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LOCAL ALGEBRAIC INVARIANTS OF Δ-SETS

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In this note we define homology and homotopy groups at a vertex of a Δ-set. The theory is developed parallel to S. T. Hu's theory [1] of homology and homotopy groups at a point of a topological space. If one considers the space S of continuous functions of one differentiable manifold into another, it is well known that S is a differentiable manifold (usually modelled on some infinite dimensional linear space). When one attempts to put a piecewise linear structure on some piecewise linear function space, the attempt fails since the induced topology is wrong. Sacrificing the geometry, one considers these spaces as Δ-sets and the homotopy structure remains. The natural question arises to what extent can these Δ-sets be thought of as manifolds. The answer obtained in this paper for Δ-sets which naturally arise in piecewise linear topology is that the local homotopy and homology theory in the sense of Hu is similar.

In the last five sections, we calculate local invariants of some well-known Δ-sets.

1. **Δ-sets.** We recall some definitions and results from the Δ-set theory of C. P. Rourke and B. J. Sanderson [5] (or equivalently, the quasisimplicial theory of C. Morlet [4]). Note that, except in §8, we could have used semisimplicial sets; however the Δ-sets are easier to handle.

Let Δ^n denote the standard n-simplex with ordered vertices $v_0, v_1, \cdots, v_n$. The ith *face map* $\partial_i : \Delta^{n-1} \rightarrow \Delta^n$ is the order-preserving simplicial embedding which omits v_i. Δ is the category whose objects are Δ^n, $n = 0, 1, \cdots$, and whose morphisms are generated by the face maps. A *Δ-set* (*Δ-group*) is a contravariant functor from Δ to the category of sets (groups). A *Δ-map* between Δ-sets is a natural transformation between the functors.

If X is a Δ-set, $X^n = X(\Delta^n)$ is the set of n-simplexes and the maps $\partial_i \equiv X(\partial_i)$ are called *face maps*. A 0-simplex will also be called a *vertex*. We shall be interested in pointed Δ-sets $(X, *)$ in which we distinguish a simplex $*^n \in X^n$ for each n and designate $* \subset X$, the

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base point of X, as the Δ-subset of X consisting of these simplexes and whose face maps are defined by $\partial_i *^n = *^{n-1}$.

With each ordered simplicial complex K, we associate a Δ-set, also designated by K, whose n-simplexes are order-preserving simplicial embeddings of Δ^n into K.

Let $\Lambda_{n,i} = \mathrm{Cl}\,(\mathrm{bdry}\ \Delta^n - \partial_i \Delta^{n-1})$. A Δ-set X is called a *Kan Δ-set* if every Δ-map $f : \Lambda_{n,i} \to X$ can be extended to a Δ-map $f_1 : \Delta^n \to X$.

If X is a Kan Δ-set and P is a polyhedron, a *map* $f: P \to X$ is a Δ-map $f : K \to X$ where K is an ordered triangulation of P. $f_0, f_1 : P \to X$ are *homotopic* if there is a map $F : P \times [0, 1] \to X$ such that $F \mid P \times \{i\} = f_i$, $i = 0, 1$. $[P, X]$ denotes the set of homotopic classes. We need the following proposition which was proved by Rourke and Sanderson [5].

Proposition 1. *Any homotopy class in* $[P; X]$ *is represented by a* Δ*-map* $f : K \to X$ *where* K *is any ordered triangulation of* P.

If $(X, *)$ is a pointed Kan Δ-set, then the nth *homotopy group* of X, $\pi_n(X, *) = [I^n, \mathrm{bdry}\ I^n; X, *]$, the homotopy classes of Δ-maps of pairs, where I^n is the PL-n-cell.

If X is a Δ-set, let $C_n(X)$ be the free Abelian group generated by the elements of X^n. Define $\partial : C_n(X) \to C_{n-1}(X)$ by $\partial = \sum_{i=0}^{n} (-1)^i \partial_i$. Now one can proceed to define in the usual way the homology groups of X, $H_n(X)$, and the reduced homology groups $\tilde{H}_n(X)$.

A Δ-map $\pi : E \to B$ is called a *Kan fibration* if for all i, n, given a Δ-map $f : \Lambda_{n,i} \to E$ and an extension $f_1 : \Delta^n \to B$ of πf, then there is an extension f' of f such that $f_1 = \pi f'$.

