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A SINGULAR PERTURBATION APPROACH TO NONLINEAR SHELL THEORY

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Introduction. The purpose of the present paper is twofold: On the one hand we want to investigate a problem naturally arising in the nonlinear theory of shallow shells; and on the other hand we hope that the method we are going to employ may prove useful in other applications, where a constructive existence proof for differential equations with small parameters is to be given by asymptotic expansions.

We shall deal with the basic system of nonlinear fourth order partial differential equations describing the behaviour of a gently sloping shell subject to an external load. These are known as Marguerre's equations, a derivation of which is given by Weinitschke in [**19**]. We shall utilize them in nondimensionalized form, given in §1, thus introducing a parameter ε which characterizes the thickness of the shell and multiplies the highest order derivatives (see [**15**]).

Now let the shell with its edge simply supported be exposed to a sufficiently strong vertical pressure. The question is whether the shell returns to its initial state, or remains in a deflected position after the load is removed. To put it mathematically: Do Marguerre's equations possess stable nontrivial solutions besides the trivial one for vanishing external load?

This problem was investigated by Srubshchik in [**15**]. The above mentioned feature suggests, for small ε, viewing it as a singular perturbation problem which is solved formally by an asymptotic method invented in the pioneering work of Višik and Lyusternik [**18**]. Since then, this method has been successfully used by several authors dealing with second order equations, more recently even in the nonlinear case (cf. Fife [**5**]).

The asymptotic expansions for a nontrivial solution constructed in [**15**] of course satisfy the differential equations approximately, the same being true for the boundary conditions. Yet they do not satisfy the boundary conditions corresponding to a simple support of the shell exactly, a defect which is typical for those conditions comprising derivatives of different order. This is the reason the justification of the formal approximations given in [**15**], i.e., the proof of the existence of an actual solution in their

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neighbourhood, does not seem to be conclusive. That is, these approximations are used as initial values for Newton-Kantorovič iterations performed in a space of functions they do not belong to, namely a space of functions satisfying the boundary conditions exactly. The requirement that the approximations already do belong to such a space is common to other known techniques as well, such as fixed point theorems and variants of the implicit function theorem.

To surmount this basic difficulty, we utilize the well known variational structure of Marguerre's equatinos (cf. **[13]**, **[2]**), together with the fact that the boundary conditions divide into so called geometrical conditions and natural conditions. By a slight modification of the expansions given in **[15]**, we ensure that the geometrical conditions are satisfied exactly by the formal approximations. Hereafter, we consider the energy functional associated with the variational formulation of the problem on a space of functions which satisfy just those conditions, and show that this functional attains a strict minimum close to the approximation. It follows from a regularity theorem that the minimizing function satisfies the natural conditions of its own accord.

We shall continue our investigation with a proof that 'thick' shells always snap back to their initial state, that is to say that the trivial solution is unique for big enough ε.

In the last paragraph, we show that this result implies the existence of a third solution of Marguerre's equations. To this end, we use a topological degree argument.

We should not fail to mention that the above question remains open when the shell is clamped along its edge, a constraint corresponding to Dirichlet conditions in the mathematical model. In this case not even the method given here to construct the formal expansions seems to be feasible. However, this might not be unexpected if one considers the following example:

$$-\varepsilon^2 u'' + u(u-1)(u-2) = 0,$$
$$u'(0) = 0,\ \mu u'(1) + u(1) = 0,\ u = u(t),\ t \in [0, 1].$$

In some respects, this equation is analogous to the equation (3.2) in §3; at least it exhibits the same functional analytic features. Now, it is easy to show by a phase plane analysis that this equation has no nontrivial solution for $\mu = 0$, whereas for $\mu \neq 0$ the number of solutions is constantly increasing at $\varepsilon \to 0$. With this result in mind, it is tempting to conjecture that there are no nontrivial solutions of Marguerre's equations for a clamped shell.

