ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 15, Number 1, Winter 1985

THE THEORY OF FORCED, CONVEX, AUTONOMOUS, TWO POINT BOUNDARY VALUE PROBLEMS

THEODORE LAETSCH

1. Introduction. We consider the semilinear, autonomous, forced, two point boundary value problem with a parameter λ,

$$u'' + bu' + \lambda f(u) = 0, \tag{1.1a}$$

$$\begin{aligned} \alpha_0 u(0) - \alpha_0' u'(0) &= \gamma_0 \\ \alpha_1 u(1) + \alpha_1' u'(1) &= \gamma_1. \end{aligned} \tag{1.1b}$$

Here f is a positive, convex function on the closed half-line $[0, +\infty)$, b is a real number, and α_i, α_i', and γ_i ($i = 0$ and 1) are nonnegative real numbers which satisfy additional conditions described in §2.

We give an almost complete description of the positive solutions of (1.1), with essentially no assumptions on f other than positivity and convexity, describing in detail how the structure of the solution set depends on the asymptotic behavior of the function f. It is well known that, at least for sufficiently smooth f, there exists a positive number λ^* such that (1.1) has solutions for positive $\lambda < \lambda^*$ and has no solutions for $\lambda > \lambda^*$. We show, using Leray-Schauder degree theory, that (1.1) has at most two positive solutions for each value of $\lambda > 0$ except for the special case described below. From known results, we deduce necessary and sufficient conditions on f for the existence of two solutions for certain values of λ and describe the values of λ for which two solutions exist.

The results presented here go beyond results already in the literature in several ways. First, the calculation of the fixed point index of the solutions and the proof of the existence of at most two solutions for all $\lambda \neq \lambda^*$; second, the conversion of the problem with a nonmonotonic convex nonlinearity to a problem with an isotone convex nonlinearity; third, the consideration of functions which are not strictly convex, and hence the possibility, for certain boundary conditions, of infinitely many solutions for $\lambda = \lambda^*$; finally, the lack of smoothness and monotonicity of f, with the convexity assumption only and the possibility that $f_+'(0) = -\infty$. These results show specifically how the results for the linear problem

Received by the editors on July 24, 1978, and in revised form on November 18, 1983.

generalize to the nonlinear one, since they include as a special case the situation where the "nonlinearity" f is a positive constant or a positive, increasing affine function. Also included are cases such as when f is a piecewise linear function (for example, $f(w) = |w - a| + b$ for $a > 0, b > 0$).

There is an exception to our claim of completeness, as well as to the assertion that there are at most two solutions for each $\lambda > 0$. Suppose the function f is affine on an interval $J \subseteq (0, +\infty)$ (say $f(\sigma) = m\sigma + c$ for $\sigma \in J$, with constants $m \geqq 0$ and $c \leqq 0$), and the boundary conditions are nonhomogeneous or non-Dirichlet. Then it may happen that (1.1) degenerates to an essentially linear problem and has, for one value of λ, infinitely many solutions. If J is bounded, we do not give sufficient conditions on f for this degeneracy to occur, although we do give necessary conditions and describe the solutions set in detail when this situation does occur.

The problem (1.1) has been analyzed in detail before (e.g., in **[32]** with $b = 0$ and boundary conditions $u(0) = u(1) = 0$) by ad hoc methods depending on the possibility of integrating (1.1) and getting an explicit expression for the relation between solutions u and corresponding values of λ. In contrast, the present approach studies (1.1) as a special case of more general equations. The theory presented may be viewed as the intersection of the theory of three kinds of more general problems. First, there is the convexity theory, with results valid for elliptic differential equations and Hammerstein integral equations and even more general convex operator equations in partially ordered Banach spaces **[28, 20, 30, 6]**. Second, there is the asymptotic theory, describing the behavior of solutions of large norm. This theory subdivides into two parts, viz., the very general theory of equations with asymptotically linear nonlinearities, and the more specialized theory of ordinary differential equations with superlinear nonlinearities **[2, 39, 3, 5, 25, 26]**. Finally, there are results which hold for very general autonomous, quasilinear ordinary differential equations, but do not hold for general nonautonomous ordinary differential equations and other more general equations of the type described above **[27]**; however, results of McLeod and Stuart **[35]** might permit generalization to a certain class of nonautonomous equations.

This paper is organized as follows. After describing our notation and assumptions in §2, a summary of the relation between the behavior of f and the number and behavior of solutions of (1.1) for different values of λ is given in §3. The solutions are described in more detail in §4 to §7; the fixed point index of the solutions is determined in §4 and the proof of the principal result on the existence of at most two solutions of (1.1) given in §7. §8 shows how to convert the problem (1.1) with a possibly nonmonotonic nonlinearity to an abstract problem with an increasing (isotone) nonlinearity, and §9 discusses the differentiability of the nonlinear operator we use to analyze (1.1). Some useful results on ordinary differential equations with possibly discontinuous coefficients are summarized in an Appendix.

Much of the qualitative material of sections 3, 4, 6, and part of 7 is given in Amann's review [**1**]; however, stronger assumptions on f, such as differentiability or strict convexity, are made in [**1**]. Since convexity itself implies continuity on open sets and differentiability except possibly at countably many points, it seems natural to obtain the results assuming only the convexity of f, as we have done here.

In addition to the references [**32**, **20**, **6**] already cited, closely related problems have been discussed in [**36**, **12**, **13**, **9**]. Other results on the existence of at most two solutions have been obtained in [**7**, **4**, **10**]. A general review of nonlinear problems in partially ordered spaces is given by Amann in [**1**], and we will make frequent reference to this article for results and original references.

2. Notation and Assumptions. We rewrite the differential equation (1.1a) in the form

$$Lu = \lambda f(u) \qquad \text{on } (0, 1), \tag{2.1a}$$

where $Lu = -u'' - bu'$, and the boundary conditions (1.1b) in the form

$$Bu = \gamma, \tag{2.1b}$$

where Bu and γ are two-component vectors, $Bu = (B_0u, B_1u)$ and $\gamma = (\gamma_0, \gamma_1)$, with

$$B_iu = \alpha_i u(i) - (-1)^i \alpha_i' u'(i) \qquad \text{for } i = 0 \text{ and } 1.$$

In addition to assuming that the numbers α_i, α_i', and γ_i are nonnegative, we assume that $\alpha_0\alpha_1 + \alpha_0\alpha_1' + \alpha_1\alpha_0' > 0$, and

$$\alpha_i + (-1)^i b\alpha_i' \geqq 0, \qquad \text{for } i = 0 \text{ and } 1. \tag{2.2}$$

The nonlinearity f, we recall, is assumed to be convex and never negative on $[0, +\infty)$; the convexity implies that f is continuous on $(0, +\infty)$, and we also require f to be continuous from the right at 0, i.e., $\lim_{w\to 0^+} f(w) = f(0) < +\infty$. We also need the following positivity assumption. Let g be the never negative function defined by (2.4) below. We assume that, for every $r > 0$, there exists $t \in (0, 1)$ such that

$$\inf\{f[w + g(t)] : 0 \leqq w \leqq r\} > 0. \tag{2.3}$$

This obviously holds if f is strictly positive on $[0, +\infty)$. If f is never decreasing on $(0, +\infty)$, then (2.3) holds if and only if at least one of the numbers $f(0)$, γ_0, or γ_1 is positive. The inequality (2.3) implies that (2.1) does not have the "trivial" solution $u = g$ for $\lambda > 0$ (if $u = g$ were a solution for some $\lambda > 0$, it would be a solution for every $\lambda \geqq 0$).

We make no further smoothness assumptions on f; in particular, f may have a negatively infinite derivative from the right, $f_+'(0)$, at 0. Furthermore, we do not assume that f is strictly convex (as is done, e.g., in **[20]** and **[6]**).

We extend f to a never negative (but not necessarily convex), continuous function on $(-\infty, +\infty)$ by defining $f(w) = f(0)$ for $w < 0$.

Let g be the unique (positive) solution of

$$\begin{aligned} Lg &= 0 \text{ on } (0, 1), \\ Bg &= \gamma. \end{aligned} \tag{2.4}$$

Explicitly, if $b \neq 0$,

$$g(t) = -\frac{\gamma_0(\alpha_1 - b\alpha_1')e^{-b} - \gamma_1(\alpha_0 + b\alpha_0') + (\alpha_0\gamma_1 - \alpha_1\gamma_0)e^{-bt}}{\alpha_0\alpha_1(1 - e^{-b}) + b(\alpha_0\alpha_1' e^{-b} + \alpha_1\alpha_0')}.$$

Then the boundary value problem (1.1) or (2.1) is equivalent to the equation

$$u = g + \lambda A u, \tag{2.5}$$

where A is the Hammerstein integral operator

$$Au(t) = \int_0^1 G(t, s) f(u(s)) ds \tag{2.6}$$

for a suitable positive Green function G. Since f is never negative, A maps every function u in the space $C[0, 1]$ of continuous functions on $[0, 1]$ into a never negative, twice continuously differentiable function Au. We denote the operator defined by the right hand side of (2.5) by $g + \lambda A$.