Proposition 2. *A Kan fibration of Kan* Δ*-sets has the homotopy lifting property for maps of polyhedra.*

2. **Local homology and homotopy groups.** If X is a topological space and $x \in X$, Hu defined the local homotopy and homology at x by considering the space of all paths $f: [0, 1] \to X$ such that $f^{-1}(x) = 0$. We propose to define an analogous set; however, there is a difficulty in getting a condition which corresponds to $f^{-1}(x) = 0$ for arbitrary Δ-sets. We shall adopt a formal device which will naturally arise in the Δ-sets we shall consider later.

Let X be a Δ-set and let x be a vertex of X. For each polyhedron P, suppose that there is defined an equivalence relation α $(\sim_{\alpha(P)})$ on the collection of maps from P to X. Let $\mathcal{L}^\alpha(X, x)$ be the Δ-set in which an n-simplex is a Δ-map f: $\Delta^{n+1} \to X$ such that

L1. $f^{-1}(x) = v_0$;

L2. If we consider f as a map from the polyhedron underlying Δ^{n+1} to X and $f \sim_\alpha g$, then $g^{-1}(x) = v_0$. $\partial_i f$ is defined to be $f \mid \partial_{i+1} \Delta^n$, $i = 0, 1, \cdots, n$. X is *Kan at* x if $\mathcal{L}^\alpha(X, x)$ is a Kan Δ-set.

The (*reduced*) *local homology groups of* X *at* x are defined by $L_n{}^\alpha(X, x) = H_n(\mathcal{L}^\alpha(X, x))$ $(\bar{L}_n{}^\alpha(X, x) = \bar{H}_n(\mathcal{L}^\alpha(X, x)))$. If X is Kan at x, the *local homotopy groups of* X *at* x are defined by $\lambda_n{}^\alpha(X, x; *) = \pi_n(\mathcal{L}^\alpha(X, x), *)$.

If Y is a Δ-subset of X which contains x, then we can define $\mathcal{L}^\alpha(Y, x)$. Note that $\mathcal{L}^\alpha(Y, x)$ is a Δ-subset of $\mathcal{L}^\alpha(X, x)$ and hence we can define the local relative homology and homotopy groups of X mod Y at x in the usual way. Since these groups are, in fact, homology and homotopy groups of a Δ-set, many results in homology and homotopy theory can be restated in terms of the local groups. In particular, we have the following theorem which follows from [5, Theorem 8.1] and [3, Theorem 13.6].

THEOREM 1 (HUREWICZ). *If* X *is a* Δ*-set which is Kan at* x *and if* $\lambda_k{}^\alpha(X, x; *)$ *is trivial for* $k \leqq n$, $n \geqq 1$, *then* $\bar{L}_k{}^\alpha(X, x)$ *is trivial for* $k \leqq n$ *and* $\lambda_{n+1}^\alpha(X, x; *)$ *is isomorphic to* $\bar{L}_{n+1}^\alpha(X, x)$.

It should be noted that two different equivalence relations can give rise to different groups. For example, if β is the trivial equivalence relation $f_0 \sim_\beta f_1$ if and only if $f_0 = f_1$, then the groups corresponding to β in Theorem 6 are all trivial.

3. **Admissible maps.** Let X and Y be Δ-sets with vertices x and y, respectively, and with equivalence relations α and β, respectively. A Δ-map $f: X \to Y$ is *admissible* if

(i) $f^{-1}(y) = x$,

(ii) $g \sim_\alpha h$ implies $fg \sim_\beta fh$,

(iii) if $g: \Delta^n \to X$ is a Δ-map and if $g \sim_\alpha h$, then $h^{-1}(x) = (fg)^{-1}(y)$. If $f: X \to Y$ is an admissible map, then f naturally induces a Δ-map $f_{\mathcal{L}}: \mathcal{L}^\alpha(X, x) \to \mathcal{L}^\beta(Y, y)$ defined by $f_{\mathcal{L}}(g) = gf$. Hence we have induced maps

$$f_L : L_n{}^\alpha(X, x) \to L_n{}^\beta(Y, y),$$

$$f_{\bar{L}} : \bar{L}_n{}^\alpha(X, x) \to \bar{L}_n{}^\beta(Y, y),$$

$$f_\lambda : \lambda_n{}^\alpha(X, x; *) \to \lambda_n{}^\beta(Y, y; f_{\mathcal{L}}(*)).$$

