exists some $0 < y_0 \in J$ such that $K(y_0) \neq 0$. Since $(T_1, \ldots, T_N)$ is a uniform joint local quasinilpotent at x_0 and $0 < y_0 \leq kAx_0$ for some k, it is easy to verify that $(T_1, \ldots, T_N)$ is also a uniform joint local quasinilpotent at y_0, and, therefore, $T = T_1 + \cdots + T_N$ is quasinilpotent at y_0 (see Proposition 2.2), T commutes with S and S dominates a compact nonzero operator. Then, by Theorem 3.5, S, T have a common nontrivial closed invariant ideal, and this ideal is invariant under $\{S, T_1, \ldots, T_N\}$. $\square$

To make the work more legible to the reader, we will remember some basic notions and principal results of [**1**].

Definition 3.7. Let $T : E \to E$ be a positive operator defined on a Banach lattice E. The super right-commutant $[T\rangle$ is the collection of all positive operators $S : E \to E$ such that $ST - TS \geq 0$. That is,

$$[T\rangle = \{S \in \mathcal{L}(E)_+ : ST - TS \geq 0\}.$$

The proof of the following theorem can be found in [**1**, page 426].

Theorem 3.8. *If a nonzero compact-friendly operator $B : E \to E$ on a Banach lattice is quasinilpotent at some $x_0 > 0$, then B has a nontrivial closed invariant ideal. Moreover, for each sequence $\{T_n\}$ in $[B\rangle$, there exists a nontrivial closed ideal which is invariant under B and under each T_n.*

Theorem 3.9. *Let $(T_1, \ldots, T_N)$ be an N-tuple of nonzero positive operators on a Banach lattice E. If the N-tuple is joint compact-friendly and uniform joint local quasinilpotent at some $x_0 > 0$, then there exists a common nontrivial closed invariant ideal for $(T_1, \ldots, T_N)$. Moreover, for each sequence $\{S_n\}$ in $\cap_{j=1}^N [T_j\rangle$ there exists a nontrivial closed ideal which is invariant under $T_1, \ldots, T_N$ and each S_n.*

Proof. Let $(T_1, \ldots, T_N)$ be an N-tuple satisfying the hypothesis of the theorem. Let us consider the operator $S = T_1 + \cdots + T_N$. Clearly, S is compact-friendly and locally quasinilpotent at x_0. If $\{S_n\} \subset \cap_{j=1}^N [T_j\rangle$, then $\{S_n\} \subset [S\rangle$. Then, by Theorem 3.8, there exists a nontrivial closed invariant ideal under S and S_n for each $n \in \mathbf{N}$, and as in the proof of Theorem 3.7, we can see that this ideal is invariant under $(S_n, T_1, \dots, T_N)$. $\square$

For Dedekind complete Banach lattices, Theorem 3.8 was improved by proving that there always exists a nontrivial closed $[B\rangle$−invariant ideal, as the following result shows, whose proof can be found in [**1**, page 428].

Theorem 3.10. *If a nonzero compact-friendly operator $B : E \to E$ on a Dedekind complete Banach lattice is quasinilpotent at some $x_0 > 0$, then there exists a nontrivial closed ideal invariant under $[B\rangle$.*

Now we see an extension of this result to N-tuples of operators.

Theorem 3.11. *Let $(T_1, \dots, T_N)$ be an N-tuple of nonzero operators on a Dedekind complete Banach lattice E. If $(T_1, \dots, T_N)$ is joint compact-friendly and $(T_1, \dots, T_N)$ is uniform joint local quasinilpotent at some $x_0 > 0$, then there exists a nontrivial closed ideal that is invariant under each operator $B \in \cap_{j=1}^N [T_j\rangle$.*

Proof. Let us consider the operator $S = T_1 + \cdots + T_N$. Clearly S is compact-friendly and quasinilpotent at $x_0 > 0$. Therefore, by Theorem 3.10, there exists a nontrivial closed $[S\rangle$-invariant ideal. Since $\cap_{j=1}^N [T_j\rangle \subset [S\rangle$, this ideal is B-invariant for each $B \in \cap_{j=1}^N [T_j\rangle$. $\square$

4. Concluding remarks and open problems. We have introduced the concept of joint compact friendly to extend some results about invariant subspaces to the case of several variables. It would be interesting to investigate if these results are true if the N-tuple $\{T_1, \dots, T_N\}$ is compact-friendly and $T = (T_1, \dots, T_N)$ is not joint compact-friendly.

On the other hand, in [**6**] the notion of joint local quasinilpotence was introduced. An N-tuple $T = (T_1, \dots, T_N)$ is said to be joint locally quasinilpotent at $x_0 \in X$ if

$$\lim_{n\to\infty} \|T_{i_1}T_{i_2}\cdots T_{i_n}x_0\|^{1/n} = 0,$$

for all $\{i_1,\ldots,i_n\} \subset \{1,\ldots,N\}$.

It would be interesting to investigate whether the above results are true if we suppose that $T = (T_1,\ldots,T_N)$ is joint locally quasinilpotent instead of uniform joint locally quasinilpotent.

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