Any solution u of (2.1) is associated with exactly one corresponding value of λ, which we denote by $\boldsymbol{\lambda}[u]$. The set of all values of λ for which (2.1) has a positive solution is denoted by Λ, the set of solutions u is denoted by $\mathscr{U}$, and we define $\lambda^* = \sup(\Lambda)$. Thus $\boldsymbol{\lambda}$ is a mapping with domain $\mathscr{U}$ and range Λ.

Because of its convexity, f has a finite derivative from the right, $f_+'(w)$, and from the left, $f_-'(w)$, for each $w \in (0, +\infty)$. In §9, we show that the operator A, considered on the Banach space $C[0, 1]$, is Fréchet differentiable at every solution $u \in \mathscr{U}$. The Fréchet derivative $A'(u)$ is the linear operator on $C[0, 1]$ defined by

$$A'(u)h(t) = \int_0^1 G(t, s) f_+'(u(s)) h(s) ds. \tag{2.7}$$

The characteristic value problem

$$\phi = \mu A'(u)\phi \tag{2.8}$$

for $A'(u)$ may be written as a Sturm-Liouville eigenvalue problem

$$\begin{aligned} L\phi &= \mu f_+'(u)\phi \text{ on } (0, 1), \\ B\phi &= 0. \end{aligned} \tag{2.9}$$

Any (nonconstant) solution u of (1.1) for $\lambda > 0$ has exactly one relative maximum on $[0, 1]$; hence there exists $t_0 \in [0, 1]$ such that u is never decreasing on $[0, t_0]$ and never increasing on $[t_0, 1]$. Since f_+' is a never decreasing function on $(0, +\infty)$, the composite function $f_+'(u)$ is never decreasing on $[0, t_0]$ and never increasing on $[t_0, 1]$. Thus either $f_+'(u) \leqq 0$ almost everywhere on $[0, 1]$, or there exists a nontrivial subinterval of $(0, 1)$ on which $f_+'(u)$ is bounded below by a positive constant (cf. hypotheses of Proposition A-2).

The well known spectral properties [**18**, Ch. 10] of the Sturm-Liouville problem (2.9) hold, even though, in general, $f_+'(u)$ will only be integrable instead of continuous as usually assumed in the treatment of the Sturm-Liouville problem. In particular (see Appendix), if $f_+'(u)$ is positive on a nontrivial subinterval of $(0, 1)$, then (2.9) has a simple, smallest positive eigenvalue $\boldsymbol{\mu}_1[u]$, and the eigenfunction corresponding to $\boldsymbol{\mu}_1[u]$ has no zeros on $(0, 1)$. If $f_+'(u) \leqq 0$ on $(0, 1)$, then (2.9) has no positive eigenvalues, and we set $\boldsymbol{\mu}_1[u] = +\infty$.

We denote by $\boldsymbol{\mu}_2[u]$ the smallest positive eigenvalue of (2.9) greater than $\boldsymbol{\mu}_1[u]$; if $\boldsymbol{\mu}_1[u] = +\infty$, then $\boldsymbol{\mu}[_2u] = +\infty$.

A critical solution of (2.5) is a solution $u \in \mathscr{U}$ for which $\boldsymbol{\lambda}[u]$ is a characteristic value of (2.8); that is, the operator $I - \boldsymbol{\lambda}[u]A'(u)$ is not invertible (where I denotes the identity operator).

Because f is convex, $f(w)/w$ has a simple behavior as w increases from 0 to $+\infty$. There exist extended real numbers ρ_1' and ρ_2', with $0 \leqq \rho_1' \leqq \rho_2' \leqq +\infty$, such that $f(w)/w$ is strictly decreasing on $(0, \rho_1']$, constant on $(\rho_1', \rho_2']$, and strictly increasing on $[\rho_2', +\infty)$. (If $\rho_2' = +\infty$, then $f(w)/w$ is never increasing.) Thus $\lim_{w\to+\infty} f(w)/w = m_\infty$ exists, with $0 \leqq m_\infty \leqq +\infty$. These facts are used especially in the summary of results in §3.

If $m_\infty < +\infty$, then the operator $A'(\infty)$ defined on $C[0, 1]$ by

$$A'(\infty)h(t) = \int_0^1 G(t, s)m_\infty h(s)ds \tag{2.10}$$

is the asymptotic derivative of A with respect to the cone K of never negative functions in $C[0, 1]$ [**22**, p. 105]. If $0 < m_\infty < +\infty$, we will be interested in the characteristic value problem for the adjoint $A'(\infty)^*$, $\phi = \mu A'(\infty)^*\phi$, which is equivalent to the Sturm-Liouville problem

$$(2.11)\qquad \begin{aligned} L^*\phi &= \mu m_\infty \phi \text{ on } (0, 1), \\ B^*\phi &= 0, \end{aligned}$$

where $L^*\phi = -\phi'' + b\phi'$ and $B_i^*\phi = [\alpha_i + (-1)^i\alpha_i']u(i) - (-1)^i\alpha_i' u'(i)$, whose eigenvalues we denote by $\mu_n[\infty]$, with $0 < \mu_1[\infty] < \mu_2[\infty] < \cdots$. For $m_\infty = 0$, we define $\mu_n[\infty] = +\infty$ for $n = 1, 2, \ldots$; for $m_\infty = +\infty$, we define $\mu_n[\infty] = 0$ for $n = 1, 2, \ldots$. For $0 < m_\infty < +\infty$, ψ_∞ is the positive eigenfunction of (2.11), normalized so that $\int_0^1 \psi_\infty = 1$, and we set $\nu = \int_0^1 \psi_\infty g$; if $m_\infty = 0$, we define $\nu = 0$ for convenience.

In the following, the norm $\|u\|$ of a function $u \in C[0, 1]$ will always mean the usual $C[0, 1]$ maximum norm.

3. Summary of results. It is known that (2.5) possesses an unbounded continuum C of positive solution pairs (λ, u) in $\mathbf{R} \times C_1[0, 1]$ (see [**1**, §17]). It follows from results quoted or proved in the remainder of the paper that $\mathscr{C}$ is a curve in $\mathbf{R} \times C[0, 1]$; in a neighborhood of the origin, at least, this curve is given by the mapping $\lambda \to (\lambda, u_0(\lambda))$, $0 < \lambda < \lambda^* \leqq +\infty$, where $u^0(\lambda)$ is the smallest positive solution of (2.5) corresponding to the parameter λ, and the mapping u^0: $(0, \lambda^*) \to C[0, 1]$ is isotone (i.e., increasing) (cf. Theorem 4–1 below).

With the assumptions and notation of the preceding section (in particular, $m_\infty = \lim_{w\to+\infty} f(w)/w$ and $\nu = \int_0^1 \psi_\infty g$, so that $\nu = 0$ if the boundary conditions are homogeneous), the behavior of the curve $\mathscr{C}$ as determined by the function f is as follows (cf. Figures 1 and 3 in [**32**]).

Case 1. $0 \leqq m_\infty < +\infty$ and $[f(w) + m_\infty \nu]/w$ is strictly decreasing for all $w > 0$. Then $\lambda^* (= \sup(\Lambda)) = \mu_1[\infty]$, there exists a unique solution of (1.1) for each $\lambda \in (0, \lambda^*)$, and there are no solutions for $\lambda \geqq \lambda^*$. The solutions are increasing functions of the parameter λ. The curve $\mathscr{C}$ is given by $\{(\lambda, u^0(\lambda)): 0 < \lambda < \lambda^*\}$, and $\lim_{\lambda\to\lambda^*-}\|u^0(\lambda)\| = \infty$.

