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SPACES WITH COMPACT SUBTOPOLOGIES

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Introduction. In [1], Banach posed the problem of characterizing metric spaces which have a coarser compact metrizable topology. Banach asked if the space c_0 has the property. Klee [5] answered the question affirmatively. The purpose of this paper is to answer Banach's question in some special cases and to study a class of spaces containing all those with compact metrizable subtopologies. A γ space X is a topological space whose topology is finer than a compact Hausdorff topology.

§1 consists of a theorem which allows us to restrict our attention to Tychonoff spaces and several examples. In §2 we show that the class of γ spaces is closed under sums and products, but not under quotients. In §3 it is proved that an example of Sierpinski of a non-γ space admits a complete metric. Finally in §4 we prove a theorem which shows the abundance of non-γ spaces.

1. DEFINITIONS 1.1. A topological space X is a γ *space* if there is some compact Hausdorff space K and a continuous bijection from X onto K. A space X has *property* Γ if it is metrizable and its topology is finer than some compact metrizable topology. A topological space X is an *s space* if the family $C(X)$ of real continuous functions on X separates the points of X. A completely regular space X is a *Baire space* if the intersection of countably many dense open subsets of X is necessarily dense in X.

EXAMPLE 1.2. Every γ space is an s space. Hence every γ space is Hausdorff. However, the family $C(X)$ need not separate points and closed sets. That is, a γ space X need not be completely regular. Let $\{Z \mid |Z| \leqq 1\}$ be the closed unit disc in the plane. Let $\mathcal{U}$ be the usual topology for X and B the boundary of X in the plane. Topologize X as follows: A set U is open if

(1) $U \subset X \backslash B$ and $U \in \mathcal{U}$ or

(2) $U \cap (X \backslash B) \in \mathcal{U}$ and $x \notin \mathcal{U} - \mathrm{cl}(X \backslash (B \cup U))$ for $x \in U$. Thus, one sees that open sets contained in $X \backslash B$ are as usual and open sets about a point p of B consist of all points in some $\mathcal{U}$-open set U about p except for the points of $B \backslash \{p\}$ in U and unions of sets of this type. Call this topology τ. Now, (X, τ) is not completely regular. In

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fact, if $Z \in B$ and $Z \in U \in \tau$, then no pair of disjoint τ-open sets can be found which separate Z and $X \backslash U$. Specifically, if V is any τ-open set containing Z, then either V must contain points of B different from Z or some points of B different from Z must not be contained in any open set which misses V. Thus (X, τ) is not even regular. It is easy to see that $i : (X, \tau) \rightarrow (X, \mathcal{U})$ is a continuous bijection of (X, τ) onto the compact space $(X, \mathcal{U})$.

However, in attempting to characterize the γ space, one need not be concerned with noncompletely regular spaces. It turns out that for each s space X there is a Tychonoff topology τ such that $C(X) = C(X, \tau)$. This result is due to Hewitt [4, p. 51]. The analogous proposition holds for γ spaces.

THEOREM 1.3. *Let (X, σ) be a γ space. Then there exists a Tychonoff topology τ for X which is coarser than σ such that (X, τ) is also a γ space and $C(X, \sigma) = C(X, \tau)$.*

PROOF. From the theory of rings of continuous functions, it can be seen that a Tychonoff space X is compact if and only if every maximal ideal of $C(X)$ is fixed at a point of X. Thus, a space X is a γ space if and only if the ring $C(X)$ contains a ring F with the property that the map $x \rightarrow M_x \cap F$ of X into the family of ideals of F is a bijection of X onto the family of maximal ideals of F. Now if (X, σ) is a γ space, let τ be the Tychonoff topology such that $C(X, \sigma)$ is isomorphic (as a ring) with $C(X, \tau)$. Now, since $C(X, \sigma)$ has a subring F such that $x \rightarrow M_x \cap F$ is a bijection between X and the maximal ideals of F, $C(X, \tau)$ also has this property. Thus (X, τ) is a γ space.