PROPOSITION 3. *If* f *is an* Δ*-automorphism of* X *such that* $g \sim_\alpha h$ *implies* $fg \sim_\alpha fh$ *and if* X *is Kan at* x, *then* X *is Kan at* $f(x)$, f *is an admissible map, and the induced maps* f_L, $f_{\bar{L}}$, f_λ *are isomorphisms.*

4. **Local invariants of spaces of PL-automorphisms.** We shall assume familiarity with either [2] or [6]. Let M be a PL (= piecewise linear) manifold and let Aut M be the Δ-group in which an n-simplex is a PL automorphism $f: M \times \Delta^n \to M \times \Delta^n$ which is *level-preserving* (i.e., f commutes with the projection along the second factor). If P is a polyhedron, K_1 and K_2 are ordered triangulations of P and $f_i: K_i \to \text{Aut } M$ are Δ-maps, then the f_i's are PL-automorphisms of $M \times P$ which are level-preserving. We define $f_1 \sim {}_\alpha f_2$ if and only if $f_1(x) = f_2(x)$ for all $x \in M \times P$. Define $\partial_i f = f|M \times \partial_i \Delta^{n-1}$. If P is a subpolyhedron of M, then Aut (M mod P) is a Δ-subgroup consisting of the n-simplexes f of Aut M such that $f|P \times \Delta^n$ is the identity map.

THEOREM 2. *Let M be a PL-manifold.*

(a) $\bar{L}_n^\alpha(\text{Aut } M, f) = \lambda_n^\alpha(\text{Aut } M, f; *) = 0$ *for all vertices* $f \in \text{Aut } M$ *and* $n \geqq 0$.

(b) $\bar{L}_n^\alpha(\text{Aut }(M \text{ mod } P), f) = \lambda_n^\alpha(\text{Aut }(M \text{ mod } P), f; *) = 0$ *for all vertices* $f \in \text{Aut }(M \text{ mod } P)$, $n \geqq 0$ *when P is a compact subpolyhedron of M of codimension at least three.*

(c) $\bar{L}_n^\alpha(\text{Aut }(M \text{ mod bdry } M), f) = \lambda_n^\alpha(\text{Aut }(M \text{ mod bdry } M), f; *) = 0$ *for all vertices* $f \in \text{Aut }(M \text{ mod bdry } M)$ *and* $n \geqq 0$.

We shall prove $\lambda_n^\alpha(\text{Aut }(M \text{ mod } P), f; *) = 0$; the proofs that the other local homotopy groups are trivial are similar. The proof of the theorem is then completed by referring to the Hurewicz Theorem.

Let $m_1, m_2, \cdots, m_r$ be r distinct interior points of $M - P$ where $rm \geqq 2n + 5$. Let E be the set of embeddings of $\{m_1, m_2, \cdots, m_r\}$ into the interior of $M - P$ with the compact-open topology. E is homeomorphic with an open subset of the product of r copies of M and hence is an rm-dimensional PL-manifold where m = dimension M.

Let E_0 be the Δ-set consisting of n-simplexes f where f: $\{m_1, m_2, \cdots, m_r\} \times \Delta^n \to \text{int }(M - P) \times \Delta^n$ is a level-preserving PL-embedding and $\partial_i f = f|\partial_i \Delta^{n-1}$. Let K be an ordered simplicial complex and let $f: K \to E$ be a piecewise linear map. f induces a level-preserving PL-map f: $\{m_1, m_2, \cdots, m_r\} \times K \to \text{int }(M - P) \times K$. Note that f' induces a Δ-map $\xi(f): K \to E_0$. The following proposition is easy to prove.

PROPOSITION 4. *ξ is a bijection between piecewise linear maps of K into E and Δ-maps of K into E_0 and induces a bijection between the homotopy classes of such maps.*

Define a Δ-map $\rho: \text{Aut }(M \text{ mod } P) \to E_0$ by restriction; — i.e., if

f is an n-simplex, then $\rho(f) = f \mid \{m_1, m_2, \cdots, m_r\} \times \Delta^n$. From [4, p. 379], we have the following.