Case 2. $0 \leqq m_\infty < +\infty$ and $[f(w) + m_\infty \nu]/w$ is first strictly decreasing, but is eventually constant, say for $w \geqq \rho_1$. Then the conclusions are the same as in Case 1, except in the special case that f has the form $f(w) = m_\infty(w - \nu)$ for $w \geqq \rho_1$, and $(\alpha_0' + \gamma_0)(\alpha_1' + \gamma_1) > 0$. In this special case, there exists a unique solution for each $\lambda \in (0, \lambda^*)$, and infinitely many solutions for $\lambda = \lambda^*$. The solutions are increasing functions of the parameter λ for $\lambda \in (0, \lambda^*)$ (or$(0, \lambda^*]$). The curve $\mathscr{C}$ is given by $\{(\lambda, u^0(\lambda)): 0 < \lambda \leqq \lambda^*\} \cup \{(\lambda^*, u^0(\lambda^*)) + \sigma\phi^*: \sigma \geqq 0\}$, where ϕ^* is the positive eigenfunction defined in §5.

Case 3. $0 < m_\infty < +\infty$ and $[f(w) + m_\infty \nu]/w$ is eventually strictly increasing. Then $\lambda^* > \mu_1[\infty] > 0$ and there exists a unique solution for each $\lambda \in (0, \mu_1[\infty]]$, namely $u^0(\lambda)$, exactly two solutions, $u^0(\lambda)$ and $u^1(\lambda)$, for each $\lambda \in (\mu_1[\infty], \lambda^*)$, and either exactly one solution or an infinite number of solutions for $\lambda = \lambda^*$. The curve $\mathscr{C}$ has the form $\{(\lambda, u^0(\lambda)): 0 < \lambda < \lambda^*\} \cup \{(\lambda^*, u^0(\lambda^*) + \delta\phi^*): 0 \leqq \delta \leqq \delta^*\} \cup \{(\lambda, u^1(\lambda)): \lambda^* > \lambda > \mu_1[\infty]\}$, where the positive eigenfunction ϕ^* is defined in §5, δ^* is a nonnegative constant, $\lim_{\lambda\to\lambda^*-} u^0(\lambda) = u^0(\lambda^*)$, $\lim_{\lambda\to\lambda^*-} u^1(\lambda) = u^0(\lambda^*) + \delta^*\phi^*$, and $\lim_{\lambda\to\mu_1[\infty]+} \|u^1(\lambda)\| = \infty$. The case of infinitely many solutions for $\lambda = \lambda^*$ (i.e., $\delta^* > 0$) can occur only if there exists an interval on which f is affine, say $f(w) = mw + c$ for some constants $m > 0$ and $c \leqq 0$, and the parameters in the boundary conditions satisfy $(\alpha_0' + \gamma_0)(\alpha_1' + \gamma_1) > 0$. (For more details of this latter case, see §5.)

Case 4. $m_\infty = +\infty$. Then there exist exactly two solutions, $u^0(\lambda)$ and $u^1(\lambda)$, for each $\lambda \in (0, \lambda^*)$, and either exactly one or an infinite number of solutions (just as in Case 3) for $\lambda = \lambda^*$. The curve $\mathscr{C}$ is as in Case 3 with $\mu_1[\infty] = 0$.

The existence of at most two solutions is proved in §7 by Leray-Schauder theory using Theorem 4–5, which gives the fixed point index of each solution for $\lambda < \lambda^*$. A detailed analysis of the effect of an interval on which f is affine (not strictly convex) is given in §5. We do not give all the details of the arguments used to show that convexity alone, not differentiability or monotonicity, is sufficient to carry over results usually established under the stronger assumptions. In this regard, there are three essential points: (1) it is possible to convert the boundary value problem (2.1) with convex but not monotonic f to an operator equation $u = \mathfrak{B}(\lambda, u)$ with an operator $\mathfrak{B}$ which it isotone in (λ, u) and convex in u (see §8); (2) the convexity of f alone is sufficient to guarantee the Fréchet differentiability of the nonlinear operator A (equation (2.6)) at all solutions of (2.1) (see §9); and (3) the spectral theory of Sturm-Liouville boundary value problems which is used to study the nonlinear problem is valid for nonsmooth coefficients (see the remarks in the Appendix).

4. The fundamental solutions. In this section, we compute (Theorem 4.5) the Leray-Schauder fixed point index of solutions u of (1.1) or (2.1) with $0 < \lambda < \lambda^*$ (where λ^* is defined below). Using the notation and assumptions of §2, we first recall the properties of the smallest positive solutions, or fundamental solutions, of (2.1) or (2.5).

THEOREM 4–1. *The set Λ of values of λ for which* (2.1) *has a positive solution is nonempty and is a bounded or unbounded interval of the form* $(0, \lambda^*)$ *or* $(0, \lambda^*]$ (*either case may occur*). *For each $\lambda \in \Lambda$, equation* (2.5) *has a smallest positive solution $u^0(\lambda)$. The mapping $\lambda \to u^0(\lambda)$ is strictly isotone on Λ; if f is monotone, this mapping is strictly convex on Λ. In fact, for* (2.1), *$u^0(\lambda_1; t) < u^0(\lambda_2; t)$ for all $t \in (0, 1)$ if $\lambda_1 < \lambda_2$; the mapping $\lambda \to u^0(\lambda)$ is continuous from Λ into the Banach space $C_2[0, 1]$ of twice continu-*

ously differentiable functions on $[0, 1]$ *with the usual norm, and* $\lim_{\lambda \to 0+} u^0(\lambda) = g$ *in* $C_2[0, 1]$.

Most of this theorem follows as in [**21**, **28**, **24**, **40**, **1**, Theorem 21.2]. Here it is useful to observe that (2.1) with a convex but not necessarily increasing nonlinearity can be written in the form $u = B_\lambda u$, where $\{B_\lambda: \lambda > 0\}$ is an increasing family of compact, isotone, convex operators, even in the case $f'_+(0) = -\infty$; we prove this in §8. The fact that $\lambda_1 < \lambda_2$ implies $u^0(\lambda_2; t) - u^0(\lambda_1; t) > 0$, for all $t \in (0, 1)$, even when f is not necessarily increasing, can be proved using the maximum principle.

The next theorem characterizes the fundamental solutions in terms of the least positive eigenvalue of the associated linear problem (2.8). This is essentially a consequence of convexity. (Cf. [**21**, **31**, **20**, **14**].)

THEOREM 4–2. *Let* u *be a solution of* (2.1). *Then* u *is a fundamental solution, say* $u = u^0(\lambda)$, *with* $\lambda = \boldsymbol{\lambda}[u] \in (0, \lambda^*)$, *if and only if* $\boldsymbol{\lambda}[u] < \boldsymbol{\mu}_1[\mu]$.

The proof of this theorem follows the arguments given in the references just cited, making use of the inequality

$$f'_+(w_0)(w - w_0) \leqq f(w) - f(w_0) \leqq f'_+(w)(w - w_0) \tag{4.1}$$

for convex functions and Proposition A-2 in the Appendix.

The following theorem gives a similar characterization of the solutions (if any) corresponding to the supremum λ^* of Λ.

THEOREM 4–3. *Let* u *be a solution of* (2.1). *Then* $\boldsymbol{\lambda}[u] = \boldsymbol{\mu}_1[u]$ *if and only if* $\boldsymbol{\lambda}[u] = \lambda^*$ $(= \sup(\Lambda))$. *In particular, if* $\lambda^* \in \Lambda$, *then* $\lambda^* = \boldsymbol{\mu}_1[u^0(\lambda^*)]$.

PROOF. If $\boldsymbol{\lambda}[u] = \lambda^*$, then by Lemma 7–3, u must be a critical solution of (2.5). For the autonomous case, $\lambda^* = \boldsymbol{\mu}_1[u]$ then follows from the fact that $\boldsymbol{\lambda}[u] < \boldsymbol{\mu}_2[u]$ for all solutions u [**27**, Remark 2–6]. (The fact that for $u = u^0(\lambda^*)$ we have $\lambda^* = \boldsymbol{\mu}_1[u^0(\lambda^*)]$ can also be verified by a continuity argument based on the inequality $\lambda < \boldsymbol{\mu}_1[u^0(\lambda)]$, which holds for $\lambda \in (0, \lambda^*)$; cf. [**28**, **31**, **30**, **3**].) The converse is proved by using the left hand inequality in (4.1) and Proposition A-2 again.

As remarked in the preceding proof, for any positive solution u of (1.1), we have $\boldsymbol{\lambda}[u] < \boldsymbol{\mu}_2[u]$. (This is a consequence of the autonomy of (1.1) and the restriction (2.2) on b; see especially Remarks 2–6 and 2–7 of [**27**].) Combining this with the result of Theorems 4–2 and 4–3 that $\boldsymbol{\lambda}[u] \leqq \boldsymbol{\mu}_1[u]$ if and only if u is a fundamental solution or $\boldsymbol{\lambda}[u] = \lambda^*$, we obtain

THEOREM 4–4. *Let* u *be a solution of* (1.1) *with* $0 < \boldsymbol{\lambda}[u] < \lambda^*$. *Then* u *is larger than the fundamental solution* $u^0(\lambda)$ *of* (1.1) *with* $\lambda = \boldsymbol{\lambda}[u]$ *if and only if* $\boldsymbol{\mu}_1[u] < \boldsymbol{\lambda}[u] < \boldsymbol{\mu}_2[u]$.