I wish to express my warmest gratitude to Professor Klaus Kirchgässner for stimulating discussions during this work.

1. **Construction of a nontrivial formally approximate solution.** We write Marguerre's equations for the case of vanishing external load in non-dimensionalized form:

$$
\begin{aligned}
&N_1[F, W] \equiv \varepsilon^2 \triangle\triangle F + \tfrac{1}{2}[W, W] + [z, W] = 0,\\
(1.1) \qquad &N_2[F, W] \equiv \varepsilon^2 \triangle\triangle W - [W, F] - [z, F] = 0,\\
&\triangle u = u_{xx} + u_{yy},\ [u, v] = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}.
\end{aligned}
$$

Here $z = z(x, y)$ is the equation of the median surface of the shell, $W(x, y)$ its vertical displacement, and $F(x, y)$ the stress function. ε is a parameter characterizing the thickness of the shell. $(x, y) \in \Omega$ are plane cartesian coordinates, where Ω is the region occupied by the shell. Its boundary Γ coincides with the edge of the shell, i.e., $z|_\Gamma = 0$. The functions F and W are subject to the following boundary conditions:

$$
(1.2) \quad
\begin{aligned}
&B_1[F] \equiv F|_\Gamma = 0,\ B_2[F] \equiv F_\rho|_\Gamma = 0,\\
&B_3[W] \equiv W|_\Gamma = 0,\ B_4[W] \equiv W_{\rho\rho} - \mu\kappa W_\rho|_\Gamma = 0,\ 0 < \mu < 1/2.
\end{aligned}
$$

Here u_ρ denotes the directional derivative of the function u along the inner normal of Γ, κ is the curvature of Γ, μ is Poisson's ratio.

For simplicity we shall assume that z as well as Γ are sufficiently smooth, the degree of smoothness actually required depending on the accuracy of the approximations desired. For technical reasons we need z and Ω to be strictly convex, i.e.,

$$(1.3) \qquad \kappa > 0,$$

$$(1.4) \qquad z_{xx}m^2 + z_{yy}n^2 - 2z_{xy}nm < -\beta(m^2 + n^2),\ \beta > 0, n, m \in \mathbf{Z}.$$

We shall construct a formally approximate solution (F_n, W_n) of (1.1), (1.2) in the following sense:

$$
\begin{aligned}
&(1.5)_1 \qquad |N_1[F_n, W_n]|_0^\Omega < c_\alpha \varepsilon^{n+1};\\
&(1.5)_2 \qquad |N_2[F_n, W_n]|_0^\Omega < c_\alpha \varepsilon^{n+1};\\
&(1.6)_1 \qquad |B_1[F_n]|_0^\Gamma = 0,\ |B_2[F_n]|_0^\Gamma < c_\alpha \varepsilon^{n+2};\\
&(1.6)_2 \qquad |B_3[W_n]|_0^\Gamma = 0,\ |B_4[W_n]|_0^\Gamma < c_\alpha \varepsilon^{n+1};
\end{aligned}
$$

here c_α is a positive constant, which depends only on Ω and z. We employ the notation $|u|_0^\Omega = \max_{\mathbf{x}\in\Omega}|u(\mathbf{x})|$, $|v|_0^\Gamma = \max_{\mathbf{x}\in\Gamma}|v(\mathbf{x})|$ accordingly, with $\mathbf{x} = (x, y)$.

We shall obtain a representation of the functions F_n, W_n in the following form:

$$
\begin{aligned}
F_n &= F_{in} + \varepsilon^2 F_{ou}\omega,\\
W_n &= W_{in} + \varepsilon^2 W_{ou}\omega.
\end{aligned}
$$

We think of F_{in}, W_{in} as being 'inner' approximations and of F_{ou}, W_{ou}

as being 'outer' approximations to solutions of (1.1), (1.2), a terminology commonly used in the theory of singular perturbations. Here ω is a cut-off function, to be defined later, matching those two types in order to obtain an approximation valid in the whole of Ω.