PROPOSITION 5. *ρ is a Kan fibration of Kan Δ-sets.*

PROPOSITION 6. *Let M and P be as in Theorem* 2. Aut (M mod P) *is Kan at each vertex of* Aut (M mod P).

PROOF. Let f be a vertex of Aut (M mod P) and let $g : \Lambda_{n,i} \to \mathcal{L}^\alpha(\text{Aut}\,(M \bmod P), f)$ be a Δ-map. Hence g is a Δ-map $\Lambda \to \text{Aut}\,(M \bmod P)$ where Λ is the join of v_0 and $\partial_0\Lambda_{n,i}$ in Δ^{n+1} and such that g satisfies L1 and L2. Therefore, g induces a level-preserving PL-automorphism of $M \times \Lambda$ which we shall also call g such that $g \mid P \times \Lambda$ is the identity and such that $g_x = f$ if and only if $x = v_0$ where g_x is the PL-automorphism of M defined by $g(m, x) = (g_x(m), x)$, $(m, x) \in M \times \Lambda$. We wish to extend g to a level-preserving PL-automorphism of $M \times \Delta^{n+1}$ with similar properties.

Consider the PL-map $g' = \xi^{-1}\rho(g) : \Lambda \to E$. Since $rm \geqq 2n + 5$, by general position, there exists a PL-homotopy $k' : \Lambda \times I \to E$ such that $k_0' = g'$, k_t' is a PL-homeomorphism for $t > 0$, and image $k_t' \cap \text{image } k_s' = \emptyset$ for $t \neq s$. By Proposition 4, we have a Δ-map $\xi(k') : \Lambda \times I \to E_0$ such that we have the following commutative diagram.

$$\begin{array}{ccc}
\Lambda & \xrightarrow{\ g\ } & \text{Aut}\,(M \bmod P) \\
\Big\downarrow{\scriptstyle \text{Id} \times 0} & & \Big\downarrow{\scriptstyle \rho} \\
\Lambda \times I & \xrightarrow{\ \xi(k')\ } & E_0
\end{array}$$

Hence by Propositions 2 and 5, there is a Δ-homotopy $G : \Lambda \times I \to \text{Aut}\,(M \bmod P)$ such that $G_0 = g$ and $\rho G = \xi(k')$. Let $h : (\Lambda \times I, \Lambda \times \{0\}) \to (\Delta^{n+1}, \Lambda)$ be a PL-homeomorphism of pairs such that $h(y, 0) = y$ for each $y \in \Lambda$.

$k = (\text{id} \times h)\ G\ (\text{id} \times h^{-1})$ is the desired PL-automorphism of $M \times \Delta^{n+1}$. For it $(x, q) \in M \times \Lambda$, $k(x, q) = (\text{id} \times h)G(x, q, 0) = (\text{id} \times h)(G_0(x, q), 0) = (\text{id} \times h)(g(x, q), 0) = g(x, q)$. Suppose $k_x = f$ for some $x \in \Delta^{n+1}$. Therefore $k(m_i, x) = (f(m_i), x)$ and $G(m_i, q, t) = (f(m_i), q, t)$, $i = 1, 2, \cdots, r$, where $h(q, t) = x$. Hence $\rho f = \rho G_{(q,t)} = \xi(k'_{(q,t)})$ and $\xi^{-1}\rho f = k'_{(q,t)}$. Since image $k_t' \cap \text{image } k_0' = \emptyset$ unless $t = 0$, we must have $t = 0$ and hence $x = v_0$. k induces the desired Δ-map $\Delta^n \to \mathcal{L}^\alpha(\text{Aut}\,(M \bmod P), f)$.

Define a base point $*$ for $\mathcal{L}^\alpha(\text{Aut}\,(M \bmod P), f)$ by letting $*^0 : \Delta^1 \to$

Aut (M mod P) be any vertex of $\mathcal{L}^\alpha$(Aut (M mod P), f) (the proof of the existence of $*^0$ follows from the fact that f represents an interior point of the PL-manifold E defined above). $*^i$, for $i \geqq 1$, is defined inductively by using the Kan condition.