We recall that the Leray-Schauder fixed point index of a noncritical fixed point $u = g + \lambda Au$ is $(-1)^\beta$, where β is the sum of the multiplicities of the real characteristic values (reciprocal eigenvalues) of $A'(u)$ lying between 0 and λ [**23**, 136]. Since the eigenvalue $\mu_1[u]$ of (2.1) is simple, we obtain from Theorems 4–2, 4–3, and 4–4 the following fundamental result.

THEOREM 4–5. *Let u be a solution of* (1.1) *with* $0 < \lambda[u] < \lambda^*$. *Then u is not a critical solution. The Leray-Schauder fixed point index of u is* $+1$ *if and only if u is a fundamental solution, and it is* -1 *if and only if u is not a fundamental solution.*

5. Critical Solutions. Suppose that $\lambda^*(= \sup(\Lambda)) \in \Lambda$. We have seen (Theorem 4–3) that the solutions u of (2.1) corresponding to $\lambda = \lambda^*$ are characterized by the fact that $\lambda(u) = \mu_1[u] = \lambda^*$. We now investigate the structure of the set U^* of solutions corresponding to $\lambda = \lambda^*$; from Theorem 4–1, we know there exists a smallest such solution, $u^0(\lambda^*) = u^*$.

As shown by the following theorems, the solution of (2.1) for $\lambda = \lambda^*$ will be unique unless f is an affine function (of the form $f(w) = mw + c$); in the case of nonuniqueness, we have the following results.

THEOREM 5–1. *If* (2.5) *has more than one solution corresponding to* $\lambda = \lambda^*$, *then it has an infinite number of solutions for* $\lambda = \lambda^*$. *All of these have the form* $u^* + \delta\phi^*$, *where* $u^* = u^0(\lambda^*)$, δ *is a nonnegative constant, and* ϕ^* *is a positive eigenfunction of* (2.8) *with* $u = u^*$, *corresponding to the characteristic value* $\lambda^* = \mu_1[u^*]$. *If, in addition,* $\gamma_0 = \gamma_1 = 0$ *in* (1.1b) (*homogeneous boundary conditions*), *then it is necessary that* $\alpha_0'\alpha_1' > 0$, *and* u^* *is a positive multiple of* ϕ^*.

DEFINITION 5.1. $\Delta = \{\delta \geqq 0 \colon u^* + \delta\phi^* \text{ is solution of (2.1) with } \lambda = \lambda^*\}$,
$U^* = \{u \geqq 0 \colon u = u^* + \delta\phi^* \text{ for some } \delta \in \Delta\}$,
$\sigma_1 = \operatorname{Inf}\{u(t) \colon u \in U^*, 0 \leqq t \leqq 1\}$,

and

$$\sigma_2 = \sup\{u(t) \colon u \in U^*, 0 \leqq t \leqq 1\}.$$

It is easily seen that Δ is an interval of the form $[0, \delta^*]$ or $[0, +\infty)$. The situation described in Theorem 5–1 can occur only if f is of the form described in the following theorem.

THEOREM 5–2. *Suppose that* (2.1) *has more than one solution for* $\lambda = \lambda^*$. *Then f has the form*

$$f(w) = mw + c \tag{5.2}$$

for some constants $m \geqq 0$ *and* $c \leqq 0$, *and all* $w \in [\sigma_1, \sigma_2]$. *The constant c must satisfy*

(5.3) $$\int_0^1 \psi^*(t)\,[mg(t) + c]dt = 0,$$

where g is the function defined by (2.4) *and* ψ^* *is a positive eigenfunction of the adjoint of* (2.9), *with* $u = u^*$, *corresponding to the eigenvalue* λ^*; *in addition, we must have* $\alpha_0' + \gamma_0' > 0$ *and* $\alpha_1' + \gamma_1' > 0$. *The constant c in* (5.2) *is zero if and only if* $\gamma_0 = \gamma_1 = 0$.

These theorems are a direct consequence of the Fredholm alternative or Lagrange's identities [**18**, p. 124] and the inequality for convex functions

$$f(w') - f(w) \geqq f_+'(w)\,(w' - w).$$

For more general situations than that considered here, see [**28**, Chapters I.10 and II.1; **33**]. For uniqueness in the case f is strictly convex, see Keener and Keller [**20**, Theorem 3.2].

We see from Theorem 5–2 that U^* is unbounded only if (5.2) holds for all sufficiently large w; conversely, if (5.2)–(5.3) hold for all sufficiently large w, then it is easily seen that U^* is unbounded.

Suppose we choose $\delta \in \Delta$, $\delta > 0$. Then for $0 < w \leqq w' \leqq \|u^* + \delta\phi^*\| = \rho$, we have

$$f(w') - f(w) \leqq f_-'(\rho)(w' - w) = m(w' - w),$$

where $m = f_+'(u^*) = f_-'(\rho)$ is the constant in (5.2). This inequality yields (cf. [**6**, Proposition 3.2)] the following theorem.

THEOREM 5–3. *Let* $\rho^* = \sup\{\|u\|\colon u \in U^*\}$. *Then for each* $\lambda \in (0, \lambda^*)$, *the fundamental solution* $u^0(\lambda)$ *of* (2.1) *is the only solution u of* (2.1) *with* $\|u\| \leqq \rho^*$.

6. Solutions of large norm and uniqueness. It is clear from the results outlined in §3 that the behavior of the solutions of large norm, as well as the multiplicty of solutions, depends on the behavior of $[f(w) + m_\infty\nu]/w$. Because of the convexity of f and the definition of m_∞, the condition that $[f(w) + m_\infty\nu]/w$ never increases for $w > 0$ is equivalent to each of the following conditions:

(i) $f(w) + m_\infty\nu - m_\infty w \geqq 0$ for all $w > 0$;
(ii) $f(w) + m_\infty\nu - wf_+'(w) \geqq 0$ for all $w > 0$.

On the other hand, if these conditions do not hold, then there exists $w_0 > 0$ such that $[f(w) + m_\infty\nu]/w$ is strictly increasing for $w \geqq w_0$, and this is equivalent to $f(w) + m_\infty\nu - m_\infty w < 0$ for $w \geqq w_0$. We shall use condition (i) in the comments on the proof of Theorem 6–1.

When the boundary conditions are homogeneous, so that $\nu = \int_0^1 \psi_\infty g = 0$, then it is the behavior of $f(w)/w$ which is important. When $f(w)/w$ is never increasing, we have a "sublinear" nonlinearity, i.e., $\tau f(w) \leqq f(\tau w)$

for all $w \geqq 0$ and all $\tau \in [0, 1]$, and for sufficiently strict sublinearity, uniqueness results are well-known [**22**, **34**, **1**, §25]. Theorem 6–1(b) includes the extension of these results to nonhomogeneous boundary conditions and nonstrict sublinearities.

THEOREM 6–1. (a) *If* $\{u_n\}$ *is any sequence of solutions of* (1.1) *with* $\|u_n\| \to +\infty$, *then* $\lambda[u_n] \to \mu_1[\infty]$. *If* $\lambda < \mu_1[\infty]$, *then the solution* $u^0(\lambda)$ *of* (1.1) *is unique.*

(b) *If* $0 \leqq m_\infty < +\infty$ *and* $[f(w) + m_\infty \nu]/w$ *is never increasing as* w *increases, then all solutions* u *of* (1.1) *satisfy* $\lambda[u] \leqq \mu_1[\infty]$. *If* $[f(w) + m_\infty \nu]/w$ *is strictly decreasing as* w *increases for all* $w > 0$, *then* $\lambda[u] < \mu_1[\infty]$ *for any solution* u *of* (1.1); *hence these solutions are the unique solutions* $u^0(\lambda)$, *and* $\lim_{\lambda \to \mu_1[\infty]-} \|u^0(\lambda)\| = +\infty$.

(c) *If* $m_\infty < +\infty$ *and* $[f(w) + m_\infty \nu]/w$ *is eventually strictly increasing, then* $m_\infty > 0$ *and there exists a number* $\rho > 0$ *such that* (1.1) *has no solutions with* $\lambda[u] \leqq \mu_1[\infty]$ *and* $\|u\| > \rho$.