EXAMPLE 1.4. Having seen in Example 1.2 that not all γ spaces are Tychonoff (even regular), it is interesting to note that not all Tychonoff spaces are γ spaces. Let Q be the space of rational numbers with the usual topology. If Q were a γ space, one would necessarily have a countable, compact Hausdorff space. According to a theorem of R. Baire, such a space must have isolated points. In fact, the intersection of any countable family of open dense sets in a locally compact space is dense. But no continuous image of Q can have isolated points. Theorem 4.1 will generalize this example.

EXAMPLE 1.5. There is a Hausdorff space whose only compact Hausdorff continuous image is the one-point space. Let Z denote the positive integers. Topologize Z by choosing for an open basis all sets of the form $\{an + b \mid (a, b) = 1\}$. This is a connected topology [3] and by the same reasoning as above, one sees that its only compact continuous image is a singleton.

One is tempted to conjecture that there is some inclusion relation between real compact spaces and completely regular γ spaces or between Baire spaces and metrizable γ spaces. These, however, are false.

EXAMPLE 1.6. The Tychonoff Plank T is a γ space as will be seen by the next theorem (T is locally compact). However, T is not real-compact (see [2, p. 123]).

EXAMPLE 1.7. Let I denote the irrational numbers and Q the rational numbers each with the usual topology. Then $I \oplus Q$ is a metrizable non-Baire space which has figure "8" as a continuous bijective image.

2. In this section we show that the class of γ spaces is quite large. In fact it includes all sums and products of locally compact spaces.

THEOREM 2.1. *Every locally compact Hausdorff space is a γ space.*

PROOF. Let (X, τ) be a locally compact Hausdorff space and let $(\delta X, h)$ be its (unique) one-point compactification with ideal point ω. Let y_0 be a point of X. Construct a new space out of the points of X by giving X the quotient topology σ determined by the map $f: \delta X \to X$ defined by

$$f(x) = h^{-1}(x), \quad \text{if } x \neq \omega,$$
$$= y_0, \quad \text{if } x = \omega.$$

That is, a subset U of X is σ-open if and only if $f^{-1}(U)$ is open in δX. To see that (X, σ) is Hausdorff, let x and y be any two points of X. If $y = y_0$, find disjoint open sets U, V and W containing respectively $h^{-1}(x)$, $h^{-1}(y)$ and ω. Now $f(V \cup W)$ and $f(U)$ are clearly seen to be disjoint and to contain respectively x and y. Also both $f(V \cup W)$ and $f(U)$ are open since $f^{-1}f(V \cup W) = V \cup W$ and $f^{-1}f(U) = U$. The case in which both x and y are different from y_0 is trivial. Thus (X, σ) is Hausdorff.

Now, the map $fh: (X, \tau) \to (X, \sigma)$ is a continuous bijection and, of course (X, σ) is compact. Hence, (X, τ) is a γ space.

COROLLARY 2.2. *The sum of any family of γ spaces is a γ space.*

PROOF. Let $\{X_\alpha : \alpha \in A\}$ be a family of γ spaces, and for each α let K_α be a compact continuous bijective image of X_α. Now $\{\sum K_\alpha : \alpha \in A\}$ is a locally compact Hausdorff space (hence a γ space) which is a continuous bijective image of $\sum\{X_\alpha : \alpha \in A\}$. But any space which has a γ space for a continuous bijective image is clearly itself a γ space.

The product of γ spaces is also a γ space. If $\{X_\alpha : \alpha \in A\}$ is a family of γ spaces with K_α compact spaces and f_α a family of injections each from X_α onto K_α, then the function $f: \prod \{X_\alpha : \alpha \in A\} \to \prod \{K_\alpha : \alpha \in A\}$ defined by $[f(x)]_\alpha = f_\alpha(x_\alpha)$ is an injective continuous mapping of $\prod \{K_\alpha : \alpha \in A\}$ onto $\prod \{K_\alpha : \alpha \in A\}$. Further, if A is countable and each X_α satisfies property Γ then so does $\prod \{X_\alpha : \alpha \in A\}$.

EXAMPLE 2.3. There are completely regular γ spaces which are not products of locally compact spaces. Let

$$X = \{(x, y) \mid 0 < x < 1, 0 < y < 1\} \cup \{(0, 0)\}$$

and let X have the relative topology from the plane. X is not a finite product because it is not locally compact and X cannot be an infinite product because it has dimension 2.