We first seek F_{in} and W_{in} in the form

$$F_{in}(\mathbf{x}, \varepsilon) = \sum_{i=0}^{n+2} f_i(\mathbf{x})\varepsilon^i, \; W_{in}(\mathbf{x}, \varepsilon) = \sum_{i=0}^{n+2} w_i(\mathbf{x})\varepsilon^i,$$

stipulating for

$$(1.5)_{in} \qquad N_1[F_{in}, W_{in}] = O(\varepsilon^{n+3}), \; N_2[F_{in}, W_{in}] = O(\varepsilon^{n+3}).$$

To this end the functions $N_i(\varepsilon)$, $i = 1, 2$, shall be expanded in Taylor series. For the f_i, w_i, $0 \leqq i \leqq n + 2$, one obtains the equations

$$\frac{\partial^i}{\partial \varepsilon^i} N_1\Big|_{\varepsilon=0} = 0, \frac{\partial^i}{\partial \varepsilon^i} N_2\Big|_{\varepsilon=0} = 0,$$

taking

$$\frac{\partial^i}{\partial \varepsilon^i} F_{in}\Big|_{\varepsilon=0} = i!f_i, \frac{\partial^i}{\partial \varepsilon^i} W_{in}\Big|_{\varepsilon=0} = i!w_i$$

into account.

In this way we get for f_0, w_0 the reduced system

$$(1.7)_1 \qquad \begin{aligned} \frac{1}{2}[w_0, w_0] + [z, w_0] &= 0, \\ [w_0, f_0] + [z, f_0] &= 0. \end{aligned}$$

We choose the nontrivial solution $w_0 = -2z$, $f_0 = 0$. This reflects our expectation that the solution of (1.1), (1.2) will yield a result close to the mirror image $z + w_0 = -z$ of the shell.

For f_1, w_1 we have

$$(1.7)_2 \qquad \begin{aligned} [z, w_1] &= 0, \\ [z, f_1] &= 0. \end{aligned}$$

We solve $(1.7)_1$ with $w_1 = f_1 = 0$. We shall see later that this choice is mandatory in order to fulfil the boundary conditions $B_1[F_n] = B_3[W_n] = 0$.

Taking $w_0 = -2z$ into account one obtains the systems for $2 \leqq i \leqq n + 2$:

$$\begin{aligned} -[z, w_i] &= -\frac{1}{2}\sum_{j+k=1} [w_j, w_k] - \triangle\triangle f_{i-2}, \\ -[z, f_i] &= -\sum_{j+k=i} [f_j, w_k] + \triangle\triangle w_{i-2}, \; k, j \neq 0, \\ f_i|_\Gamma &= a_i, \;\; w_i|_\Gamma = b_i. \end{aligned}$$

The functions $a_i, b_i \in C^\infty(\Gamma)$ will be uniquely defined at a later stage of the construction.

It follows from (1.4) that the differential operator $-[z, \cdot]$ is uniformly elliptic. Furthermore it consists of derivatives of second order only and therefore possesses a strongly coercive Dirichlet form. It is well known that this implies existence and uniqueness of the solutions of the systems $(1.7)_i$. These solutions are infinitely differentiable, – we write f_i, $w_i \in C^\infty(\bar{\Omega})$ –, since the right hand sides of the equations comprise C^∞-functions previously obtained.

Although the functions F_{in}, W_{in}, whatever a_i, b_i may be, satisfy the equations (1.1) approximately this is not the case for the boundary conditions (1.2). We shall remedy this defect by adding the boundary layer functions $\varepsilon^2 F_{ou}$, $\varepsilon^2 W_{ou}$.