Since (2.5) has solutions of arbitrarily large norm, Theorem 6–1 implies that if $0 < m_\infty$ and $f(w) + m_\infty \nu - m_\infty w \geqq 0$ for all w, then $\lambda[u]$ approaches $\mu_1[\infty] (\leqq +\infty)$ from below as $\|u\| \to +\infty$. But if $m_\infty < +\infty$ and $f(w) + m_\infty \nu - m_\infty w$ is eventually negative, or if $m_\infty = +\infty$, then $\lambda[u]$ approaches $\mu_1[\infty]$ ($\geqq 0$) from above as $\|u\| \to +\infty$. (Cf. [**25**])

Theorem 6–1 is proved as in [**6**; **1**, §19 and §26]. The fact that (a) is true also when $m_\infty = +\infty$ and $\mu_1[\infty] = 0$ is a consequence of the property of the operators (L, B) mentioned below. Since the references given generally assume strict sublinearity or convexity and smooth f, we outline a proof of (b) and (c) of Theorem 6–1 based on the inequality (i) above.

Suppose $0 < m_\infty < +\infty$. If u satisfies (2.1), and ψ_∞ is defined as in connection with (2.11), then, using the identity

$$\int_0^1 [\psi_\infty(t)\, Lu(t)]dt = \int_0^1 [L^*\psi_\infty(t)\, (u(t) - g(t))]dt,$$

we obtain

$$\lambda \int_0^1 [\psi_\infty(t) f(u(t))]dt = \mu_1[\infty]\, m_\infty \int_0^1 [\psi_\infty(t)\, (u(t) - g(t))]dt.$$

With the use of the normalization $\int_0^1 \psi_\infty = 1$ and the definition $\nu = \int_0^1 \psi_\infty g$, this becomes

$$\lambda \int_0^1 [\psi_\infty(t)(f(u(t)) + m_\infty \nu - m_\infty u(t))]dt \tag{6.1}$$
$$= m_\infty(\mu_1[\infty] - \lambda) \int_0^1 [\psi_\infty(t)\, (u(t) - g(t))]dt.$$

Note that $u - g = \lambda Au \geqq 0$. Part (b) of Theorem 6–1 follows immediately.

Part (c) of Theorem 6–1 follows from this identity together with the fact that the operator L with the boundary operator B (with condition (2.2) satisfied) has the property that there exists a continuous function γ, with $\gamma(x) > 0$ for $x \in (0, 1)$, such that if $u \in C^2[0, 1]$ satisfies $Lu \geqq 0$, $Bu \geqq 0$, then $u(x) \geqq \|u\|\gamma(x)$ (cf. **[29]**). By convexity and the remarks in connection with inequality (i) above, under the hypothesis of (c) there is a $w_1 > 0$ and $\varepsilon > 0$ such that $f(w) + m_\infty \nu - m_\infty w < -\varepsilon$ for all $w \geqq w_1$. By the property of (L, B) just mentioned, for any closed subinterval I of $(0, 1)$ we can find a number ρ_I such that for any solution $u \in U$ with $\|u\| \geqq \rho_I$ we have $u(x) \geqq w_0$ for all $x \in I$. The assumption of the existence of a sequence of solutions $\{u_n\}$ with $\|u_n\| \to \infty$ and $\boldsymbol{\lambda}[u_n] \leqq \boldsymbol{\mu}_1[\infty]$ is then easily seen to contradict the identity (6.1).

7. Existence and uniqueness of the second solution. Suppose that Δ (see equation (5.1)) is a nonempty, bounded interval, so that the set $U^* = \{u^* + \delta\phi^*: \delta \in \Delta\}$ (where $u^* = u^0(\lambda^*)$) is a compact subset of the positive cone K. Choose a bounded open set B containing U^* in the Banach space $C[0, 1]$ whose boundary ∂B is a positive, finite distance from U^*. Since the operator A is compact, it is possible to choose ε_0 sufficiently small that, for $\lambda \geqq \lambda^* - \varepsilon_0$, we have $u^0(\lambda) \in B$, and that $g + \lambda A$ has no fixed points on ∂B.

For $\lambda > \lambda^*$, $I - (g + \lambda A)$ has no zeroes in B, so the Leray-Schauder degree of $I - (g + \lambda A)$ relative to B is zero for $\lambda > \lambda^*$, and therefore, by homotopy invariance, also for $\lambda \geqq \lambda^* - \varepsilon_0$. For each $\lambda \in [\lambda^* - \varepsilon_0, \lambda^*)$, there is a fixed point of $g + \lambda A$, namely $u^0(\lambda)$, in B, with index $+1$ (Theorem 4–5). For each such λ, the sum of the indices of the fixed points in B is zero, so there must be at least one other fixed point of $g + \lambda A$ in B, and the sum of the indices of these other fixed points is -1. But by Theorem 4–5, each of these other fixed points has index -1, and therefore there is exactly one such nonfundamental solution in B, say $u^1(\lambda)$.

Since the distance from ∂B to U^* can be taken arbitrarily small, the distance from $u^1(\lambda)$ to U^* approaches zero as $\lambda \to \lambda^*$. Thus, by the results of §5, if the solution of $u = g + \lambda Au$ is unique for $\lambda = \lambda^*$, so that $U^* = \{u^*\}$, then $\lim_{\lambda\to\lambda^*-} u^1(\lambda) = u^*$. In general, $\lim_{\lambda\to\lambda^*-} u^1(\lambda) = u^* + \delta^*\phi^*$, where $\delta^* = \sup(\Delta)$.

We have thus established, for each $\lambda \in [\lambda^* - \varepsilon_0, \lambda^*)$, the existence of a second solution $u^1(\lambda) > u^0(\lambda)$, which is the only nonfundamental solution of $u = g + \lambda Au$ in B, and which satisfies $\lim_{\lambda\to\lambda^*-}u^1(\lambda) = u^0(\lambda^*) + \lambda^*\phi^*$. (On existence of this second solution, cf. **[1**, §20**]**).

REMARK 7–1. This result was based on Theorem 4–5; however, this theorem was used only for solutions u close to U^*. For the existence of the second fixed point $u^1(\lambda)$ near U^*, we need only the fact that $u^0(\lambda)$ has index $+1$, which can be established for the fundamental solutions of (2.5) for any compact, positively-convex, forced, isotone operator A. Moreover, with the use of the known properties of continuous dependence of the eigenvalues of a linear operator on the operator, the uniqueness result can be similarly extended if, in addition to the properties just stated, it is assumed that A is continuously differentiable on the set of positive solutions u of (2.5), and that the smallest real characteristic value, $\boldsymbol{\mu}_1[u]$, of $A'(u)$ is simple.

It is now a simple matter to use Theorem 6–1 and Lemma 7–3 below to extend the existence and uniqueness to arbitrary $\lambda \in (\boldsymbol{\mu}_1[\infty], \lambda^*)$. Given any $\lambda_1 \in (\boldsymbol{\mu}_1[\infty], \lambda^*)$, there exists, by Theorem 6–1, a positive number $\rho(\lambda_1)$ such that (1.1) has no solution u with $\|u\| \geqq \rho(\lambda_1)$ and $\boldsymbol{\lambda}[u] \geqq \lambda_1$. We take B in the above argument to be a ball centered at the origin with radius at least $\rho(\lambda_1)$ and containing U^* in its interior. Then the above argument shows that, for each $\lambda \in [\lambda_1, \lambda^*)$, there exists a unique second solution $u^1(\lambda) > u^0(\lambda)$. By Lemma 7–3 below, $u^1(\lambda)$ depends continuously on λ.

Thus we have

THEOREM 7–2. *Suppose that* $0 < m_\infty < +\infty$ *and* $[f(w) + m_\infty v]/w$ *is eventually strictly increasing, or* $m_\infty = +\infty$. *Then the sets* Δ *and* U^* *are bounded*; $0 \leqq \boldsymbol{\mu}_1[\infty] < \lambda^*$. *For each* $\lambda \in (\boldsymbol{\mu}_1[\infty], \lambda^*)$, *there exist exactly two solutions* $u^0(\lambda)$ *and* $u^1(\lambda)$ *of* (1.1), *and*

(a) $u^1(\lambda) > u^0(\lambda)$;

(b) $\lim_{\lambda \to \mu_1[\infty]+} \|u^1(\lambda)\| = +\infty$;

(c) $\lim_{\lambda \to \lambda^*-} u^1(\lambda) = u^0(\lambda^*) + \delta^* \phi^*$ (where $\delta^* = \sup(\Delta)$);

(d) the mapping $\lambda \to u^1(\lambda)$ is continuous from $(\boldsymbol{\mu}_1[\infty], \lambda^*)$ into $C[0, 1]$.