PROOF OF THEOREM 2. Let g be a level-preserving PL-automorphism of $M \times \Delta^{n+1}$ which represents an element of λ_n^α(Aut (M mod P), f; $*$). Hence $g \mid M \times \Lambda_{n+1,0} = *^{n+1} | M \times \Lambda_{n+1,0}$. To show that g is trivial, it suffices to find a level-preserving PL-automorphism k of $M \times (\Delta^{n+1} \times [0, 1])$ such that

(i) $k(m, x, t) = (*^{n+1}(m, x), t)$ for $x \in \Lambda_{n+1,0}$, $m \in M$, $t \in [0,1]$;

(ii) $k(m, x, 0) = (g(m, x), 0)$ for $m \in M$, $x \in \Delta^{n+1}$;

(iii) $k \mid P \times (\Delta^{n+1} \times [0, 1]) =$ identity;

(iv) $k_{(x,t)} = k_{(v_0,0)}$ only if $(x, t) = (v_0, 0)$.

Using the fact that the pair $(\Delta^{n+1} \times [0, 1], \Delta^{n+1} \times \{0, 1\} \cup \Lambda_{n+1,0} \times I)$ is PL-homeomorphic to $(\Delta^{n+1} \times [0, 1], \Delta^{n+1} \times \{0\})$, k can be constructed as the k of Proposition 6 was constructed.

5. **Local invariants of spaces of PL-embeddings.** Let M be a PL-manifold and let P be a compact subpolyhedron of M. Let PL(P, M) be the Δ-set in which an n-simplex is a level-preserving PL-embedding of $P \times \Delta^n$ into $M \times \Delta^n$. If Q is a polyhedron, K_1 and K_2 are ordered triangulations of Q and $f_i : K_i \to$ PL(P, M) are Δ-maps, then the f_i's are PL-embeddings of $P \times Q$ into $M \times Q$ which are level-preserving. We define $f_1 \sim_\alpha f_2$ if and only if $f_1(x) = f_2(x)$ for all $x \in P \times Q$.

THEOREM 3. *Let M be a PL-manifold and let P be a compact subpolyhedron of codimension at least three and dimension $\geqq 1$ contained in the interior of M.* $\bar{L}_n^\alpha(PL(P, \text{int } M), f) = \lambda_n^\alpha(PL(P, \text{int } M), f; *)$ $= 0$ *for all vertices* $f \in PL(P, \text{int } M)$ *and* $n \geqq 0$. (An analogous theorem can be proved when $P \cap \text{bdry } M \neq 0$; but in this case, one must work with PL-embeddings of pairs.)

PROPOSITION 7. *Let M and P be as in Theorem* 3; *then* $PL(P, \text{int } M)$ *is Kan at each of its vertices.*

PROOF. Let f be a vertex of PL(P, int M) and let $g : \Lambda_{n,i} \to \mathcal{L}^\alpha$(PL($P$, int M),f) be a Δ-map; then g is a Δ-map $\Lambda \to$ PL(P int M) and induces a level-preserving PL-embedding $f: P \times \Lambda \to M \times \Lambda$. By [2, Remark 2, p. 154], there is a level-preserving PL-automorphism g' of $M \times \Lambda$ such that $g'_{v_0} =$ identity and $g'(f \times \text{id}) = g$. Proceed as in Proposition 6 except that one should choose $\{m_1, m_2, \cdots, m_r\}$ from $f(P)$.

6. **Local invariants of PL_m and $\mathrm{PL}_{m,n}$.** *Let* R^m be Euclidean m-space

with its usual PL-structure and let 0 be the origin. Let Aut (R^m mod $N(0)$) be the Δ-subgroup of Aut (R^m mod $\{0\}$) consisting of the n-simplexes f such that there exists a neighborhood U of 0 such that $f \mid U \times \Delta^n$ is the identity. The Δ-quotient group Aut (R^m mod $\{0\}$)/Aut (R^m mod $N(0)$) is PL_m, *the* Δ-*group of germs of PL-automorphisms of* R^m. Let α be the equivalence relation induced from the equivalence relation defined on Aut (R^m mod $\{0\}$) in §4.

THEOREM 4. $\bar{L}_n^{\alpha}(PL_m, f) = \lambda_n^{\alpha}(PL_m, f;\ *) = 0$ *for all* $n \geqq 0$ *and vertices* $f \in PL_m$.