On the boundary strip $\Omega_d = \{\mathbf{x} | \mathbf{x} \in \bar{\Omega}, |\mathbf{x} - \mathbf{y}| < d \text{ for some } \mathbf{y} \in \Gamma\}$ we introduce the curvilinear coordinates (ρ, s):

$$\mathbf{x} = \mathbf{x}_0(s) + \rho \mathbf{n}(s), \ \mathbf{x}_0 \in \Gamma,$$

s denoting the arclength on Γ, $\mathbf{n}$ the inner unit normal. We choose d equal $d(\kappa) = (2\max\kappa(s))^{-1}$. This and the convexity of Ω guarantee that the above coordinate transformation is well defined on Ω_d.

We now determine $F_{ou} = \hat{F}_{ou}(\rho, s, \varepsilon)$, $W_{ou} = \hat{W}_{ou}(\rho, s, \varepsilon)$ requiring

$$\hat{N}_1[\hat{F}_n, \hat{W}_n] = O(\varepsilon^{n+1}), \ \hat{N}_2[\hat{F}_n, \hat{W}_n] = O(\varepsilon^{n+1}),$$

where differential operators and functions of the variables (ρ, s) are denoted by $\hat{D}$ and $\hat{u}$.

Recalling $(1.5)_{in}$ we obtain

$$\tag{1.8} \begin{aligned} \hat{M}_1[\hat{F}_{ou}, \hat{W}_{ou}] &\equiv \varepsilon^4 \triangle\triangle \hat{F}_{ou} + \frac{1}{2}\varepsilon^4[\hat{W}_{ou}, \hat{W}_{ou}] + \varepsilon^2[\hat{W}_{in}, \hat{W}_{ou}] \\ &\quad + \varepsilon^2[\hat{z}, \hat{W}_{ou}] = O(\varepsilon^{n+1}), \\ \hat{M}_2[\hat{F}_{ou}, \hat{W}_{ou}] &\equiv \varepsilon^4 \triangle\triangle \hat{W}_{ou} - \varepsilon^4[\hat{F}_{ou}, \hat{W}_{ou}] - \varepsilon^2[\hat{F}_{in}, \hat{W}_{ou}] \\ &\quad - \varepsilon^2[\hat{F}_{ou}, \hat{W}_{in}] - \varepsilon^2[\hat{z}, \hat{F}_{ou}] = O(\varepsilon^{n+1}). \end{aligned}$$

A further transformation stretches the variable ρ:

$$\rho = \varepsilon t.$$

The differential operators resulting from $\hat{M}_i(\rho, s)$ under this coordinate transform are denoted by $\hat{\hat{M}}_i(t, s)$, $i = 1, 2$. Since $\partial/\partial\rho = 1/\varepsilon\, \partial/\partial t$, we have

$$\begin{aligned} \hat{\hat{M}}_1[v(t, s), u(t, s)] &= (\rho_x^2 + \rho_y^2)^2|_\Gamma \frac{\partial^4}{\partial t^4} v \\ &\quad - \hat{z}_\rho(\rho_x^2\rho_{yy} + \rho_y^2\rho_{xx} - 2\rho_x\rho_y\rho_{xy})|_\Gamma \frac{\partial^2}{\partial t^2} u + O(\varepsilon), \end{aligned}$$

$$\hat{\hat{M}}_2[v(t,s), u(t,s)] = (\rho_x^2 + \rho_y^2)^2|_\Gamma \frac{\partial^4}{\partial t^4} u$$
$$+ \hat{z}_\rho(\rho_x^2\rho_y + \rho_y^2\rho_{xx} - 2\rho_x\rho_y\rho_{xy})|_\Gamma \frac{\partial^2}{\partial t^2} v + O(\varepsilon).$$