In the preceding arguments, we have made use of the following lemma, which is an immediate consequence of the homotopy invariance of the Leray-Schauder degree of a completely continuous vector field and the continuous dependence of $A'(u)$ on solutions u of (2.1).

LEMMA 7–3. *Suppose* u_0 *is a solution of* (2.5) *corresponding to* $\lambda_0 = \boldsymbol{\lambda}[u_0] > 0$. *If* u_0 *is not a critical solution, i.e., if* λ_0 *is not a characteristic value of* $A'(u_0)$, *then for any ball* $B_\varepsilon(u_0)$ *of sufficiently small radius* ε, *there exists an interval* $I_\delta(\lambda_0) = (\lambda_0 - \delta, \lambda_0 + \delta)$ *such that, for each* $\lambda \in I_\delta(\lambda_0)$, (2.5) has *a unique solution* $u(\lambda) \in B_\varepsilon(\lambda_0)$ *with* $\boldsymbol{\lambda}[u(\lambda)] = \lambda$.

8. Conversion of the Nonisotone Case to an Isotone Problem. The assertions of Theorem 4–1 on the fundamental solutions $u^0(\lambda)$ are usually proved under the assumption that A is isotone (i.e., $0 \leqq u \leqq v$ implies $Au \leqq Av$). In this section, we show how to express the problem (2.1) as a fixed point problem for an isotone operator in $C[0, 1]$ in the case where the convex function f is initially strictly decreasing (i.e., $0 > f'_+(0) \geqq$

$-\infty$). The approach is a modification of the well-known method [**44**; **1**, §21] of handling the case where f satisfies a one-sided Lipschitz condition of the form

$$f(w_2) - f(w_1) \geqq -M(w_2 - w_1)$$

(see Lemma 8–3 below), which could be directly applied here if we assumed $f'_+(0)$ finite.

We are primarily interested in the situation where f is initially decreasing but eventually strictly increasing. The case in which f is never increasing (which falls in Case 1 of §3) is covered by Lemmas 8–2 and 8–3 below, and can also be considered a special case of the general result of Theorem 8–1.

Our results in this section can easily be generalized to the problem

$$Lu(t) = \lambda F(u(t), t), \qquad 0 \leqq t \leqq 1,$$
$$Bu = \gamma,$$

with L a general second-order ordinary differential operator to which the maximum principle can be applied and F a continuous function on $[0, +\infty) \times [0, 1]$ of the form $F(w, t) = f(w) + F^{\dagger}(w, t)$, where f is positive, convex, and never increasing on $[0, +\infty)$ and $F^{\dagger}(w, t)$ is positive and never decreasing in w.

Throughout the following, K denotes the cone of never negative functions in $C[0, 1]$.

THEOREM 8–1. *Suppose f: $[0, +\infty) \to [0, +\infty)$ is continuous, convex, strictly decreasing near* 0, *and satisfies* (2.3). *Then there exists an operator $\mathfrak{B}$: $[0, +\infty) \times K \to C_2[0, 1]$ which is positive, forced* (i.e., $\mathfrak{B}(\lambda, 0) > 0$ *for all* $\lambda > 0$), *strictly isotone, compact,* and *continuous from* $[0, +\infty) \times K$ *into* $C_2[0, 1]$, *such that* $u \in C[0, 1]$ *and* $\lambda \geqq 0$ *satisfy*

$$u \in C_2[0, 1],$$
$$Lu = \lambda f(u) \text{ on } [0, 1], \tag{8.1}$$
$$Bu = \lambda$$

if and only if $v = u - g$ *and* $\lambda \geqq 0$ *satisfy*

$$v = \mathfrak{B}(\lambda, v), \tag{8.2}$$

where g is the function defined by (2.4). *For* $\lambda \geqq 0$ *and* $v \geqq 0$, $\mathfrak{B}(\lambda, v)$ *is convex in* v, *and* $\mathfrak{B}(0, 0) = 0$.

PROOF. Clearly u satisfies (8.1) if and only if $v = u - g$ satisfies

$$v \in C_2[0, 1]$$
$$(8.3) \qquad Lv(t) = \lambda f(v(t) + g(t)), \qquad 0 < t < 1,$$
$$Bv = 0.$$

We solve (8.3) as follows. We write $f(w + g(t))$ as a sum of a decreasing, convex function of w and an increasing, convex function of w,

$$f(w + g(t)) = F^{\uparrow}(w, t) + F^{\uparrow}(w, t),$$

where $F^{\downarrow}$ will be defined below. We then find an inverse on K for the operator $v \to Lv - \lambda F^{\downarrow}(v, \cdot)$, where $Bv = 0$. As discussed below, for any given $b \in K$ and $\lambda \geqq 0$, the problem

$$(8.4) \qquad \begin{aligned} Lv(t) - \lambda F^{\downarrow}(v(t), t) &= b(t), \qquad 0 < t < 1, \\ Bv &= 0, \end{aligned}$$

has a unique nonnegative solution $v \in C_2[0, 1]$. We write $v = \mathfrak{R}(\lambda, b)$ for the solution of (8.4). We then define

$$(8.5) \qquad \mathfrak{B}(\lambda, h) = \mathfrak{R}(\lambda, \lambda F^{\uparrow}(h, \cdot)),$$

so that the equation $v = \mathfrak{B}(\lambda, v)$ is equivalent to (8.3). It follows from the properties of the operator $\mathfrak{R}$ described in Lemma 8–3 below that $\mathfrak{B}$ has the properties stated in Theorem 8–1.

We now define $F^{\downarrow}$; since the function $F^{\uparrow}$ plays no further role in the discussion, we will simplify the notation and write F in place of $F^{\downarrow}$. If f is never increasing, let $F(w, t) = f(w + g(t))$. Otherwise let $f_{\min} = \inf\{f(w)\colon w \geqq 0\}$, $w_0 = \sup\{w \geqq 0\colon f(w) = f_{\min}\}$, and define for each $t \in [0, 1]$ and $w \geqq 0$,

$$F(w, t) = \begin{cases} f(w + g(t)), & \text{if } 0 \leqq w \leqq w_0 - g(t), \\ f_{\min}, & \text{if } 0 \leqq w_0 - g(t) \leqq w, \\ f(g(t)), & \text{if } w_0 - g(t) \leqq 0 \leqq w. \end{cases}$$

For $t \in [0, 1]$ and $w < 0$, let $F(w, t) = F(0, t)$.

Then F is continuous on $\mathbf{R} \times [0, 1]$, and for each $t \in [0, 1]$, $F(w, t)$ is a never increasing, convex function of w for all $w \geqq 0$, with $F(0, t) = f(g(t))$. By (2.3), $F(w, t) > 0$ for all $w \geqq 0$, $t \in [0, 1]$.

For $t \in [0, 1]$ and $w \geqq 0$, we define $\tilde{F}(w, t)$ to be the partial derivative from the right of $F(w, t)$ with respect to w, so that

$$\tilde{F}(w, t) = \begin{cases} f'_{+}(w + g(t)), & \text{if } 0 \leqq w < w_0 - g(t), \\ 0, & \text{otherwise.} \end{cases}$$

The convexity of f implies that if $t \in [0, 1]$ and $w_2 \geqq w_1 > 0$, then

$$(8.6) \qquad 0 \geqq F(w_2, t) - F(w_1, t) \geqq \tilde{F}(w_1, t)\,(w_2 - w_1).$$

It is well known that the operator $v \to Lv - \lambda F(v, \ )$ is bijective from $\{v \in C_2[0, 1]: Bv = 0\}$ into $C[0, 1]$ (cf., e.g., [**16**, Theorem 26.19]; the uniqueness result goes back to Picard [**37**, Chaptre VII]). The following lemma establishes the isotonicity of the inverse of this operator. The proof is based on the maximum principle (cf. [**38**, Chapter 1, §9]); the proof makes clear how the general assumptions on the coefficients given in [**38**, Chapter 1] are needed to deal with the possibility that $f'_-(0) = -\infty$.