The proof of Theorem 4 is analogous to the proof of Theorem 2, but in the place of the manifold E which we used to distinguish maps, we must use a different space. Let $u_1, u_2, \cdots, u_r$ be r distinct points in $R^m - \{0\}$ $(r \geqq 3)$ such that the cone $K_r = 0\{u_1, u_2, \cdots, u_r\}$ is nonsingular. Let PL(K_r, R^m mod $\{0\}$) be the Δ-subset of PL(K_r, R^m) consisting of the n-simplexes f such that $f \mid \{0\} \times \Delta^n$ is the identity. Let Aut (K_r mod $N(0)$) be the Δ-subgroup of Aut K_r consisting of the n-simplexes f such that $f \mid U \times \Delta^n$ is the identity for some neighborhood U of 0 in K_r.

Let f and g be n-simplexes of PL(K_r, R^m mod $\{0\}$); $f \simeq g$ if and only if there exists a neighborhood U of $\{0\}$ such that $f \mid U \times \Delta^n = g \mid U \times \Delta^n$. Clearly $\simeq$ is an equivalence relation; let G be the resulting Δ-set of equivalence classes. Let A be the Δ-quotient group Aut K_r/Aut (K_r mod $N(0)$) and note that A acts on G in the following way. Let $[g] \in A$ and $[f] \in G$ be n-simplexes and define $[g]\ [f] = [gf]$ where gf is the composition of the two functions. Let $\mathcal{E}(r, m)$ be the Δ-set which is the orbit space of this action.

Let h be an n-simplex in Aut (R^m mod $\{0\}$) and let $h' = h \mid K_r \times \Delta^n$. Define ρ : Aut (R^m mod $\{0\}$) $\rightarrow \mathcal{E}(r, m)$ by $\rho(h) = [[h']]$.

PROPOSITION 8. ρ *is a Kan fibration of Kan* Δ-*sets.*

PROOF. Let $f: \Lambda_{n,i} \rightarrow$ Aut (R^m mod $\{0\}$) be a Δ-map and let $f_1 : \Delta^n \rightarrow \mathcal{E}(r, m)$ be a Δ-map which is an extension of ρf. Let g be an n-simplex of PL(K_r, R^m mod $\{0\}$) such that $[[g]] = f_1(\Delta^n)$. By [2, Remark 2, p. 154], there is a level-preserving PL-automorphism G of $R^m \times \Delta^n$ such that $G(g_{v_0} \times \mathrm{id}) = g$. Hence G is an n-simplex of Aut (R^m mod $\{0\}$). Let $\tilde{G} : \Lambda_{n,i} \rightarrow$ Aut (R^m mod $g_{v_0}(K_r)$) be the Δ-map induced from $\tilde{G} = G^{-1}f$: $R^m \times \Lambda_{n,i} \rightarrow R^m \times \Lambda_{n,i}$. Aut ($R^m$ mod $g_{v_0}(K_r)$) is a Kan Δ-set, hence $\tilde{G}$ can be extended to $\tilde{G}_1 : \Delta^n \rightarrow$ Aut (R^m mod $g_{v_0}(K_r)$). $G\tilde{G}_1$ is the desired extension.

Let $\sigma \in S_r$, the symmetric group on $\{1, 2, \cdots, r\}$, and define an action on $\mathrm{PL}(\{s_1, s_2, \cdots, s_r\}, S^{m-1})$ by $\sigma(f)\ (s_i, x) = f(s_{\sigma(i)}, x)$ where f is an n-simplex of $\mathrm{PL}(\{s_1, s_2, \cdots, s_r\}, S^{m-1})$, $x \in \Delta^n$, and $s_1, s_2, \cdots, s_r$ are r distinct points in S^{m-1}. Let $\mathcal{G}_0(r, m-1)$ be the resulting Δ-orbit space.

Let $\varphi : \Delta^m \to R^m$ be a PL-embedding such that $\varphi(\Delta^m)$ is the cone over $\varphi(\text{bdry } \Delta^m) = \varphi(S^{m-1})$ with vertex 0. Let $[g]$ be an n-simplex of $\mathcal{G}_0(r, m-1)$ and let $f(g)$ be a level-preserving PL-embedding of $K_r \times \Delta^n$ into $R^m \times \Delta^n$ defined by

$$f(g)(u_i, x) = (\varphi g_x s_i, x), \qquad i = 1, 2, \cdots, r;\ x \in \Delta^n,$$

$$f(g)(0, x) = (0, x), \qquad x \in \Delta^n,$$

and extend linearly. Define $\lambda : \mathcal{G}_0(r, m-1) \to \mathcal{E}(r, m)$ by $\lambda([g]) = [[f(g)]]$.