THEOREM 2.4. *Let (X, τ) be a space. The following three conditions are equivalent*:

(1) *(X, τ) is a γ space.*

(2) *τ contains a compact Hausdorff topology.*

(3) *(X, τ) is homeomorphic with the graph of some (not necessarily continuous) function f defined on a compact Hausdorff space K into a (not necessarily Hausdorff) space Y.*

PROOF. If (X, τ) is a γ space with $f: X \to K$ a continuous bijection to the compact Hausdorff space K, then τ contains the compact Hausdorff topology $\{f^{-1}(U) \mid U \text{ open in } K\}$. Thus (1) implies (2). To see that (2) implies (3), let τ contain a compact Hausdorff topology σ for the set X. For each $U \in \tau$, let $\{0, 1\}_U$ denote the two-point Sierpinski space (with $\{1\}$ open but not closed). Let $Y = \prod \{0, 1\}_U \mid U \in \tau$ and give Y the product topology. For each $U \in \tau$ define the function $\chi_U : X \to \{0, 1\}_U$ according to

$$\begin{aligned} \chi_U(x) &= 1, \quad \text{if } x \in U, \\ &= 0, \quad \text{if } x \notin U. \end{aligned}$$

Now define the map f required in condition (3) by $(f(x))_U = \chi_U(x)$ for $x \in (X, \sigma)$. The graph $G(f)$ of f is homeomorphic with (X, τ). In fact $P_x|_{G(f)}$ is a homeomorphism on $G(f)$ onto (X, τ), where P_x is the projection of $X \times Y$ onto X. Now clearly $P_x|_{G(f)}^{-1}(U) = [(X \times T_U) \cap G(f)]$ where $T_U = \{g \in Y \mid g(U) = 1\}$. Since T_U is open in Y, $(X \times T_U) \cap G(f)$ is open in $G(f)$ and so $P_x|_{G(f)}$ is continuous. To see that $P_x|_{G(f)}$ is an open mapping, it suffices to show that there is a basis $\mathcal{U}$ of open subsets of $G(f)$ such that $P_x|_{G(f)}(V)$ is open for each $V \in \mathcal{U}$.

Let

$$\mathcal{U} = \left\{ \left(U_0 \times \bigcap_{i=1}^{n} T_{U_i} \right) \cap G(f) \mid \{U_i\}_{i=1}^{n} \subset \tau \text{ and } U_0 \in \sigma \right\}.$$

Now

$$P_x|_{G(f)} \left[\left(U_0 \times \bigcap_{i=1}^{n} T_{U_i} \right) \cap G(f) \right] = \bigcap_{i=0}^{n} U_i$$

is τ-open. Thus $P_x|_{G(f)}$ is open. Thus $P_x|_{G(f)}$ is a homeomorphism and (2) implies (3).

To see that (3) implies (1), let (X, τ) be homeomorphic with the graph $G(f)$ of the function f defined on the compact Hausdorff space K. Then the map $(x, f(x)) \to x$ is a continuous bijection of $G(f)$ onto K.

COROLLARY 2.5. *The space Q of rationals cannot be homeomorphic with the graph of any function on a compact Hausdorff space.*

COROLLARY 2.6. *A space X is a γ space if and only if X is homeomorphic with the graph of some function f on a γ space.*

COROLLARY 2.7. *If (X, τ) is any space and σ is a topology containing τ, then (X, σ) is homeomorphic with the graph of some function defined on (X, τ) into a topological space Y.*

We have seen that the class of γ spaces is closed under several operations. It is not, however, closed under taking quotients. Every first countable Hausdorff space, being a K space, is a quotient space of a locally compact space. But Example 1.4 shows that there are first countable non-γ spaces.

3. Having seen that every locally compact Hausdorff space is a γ space, one might next ask if every complete separable metric space is a γ space. The question has a negative answer as the following example shows.