LEMMA 8–2. *Suppose* $v_1, v_2 \in C_2[0, 1]$, λ_1 and $\lambda_2 \geqq 0$, *and* $b_1, b_2 \in K$ *satisfy*

$$(8.7a) \qquad Lv_i(t) = \lambda_i F(v_i(t), t) + b_i(t), \qquad 0 \leqq t \leqq 1,$$

$$(8.7b) \qquad Bv_i = 0,$$

for $i = 1, 2$, *and* $b_2 \geqq b_1 \geqq 0$, $\lambda_2 \geqq \lambda_1 \geqq 0$. *Then* $v_2 \geqq v_1 \geqq 0$. *Furthermore*, $v_2 > v_1$ *if either* $b_2 > b_1$ or $\lambda_2 > \lambda_1$.

PROOF. It follows from the maximum principle and the nonnegativity of the right hand side of (8.7a) that $v_2 \geqq 0$, $v_1 = 0$ if $\lambda_1 = 0$ and $b_1 = 0$, and $v_1(t) > 0$ for all $t \in (0, 1)$ if either $\lambda_1 > 0$ or $b_1 > 0$. (Recall that F is strictly positive.) The assertion is thus valid in the case $\lambda_1 = 0$ and $b_1 = 0$. If $\lambda_1 > 0$ or $b_1 > 0$, then, for $t \in (0, 1)$,

$$\begin{aligned} Lv_2(t) - Lv_1(t) &\geqq \lambda_1[F(v_2(t), t) - F(v_1(t), t)] \\ &\geqq \lambda_1 \tilde{F}(v_1(t), t)\, [v_2(t) - v_1(t)] \end{aligned}$$

by (8.6) (with strict inequality if $b_2 > b_1$ or $\lambda_2 > \lambda_1$), and $B(v_2 - v_1) = 0$. Since $v_1 > 0$, $\tilde{F}(v_1(t), t)$ is bounded on all compact subsets of $(0, 1)$, and the maximum principle [**38**, Chapter 1, Theorem 3] implies that if $v_2 - v_1$ had a negative minimum, it would occur at an endpoint, say 0; then the boundary conditions imply that the parameter α_0 in (1.1b) is zero, $v_2'(0) - v_1'(0) = 0$, $v_2 - v_1$ is not constant, and (by (1.1b) applied to v_1) $v_1'(0) = 0$. The maximum principle [**38**, Chapter 1, Theorem 4] applied to v_1 implies then that $v_1(0) > 0$, so that $\tilde{F}(v_1(t), t)$ is bounded near $t = 0$, and now the same version of the maximum principle applied to $v_2 - v_1$ implies $v_2'(0) - v_1'(0) > 0$, a contradiction. Thus $v_2 - v_1 \geqq 0$. Clearly $v_2 - v_1 > 0$ if $b_2 > b_1$ or $\lambda_2 > \lambda_1$. This completes the proof.

Thus we may define an operator $\mathfrak{R}$: $[0, +\infty) \times K \to K$ by $v = \mathfrak{R}(\lambda, b)$ if and only if v satisfies (8.4) for the given λ, b. The following lemma gives the properties of $\mathfrak{R}$. This will then complete the proof of Theorem 8–1.

LEMMA 8–3. *The operator* $\mathfrak{R}$: $[0, +\infty) \times K \to K$ *defined as above has the following properties*: $\mathfrak{R}(0, 0) = 0$; $\mathfrak{R}$ *is positive, forced* ($\mathfrak{R}(\lambda, 0) > 0$ *for all* $\lambda > 0$), *isotone, convex in its second argument, compact, and maps* $[0, +\infty) \times K$ *continuously into* $C_2\,[0, 1]$.

PROOF. The fact that $\mathfrak{R}$ is positive, forced, and isotone follows from Lemma 8–2. To prove the convexity property, let $b_s = sb_1 + (1 - s)b_0$ and $v_s = \mathfrak{R}(\lambda, b_s)$ for any $s \in [0, 1]$, where $b_0, b_1 \in K$. We wish to prove that $v_s \leqq sv_1 + (1 - s)v_0$. Writing $F(v)$ in place of $F(v, \cdot)$, we have, from equations (8.4), $B(v_s - sv_1 - (1 - s)v_0) = 0$ and

$$L(v_s - sv_1 - (1 - s)v_0) = s\lambda[F(v_s) - F(v_1)] + (1 - s)\lambda[F(v_s) - F(v_0)]$$
$$\leqq s\lambda\tilde{F}(v_s)(v_s - v_1) + (1 - s)\lambda\tilde{F}(v_s)(v_s - v_0) = 0$$

Thus $v_s - sv_1 - (1 - s)v_0 \leqq 0$; this establishes the convexity of $\mathfrak{R}(\lambda, b)$ in b for each fixed $\lambda > 0$.

We now prove the continuity of $\mathfrak{R}$. Using the appropriate Green function G (cf. (2.6)), we define two operators $A^\downarrow$ and Γ from K into K as follows. For any v in K,

$$A^\downarrow v(t) = \int_0^1 G(t, s)F(v(s), s)\, ds,$$

and

$$\Gamma v(t) = \int_0^1 G(t, s)v(s)\, ds.$$

Suppose that $\{b_n\}$ is a convergent sequence in K and $\{\lambda_n\}$ is a convergent sequence of positive numbers; let $v_n = \mathfrak{R}(\lambda_n, b_n)$, $\bar{\lambda} = \lim_{n\to\infty}\lambda_n$, and $\bar{b} = \lim_{n\to\infty} b_n$. Then

$$0 \leqq v_n = \lambda_n A^\downarrow v_n + \Gamma b_n \leqq \lambda_n A^\downarrow 0 + \Gamma b_n.$$

Thus $\{v_n\}$ is bounded. Since $A^\downarrow$ is compact, every subsequence of $\{v_n\}$ has a subsequence $\{\bar{v}_n\}$ such that $\{A^\downarrow \bar{v}_n\}$ is convergent, and hence $\{\bar{v}_n\}$ converges also, say to $\bar{v}$. Then $\bar{v}$ satisfies $\bar{v} = \bar{\lambda}A\bar{v} + \Gamma\bar{b}$, or $\bar{v} = \mathfrak{R}(\bar{\lambda}, \bar{b})$. Since every subsequence of $\{v_n\}$ contains a subsequence which converges to $\bar{v}$ it follows that $\{v_n\}$ itself converges to $\bar{v}$ in $C[0, 1]$. Since $A^\downarrow$ and Γ are continuous from K into $C_2[0, 1]$, $\{v_n\}$ converges to $\bar{v}$ in $C_2[0, 1]$. Thus $\mathfrak{R}$: $(0 + \infty) \times K \to C_2[0, 1]$ is continuous. This, in turn, implies that $\mathfrak{R}$ is compact as a mapping into $C_1[0, 1]$ or $C[0, 1]$.

9. Differentiability of A. Let u be a twice differentiable function on $[0, 1]$ which satisfies $Lu \geqq 0$, $Bu = \gamma$ (for example, u might be a solution of (1.1)). We show that if u is not a constant, then the operator A defined by (2.6) is Fréchet differentiable in $C[0, 1]$ at u and the Fréchet derivative of A is given by (2.7).

We first show that the integral in (2.7) exists. Since G and h are continous, it suffices to show that $f'_+(u)$ is integrable on $(0, 1)$. It is only with the special case in which $f'_+(0) = -\infty$ and $u(0) = 0$ or $u(1) = 0$ that we need be concerned, for otherwise $f'_+(u)$ is bounded on $(0, 1)$. Suppose, for example, that $f_+'(0) = -\infty$ and $u(0) = 0$. Then $u'(0) > 0$ [**38**, p. 4] and there exists a nontrivial interval with left endpoint 0 on which f_+' is negative. It follows that we can find a nontrivial interval $[0, t_0] \subseteq [0, 1]$ and a constant $\mu > 0$ such that $u(s) \geqq \mu s$ for all $s \in [0, t_0]$ and $|f_+'(u(s))| = -f_+'(u(s)) \leqq -f_+'(\mu s)$ for all $s \in (0, t_0]$. Hence

$$\int_0^{t_0} |f_+'(u(s))| ds \leqq -\int_0^{t_0} f_+'(\mu s) ds = \mu^{-1}[f(0) - f(\mu t_0)] < +\infty.$$

A similar analysis holds near $s = 1$, and hence $f_+'(u)$ is integrable on $[0, 1]$.

Suppose now that, in addition, u is nonconstant. Let $\sum$ be the countable set of points in $[0, +\infty)$ at which f is not differentiable. Since u satisfies $Lu \geqq 0$, it assumes any given value only a finite number of times, and thus the set $u^{-1}(\sum) \subseteq [0, 1]$ is countable. Thus for almost all $s \in [0, 1]$, $f_+'[u(s)] = f_-'[u(s)] = f'[u(s)]$.