PROPOSITION 9. *λ is a Δ-isomorphism.*

PROOF. λ is a well-defined Δ-map since one can easily find $[h] \in A$ such that $[h]\ [f(g)] = [f(\sigma(g))]$. We shall construct a Δ-map $\mu : \mathcal{E}(r, m) \to \mathcal{G}_0(r, m-1)$ such that $\mu\lambda$ and $\lambda\mu$ are the identities on the respective Δ-sets. Let $[[f]]$ be an n-simplex in $\mathcal{E}(r, m)$ and let K and L be triangulations of $K_r \times \Delta^n$ and $R^m \times \Delta^n$ such that $f: K \to L$ is simplicial. Let N be a convex regular neighborhood of $\{0\} \times \Delta^r$ which is contained in the first derived neighborhood of $\{0\} \times \Delta^r$ in L and let $w_i(x) = f_x(Ou_i) \cap \text{bdry } N$, $x \in \Delta^n$, $i = 1, 2, \cdots, r$. Let $\varphi^i(x) = \varphi^{-1}(Ow_i(x) \cap \varphi(S^{m-1}))$ and $\mu([[f]]) = [y(f)]$ where $y(f)(s_i, x) = (y_i(x), x)$.

If $f \simeq g$, then $f \mid f^{-1}N = g \mid f^{-1}N$ and $y(f) = y(g)$. Let $h \in \text{Aut } K_r$; from the construction above it should be clear that image $y(f) =$ image $y(hf)$ and hence $[y(f)] = [y(hf)]$ Therefore μ is a well-defined Δ-map. It is easy to see that $\mu\lambda$ and $\lambda\mu$ are identities.

Let E be the set of embeddings of $\{s_1, s_2, \cdots, s_r\}$ into S^{m-1} with the compact-open topology; the symmetric group acts naturally on E and let $\mathcal{G}(r, m-1)$ be the resulting orbit space. It is well known that $\mathcal{G}(r, m-1)$ is an open PL-manifold of dimension $r(m-1)$.

As in §4, we can define a map β from the set of piecewise linear maps of a simplicial complex K into $\mathcal{G}(r, m-1)$ and the set of Δ-maps from the Δ-set K into $\mathcal{G}_0(r, m-1)$.

PROPOSITION 10. *β is a bijection and induces a bijection between the homotopy classes of the respective maps.*

PROPOSITION 11. *PL_m is Kan at each of its vertices.*

PROOF. Let $G : \Lambda \to \mathrm{PL}_n$ be a Δ-map satisfying L1 and L2 which represents a Δ-map $\Lambda_{n,i} \to \mathcal{L}^{\alpha}(\mathrm{PL}_m, [f])$. G can be represented by a level-preserving PL-automorphism g of $R^m \times \Lambda$ such that $g_x = f$ if and only if $x = v_0$.

Let r be chosen such that $rm \geqq 2n + 7$ and let $g' = g \mid K_r \times \Lambda$. g' induces a Δ-map $y(g') : \Lambda \to \mathrm{PL}(\{s_1, s_2, \cdots, s_r\}, S^{m-1})$ (see definition of μ above) and a Δ-map $[y(g')] : \Lambda \to \mathcal{G}_0(r, m-1)$. Consider the PL-map $\beta^{-1}[y(g')] : \Lambda \to \mathcal{G}(r, m-1)$. By general position, there exists a simplicial homotopy $k' : \Lambda \times I \to \mathcal{G}(r, m-1)$ such that $k_0' = \beta^{-1}[y(g')]$, k_t' is a homeomorphism if $t \neq 0$, and image k_t' $\cap$ image $k_s' = \emptyset$ for $t \neq s$.