EXAMPLE 3.1. For each positive integer n, let

$$A_n = \left\{ (x, y) \mid x = \frac{1}{n} \text{ and } 0 \leqq y \leqq 1 \right\}$$

$$\cup \left\{ (x, y) \mid x^2 + y^2 = \frac{1}{n^2} \text{ and not both } x \text{ and } y \text{ are positive} \right\}.$$

Let $A_0 = \{(0, 0)\}$. Set $S = \bigcup_{n=0}^{\infty} A_n$ and let S have the relative topology of the plane. The space $S \backslash \{(0, 0)\}$ was introduced by

Sierpinski. It is easy to see that S is a connected, nonlocally connected, nonlocally compact separable metric space which is the union of countably many pairwise disjoint compact continua. If S were a γ space and K were its compact continuous bijective image, then K would be decomposable into countably many pairwise disjoint subcontinua, contradicting a theorem of Sierpinski, see [6]. The purpose of this example, of course, is to show that the space S admits a complete metric consistent with its topology. To this end, let $z_n = (0, 1/n)$ if $n > 1$ and $z_0 = (0, 0)$. If p and q are two points of the plane, let $|p - q|$ denote the ordinary Euclidean distance from p to q. Let $M = \{z_n \mid n = 0, 1, 2, \cdots\}$. Now define a function $d : X \times X \rightarrow R$ by

$$d(p, q) = |p - q|, \text{if (1) } p = z_0 \text{ or } q = z_0, \text{ or}$$
$$\text{(2) if both } p \text{ and } q \text{ belong to the same } A_n,$$
$$= \inf \{|p - z_i| + |q - z_i| : z_i \in M\}, \quad \text{otherwise.}$$

Clearly the conditions $d(p, q) = 0$ if and only if $p = q$, $d(p, q) \geqq 0$, and $d(p, q) = d(q, p)$ are satisfied. Thus it must be shown that $d(p, q) \leqq d(p, r) + d(r, q)$ for any three points p, q and r of S. If p and q lie in the same A_n, the inequality is obvious. If p and q lie in different A_n, one considers two cases.

Case I. The point r lies in the same A_n as p or in the same A_n as q. For convenience, assume $r, p \in A_n$, $q \in A_m$ and that

$$\inf \{|p - z_i| + |q - z_i| : z_i \in M\} = |p - z_j| + |q - z_j|,$$
$$\inf \{|r - z_i| + |q - z_i| : z_i \in M\} = |r - z_k| + |q - z_k|.$$

Then

$$d(p, q) = |p - z_j| + |q - z_j| \leqq |p - z_k| + |q - z_k|$$
$$\leqq |p - r| + |r - z_k| + |q - z_k| = d(p, r) + d(r, q).$$

Case II. The point r lies in an A_n different from those in which p and q lie. As before, assume that

$$d(p, q) = |p - z_j| + |q - z_j|,$$
$$d(r, q) = |r - z_k| + |q - z_k|.$$

Also assume that

$$d(p, r) = |p - z_n| + |r - z_n|.$$

Now

$$\begin{aligned} d(p,q) &= |p - z_j| + |q - z_j| \leqq |p - z_k| + |q - z_k| \\ &\leqq |p - z_n| + |z_n - z_k| + |q - z_k| \\ &\leqq |p - z_n| + |z_n - r| + |r - z_k| + |q - z_k| \\ &= d(p,r) + d(r,q). \end{aligned}$$

Now it remains to show that the topology of d is the original topology and that d is complete. Convergence of a sequence in the d-topology clearly implies convergence in the usual topology. Suppose $\{x_n\} \to x_0$ in the usual topology and $x_0 \neq z_i$, $i = 0, 1, \cdots$. If $x_0 \in A_m$, then $\{x_n\}$ is eventually A_m. Then, $d(x_n, x_0) = |x_n - x_0| \to 0$. If $x_0 = z_k$, then $d(x_n, z_k) = \inf\{|x_n - z_i| + |z_k - z_i| : z_i \in M\} = |x_n - z_k| \to 0$. Therefore the topologies are the same. To show that (S, d) is complete, let $\{x_i\}$ be a Cauchy sequence which is not eventually in any arc A_n (otherwise, convergence is obvious). Let $\{x_{p_i}\}$ be a subsequence of $\{x_i\}$ satisfying