Define $A'(u)$ by (2.7); we show that $A'(u)$ is the derivative of A. Choose $h \in C[0, 1]$, $h \neq 0$, and let $\alpha = \|h\|$ and $h_0 = \alpha^{-1} h$. Then

$$\begin{aligned}
&0 \leqq \alpha^{-1}[A(u + h)(t) - Au(t) - A'(u)h(t)] \\
(9.1) \qquad &= \alpha^{-1} \int_+ G(t, s)[f(u(s) + \alpha h_0(s)) - f(u(s)) - \alpha f_+'(u(s))h_0(s)] ds \\
&+ \alpha^{-1} \int_- G(t, s)[f(u(s) + \alpha h_0(s)) - f(u(s)) - \alpha f_-'(u(s))h_0(s)] ds,
\end{aligned}$$

where $\int_+$ and $\int_-$ denote integrals over the sets on which $h \geqq 0$ and $h \leqq 0$, respectively; each of the integrands is nonnegative. We consider $\int_-$. Since $u(s) > 0$ for $s \in (0, 1)$, the measure of the set $\{s\colon u(s) - \|h\| \leqq 0\}$ goes to zero as $\|h\|$ goes to zero. Thus the integral

$$\alpha^{-1} \int_{u(s) - \|h\| \leqq 0} G(t, s)\, [f(u(s) + h(s)) - f(u(s)) - f_-'(u(s))h(s)] ds$$

can be made arbitrarily small, uniformly for $t \in [0, 1]$, by choosing $\|h\|$ sufficiently small. Thus, in $\int_-$, it suffices to consider integration only over the set $P = \{s\colon u(s) - \|h\| > 0$ and $h_0(s) < 0\}$. If $\alpha h_0(s) < 0$, then since f is convex and $-1 \leqq h_0(s) < 0$,

$$\frac{f(u(s)) - f(u(s) + \alpha h_0(s))}{-\alpha h_0(s)} \geqq \frac{f(u(s)) - f(u(s) - \alpha)}{\alpha}$$

and hence

$$0 \leqq \alpha^{-1} \int_P G(t, s)\, \{f(u(s) + \alpha h_0(s)) - f(u(s)) - \alpha f_-'(u(s))h_0(s)\} ds$$
$$\leqq \alpha^{-1} \int_P G(t, s)\, \{-[f(u(s)) - f(u(s) - \alpha)] + \alpha f_-'(u(s))\}\{-h_0(s)\} ds$$
$$\leqq \int_P G(t, s)\{-\alpha^{-1}[f(u(s)) - f(u(s) - \alpha)] + f_-'(u(s))\} ds.$$

The integrand in the last integral converges monotonically to zero as $\alpha \to 0^+$; by the Lebesgue dominated convergence theorem, the integral converges monotonically to zero for each $t \in [0, 1]$, and by Dini's theorem the convergence is uniform. Similarly, the first term on the right hand side of (9.1) converges to zero uniformly as $\alpha \to 0^+$, and hence $\|h\|^{-1}\|A(u + h) - Au - A'(u)h\|$ converges to zero as $\|h\| \to 0$. This proves that $A'(u)$ is the Fréchet derivative of A at u.

A similar argument shows that $A'(u)$ depends continuously on u for solutions u of (2.1).

Appendix

In this appendix, we discuss certain results, all of which can be derived by elementary methods, on ordinary differential equations with integrable, locally bounded coefficients. We denote by $\mathscr{L}$ the linear space $L_1(0, 1) \cap L_\infty^{\mathrm{loc}}(0, 1)$ of integrable functions on (0, 1) which are bounded on closed subintervals of (0, 1). For this appendix, we define the differential operator L by

$$Lu = -u'' - bu' - hu, \tag{A.1}$$

where b and h are functions in $\mathscr{L}$, with $h \leqq 0$.

Consider the eigenvalue problem

$$L\phi = \mu k \phi \tag{A.2}$$

with $k \in \mathscr{L}$, with homogeneous boundary condition $B\phi = 0$ (where we make the same assumptions on α_i and α_i' as in §2). By a solution of (A.2), we understand, as usual, a continuously differentiable function ϕ with an absolutely continuous derivative, such that (A.2) is satisfied almost everywhere on [0, 1]. For the special case where (A.2) reduces to (2.9), this is equivalent to requiring that any solution of (2.9) be a solution of (2.8), and conversely.

The problem (A.2) – (1.1b) with k not necessarily positive everywhere on (0, 1) is discussed by Ince [**18**, §10.61], assuming continuity of the coefficients, and he gives references to more detailed discussion. The problem with weak smoothness assumptions on the coefficients, but with

$k \geqq 0$, is discussed by Atkinson [**8**, Ch. 8]. For our purposes, it suffices to have the following result.

PROPOSITION A-1. *Suppose that $b, h, k \in \mathscr{L}$, that $h \leqq 0$, that k is bounded below by a positive constant on a nontrivial subinterval of* (0, 1), *and that for all $\lambda \geqq 0$, $b(t) + t[h(t) + \lambda k(t)]$ and $b(t) - (1 - t)[h(t) + \lambda k(t)]$ are bounded as $t \to 0^+$ and $t \to 1^-$, respectively. Then the eigenvalue problem* (A.2) – (1.1b) *has a countable number of eigenvalues, all of which are real and have no finite limit point. There is an eigenfunction, ϕ_1, corresponding to an eigenvalue, μ_1, such that*: (i) *no number $\mu \in [0, \mu_1)$ is an eigenvalue*; (ii) *ϕ_1 has no zero on* (0, 1); *and* (iii) *any eigenfunction corresponding to μ_1 is a multiple of ϕ_1.*

The fact that the set of eigenvalues is countable and has no finite limit point follows from the fact that (A.2) – (1.1b) can be written as an eigenvalue problem for a compact operator (on the space $C[0, 1]$, for example). The fact that the eigenvalues are real can be proven as in [**43**, pp. 25–26]. Finally, the existence and properties of the eigenvalue μ_1 and corresponding eigenfunction ϕ_1 can be proven in a manner analogous to a proof in [**38**, Chapter 1, §7]. The latter proof depends on the existence, uniqueness, and continuous dependence on λ of solutions of initial value problems for (A.2); cf. [**17**, p. 46, p. 54; **41**, Chapter 2].

With the existence of a "principal" positive eigenvalue μ_1 and corresponding positive eigenfunction ϕ_1 established, we turn to the result used repeatedly in this paper. When $k \geqq 0$, the result is well-known and at least part of the proof is trivial.

PROPOSITION A-2. *With the assumptions and notation of Proposition* A-1, *suppose that $h = 0$ and that there exists a twice differentiable function v and a real number λ such that*

$$\text{(4.3)} \qquad \begin{aligned} Lv &\geqq \lambda k v \text{ on } (0, 1). \\ Bv &\geqq 0. \end{aligned}$$

If $0 \leqq \lambda < \mu_1$, then $v \geqq 0$ on (0, 1); *if $v(t) > 0$ for $t \in (0, 1)$, then $\lambda \leqq \mu_1$; and $\lambda = \mu_1$ if and only if v is a multiple of ϕ_1 and $Lv = \lambda k v$. If we keep the same hypotheses except that $k \leqq 0$ almost everywhere on* (0, 1), *then any function v satisfying* (A.3) *for $\lambda \geqq 0$ is nonnegative on* (0, 1).

PROOF. The only case needing investigation here is when k is as in Proposition A-1 and $\lambda > 0$. In the usual way we can assume that the operator L in (A.2) and (A.3) is formally self-adjoint, say $L\phi = -(p\phi')'$, with $p(t) = \exp[\int_0^t b]$. When $\lambda > 0$, the results stated are obtained from Picone's identity [**42**] in the form

$$\left\{pvv'(\lambda^{-1} - \mu_1^{-1}) + \mu_1^{-1}\frac{pv}{\phi_1}(\phi_1 v' - v\phi_1')\right\}\Bigg|_{t_1}^{t_2}$$
$$= \int_{t_1}^{t_2}(\lambda^{-1} - \mu_1^{-1})pv'^2 + \int_{t_1}^{t_2}\mu_1^{-1}p\left(\frac{\phi_1 v' - v\phi_1'}{\phi_1}\right)^2 - \int_{t_1}^{t_2} vQ$$

and a similar identity with ϕ_1 and v interchanged, where $Q = \lambda^{-1}Lu - kV \geqq 0$.

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MATHEMATICS DEPARTMENT, UNIVERSITY OF ARIZONA, TUSCON, ARIZONA 85721