Consider $\lambda\beta(k') : \Lambda \times I \to \mathcal{E}(r, m)$ and note that $\lambda\beta(k') \mid \Lambda \times \{0\} = \rho(g)$. By Propositions 2 and 8, there is a Δ-homotopy $H : \Lambda \times I \to \mathrm{Aut}\,(R^m \bmod \{0\})$ such that $H_0 = g$ and $\rho H = \lambda\beta(k')$. Let $h : (\Lambda \times I, \Lambda \times \{0\}) \to (\Delta^{n+1}, \Lambda)$ be as in Proposition 6 and let $k = (\mathrm{id} \times h)H(\mathrm{id} \times h^{-1})$. Proceed as in Proposition 6 to show that $[k]$ is the desired element of PL_m.

The proof of Theorem 4 is analogous to the proof of Theorem 2.

Consider $R^n = R^n \times 0 \subseteq R^n \times R^{m-n} = R^m$ and $\mathrm{PL}_{m,n} = \mathrm{Aut}\,(R^m \bmod R^n)/(\mathrm{Aut}\,(R^m \bmod N(0)) \cap \mathrm{Aut}\,(R^m \bmod R^n))$. Let α be the equivalence relation on $\mathrm{PL}_{m,n}$ induced from the equivalence relation on $\mathrm{Aut}\,(R^m \bmod R^n)$.

THEOREM 5. *If* $n \neq m-2$, *then* $\bar{L}_i^{\alpha}(PL_{m,n}, f) = \lambda_i^{\alpha}(PL_{m,n}, f; *) = 0$ *for all* $i \geqq 0$ *and vertices* $f \in PL_{m,n}$.

The proof of this theorem is analogous to the proof of Theorem 4. The only difference in the proof is the choice of spaces and Δ-sets used in distinguishing maps. For example, one considers instead of $\mathrm{PL}(K_r, R^m \bmod \{0\})$, $\mathrm{PL}(K_r, (R^m - R^n) \cup \{0\}, \bmod \{0\})$ and instead of $\mathrm{PL}(\{s_1, s_2, \cdots, s_r\}, S^{m-1})$, $\mathrm{PL}(\{s_1, s_2, \cdots, s_r\}, S^{m-1} - S^{r-1})$.

7. **Local invariants of spaces of PL-embeddings of points.** Let $X = \{x_1, x_2, \cdots, x_r\}$ be a finite collection of distinct points and let M be an M-dimensional PL-manifold.

THEOREM 6. $\bar{L}_n^{\alpha}(PL(X, \mathrm{int}\, M), f) = 0$ *for* $n \neq mr - 1$ *and is the integers for* $n = mr - 1$; $\lambda_n^{\alpha}(PL(X, \mathrm{int}\, M), f; *) = \pi_n(S^{mr-1})$ *for all* $n \geqq 0$ *and vertices* $f \in PL(X, \mathrm{int}\, M)$.

Let E be the subspace of the product of r copies of M consisting of the r-tuples $(y_1, y_2, \cdots, y_r)$ such that if $i \neq j$, then $y_i \neq y_j$ for all i, j. E is an open mr-dimensional submanifold of the product of r copies of M. The theorem essentially follows from [1] and the following proposition.

PROPOSITION 12. *There is a bijection from the set of maps of a compact polyhedron P into $PL(X, \mathrm{int}\, M)$ and PL-maps of P into* $\mathrm{int}\, E$ *and a bijection between the homotopy classes of such maps.*

8. **Local invariants of pseudo-isotopy spaces.** Let M be a PL-manifold and let $\mathrm{Aut}_{PI}(M)$ be the Δ-group in which an n-simplex is a PL-automorphism f of $M \times \Delta^n$ such that if F is a face of Δ^n, then $f(M \times F) = M \times F$. One can define analogously $\mathrm{Aut}_{PI}(M \bmod P)$ and $\mathrm{PL}_{PI}(P, \mathrm{int}\, M)$. [$\widetilde{PL}_m = \mathrm{Aut}_{PI}(I^m \bmod (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}))$ is the Δ-group used in classifying block bundles.] We note that Theorems 2 and 3 can be generalized to these Δ-sets even though the analogue of Proposition 4 is false. Because of this difficulty, the conclusion of Theorem 6 must be changed to $\lambda_n{}^{\alpha}(\mathrm{PL}_{PI}(X, \mathrm{int}\, M), f; *) = \pi_n(\mathrm{PL}_{PI}(\{x\}, S^{mr-1}))$.

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