$$d(x_{p_i}, x_{p_{i+1}}) \leqq (\tfrac{1}{2})^i.$$

Let z_{p_i} be a sequence from $\{z_i : i = 0, 1, \cdots\}$ satisfying

$$d(x_{p_i}, x_{p_{i+1}}) = |x_{p_i} - z_{p_i}| + |x_{p_{i+1}} - z_{p_i}|.$$

The compactness of M assures that such a sequence exists. The sequence $\{z_{p_i}\}$ is Cauchy since

$$\begin{aligned} d(z_{p_i}, z_{p_j}) = |z_{p_i} - z_{p_j}| &\leqq d(x_{p_i}, x_{p_{i+1}}) + d(x_{p_{i+2}}, x_{p_{i+1}}) \\ &\leqq \frac{1}{2^i} + \frac{1}{2^{i+1}}. \end{aligned}$$

Thus the sequence $\{z_{p_i}\}$ converges (again using the compactness of M). Therefore $\{z_{p_i}\}$ is eventually a constant z' or converges to z_0. In the first case it is clear that $\{x_{p_i}\}$ converges to z'. In the second case choose N so large that $i > N \Rightarrow |z_{p_i} - z_0| < \epsilon/2$ and $d(x_{p_i}, x_{p_{i+1}}) < \epsilon/2$. Then, for $j > N$, one has

$$\begin{aligned} d(x_{p_j}, z_0) &\leqq d(x_{p_j}, z_{p_j}) + d(z_{p_j}, z_{p_0}) \\ &\leqq d(x_{p_j}, x_{p_{j+1}}) + d(z_{p_j}, z_0) < \epsilon. \end{aligned}$$

This completes the proof of the completeness of (S, d). One could also prove the complete metrizability of (S, d) by noting that S is a G_δ in its closure in the plane.

4. The following theorem shows that non-γ spaces are abundant.

THEOREM 4.1. *Let X be a space which can be expressed as the union of countably many compact sets X_i. If either one of the following pairs of conditions are met, then X is not a γ space*:

A_1. *X is separable metric and*

A_2. *each X_i is nowhere dense.*

B_1. *X is connected and*

B_2. *the X_i are pairwise disjoint.*

PROOF. Suppose $X = \bigcup_{i=1}^{\infty} X_i$ where each X_i is compact and conditions A_1 and A_2 are satisfied. If f is a continuous bijection on X to a compact Hausdorff space $f(X)$, then $f(X)$ is a separable metric space. In fact, since $f(X) \times f(X)$ is a continuous image of $X \times X$, it is hereditarily Lindelöf, and therefore (being regular) perfectly normal. But a compact Hausdorff space with a G_δ diagonal is metrizable. Since $f(X)$ is compact (hence complete) it must be of the second category. However, the $f(X_i)$ are closed nowhere dense subsets of $f(X)$ whose union is $f(X)$, a contradiction to the Baire category theorem.

Now suppose conditions B_1 and B_2 are satisfied and F is a continuous bijection on X to a compact Hausdorff space K. Then K is connected and is expressible as the union of the $f(X_i)$. But since f is one-to-one $f(X_i) \cap f(X_j) = \emptyset$ for $i \neq j$. Thus the compact continuum K is the union of countably many pairwise disjoint closed subsets of itself. This contradicts the theorem of Sierpinski [**6**]. This completes the proof.

COROLLARY 4.2. *The subspace $l_F(\omega)$ of l_1 consisting of sequences which are zero for all but finitely many coordinates is not a γ space.*

PROOF. The space $l_F(\omega)$ can be decomposed as follows:

$$l_F(\omega) = \bigcup_{i=j}^{\infty} \bigcup_{j=1}^{\infty} E_{i_j}$$

where

$$E_{i_j} = \{\{x_n\} \mid x_n = 0 \text{ if } n > i \text{ and } \max\{|x_1|, |x_2|, \cdots, |x_i|\} \leqq j\}.$$

Each E_{i_j} is compact and nowhere dense.

REMARK. The theorem above also shows that a separable metric space has property Γ if and only if it is a γ space. Thus the problem of characterizing γ spaces includes as a special case finding all separable metric spaces with property Γ.

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