A straightforward calculation yields

$$(\rho_x^2 + \rho_y^2)^2|_\Gamma = 1$$

and

$$(\rho_x^2\rho_{yy} + \rho_y^2\rho_{xx} - 2\rho_x\rho_y\rho_{xy})|_\Gamma = -\kappa,$$

and therefore

$$\begin{aligned} \hat{\hat{M}}_1[v, u] &= \frac{\partial^4}{\partial t^4} v + \kappa\hat{z}_\rho|_\Gamma \frac{\partial^2}{\partial t^2} u + O(\varepsilon), \\ \hat{\hat{M}}_2[v, u] &= \frac{\partial^4}{\partial t^4} u - \kappa\hat{z}_\rho|_\Gamma \frac{\partial^2}{\partial t^2} v + O(\varepsilon). \end{aligned} \tag{1.9}$$

We now set

$$\hat{\hat{F}}_{ou}(t, s, \varepsilon) = \sum_{i=0}^{n} h_i(t, s)\varepsilon^i, \quad \hat{\hat{W}}_{ou}(t, s, \varepsilon) = \sum_{i=0}^{n} g_i(t, s)\varepsilon^i$$

and determine the functions h_i, g_i requiring that

$$\begin{aligned} \hat{\hat{M}}_1[\hat{\hat{F}}_{ou}, \hat{\hat{W}}_{ou}] &= O(\varepsilon^{n+1}), \\ \hat{\hat{M}}_2[\hat{\hat{F}}_{ou}, \hat{\hat{W}}_{ou}] &= O(\varepsilon^{n+1}). \end{aligned} \tag{1.5$_{ou}$}$$

As is shown by formula (1.9), one obtains, for any i, a tystem of linear ordinary differential equations in the variable t, whose coefficients and right hand sides depend on the parameter s. The respective boundary conditions are chosen so as to guarantee that

$$B_2[F_n] = \sum_{i=1}^{n+2} \hat{f}_{i,\rho}|_{\rho=0}\, \varepsilon^i + \varepsilon \sum_{i=0}^{n} h_{i,t}|_{t=0}\, \varepsilon^i = O(\varepsilon^{n+2}), \text{ (recall } f_0 = 0!) \tag{1.10$_1$}$$

and

$$\begin{aligned} B_4[W_n] &= \sum_{i=0}^{n+2} (\hat{w}_{i,\rho\rho} - \mu\kappa\hat{w}_{i,\rho})|_{\rho=0}\, \varepsilon^i \\ &\quad + \sum_{i=0}^{n} g_{i,tt}|_{t=0}\, \varepsilon^i - \varepsilon \sum_{i=0}^{n} \mu\kappa g_{i,t}|_{t=0}\, \varepsilon^i = O(\varepsilon^{n+1}), \end{aligned} \tag{1.10$_2$}$$

whence

$$\begin{aligned} h_{i,t}|_{t=0} &= -\hat{f}_{i+1,\rho}|_{\rho=0}, \\ g_{i,tt}|_{t=0} &= -(\hat{w}_{i,\rho\rho} - \mu\kappa\hat{w}_{i,\rho})|_{\rho=0} - \mu\kappa g_{i-1,t}|_{t=0}, \ (g_{-1} = 0). \end{aligned} \tag{1.11$_1$}$$

To achieve a rapid decay of the functions F_{ou}, W_{ou} outside the boundary strip Ω_d we impose furthermore the conditions

$$(1.11)_2 \qquad \begin{aligned} &h_i,\, h_{i,t} \to 0 \quad \text{as } t \to \infty, \\ &g_i,\, g_{i,t} \to 0 \quad \text{as } t \to \infty. \end{aligned}$$

Utilizing (1.9), (1.11) we write the equations for the leading boundary layer terms h_0, g_0 explicitly:

$$(1.12) \qquad \begin{aligned} &\frac{\partial^4}{\partial t^4} h_0 + \kappa \hat{z}_\rho|_\Gamma \frac{\partial^2}{\partial t^2} g_0 = 0, \\ &\frac{\partial^4}{\partial t^4} g_0 - \kappa \hat{z}_\rho|_\Gamma \frac{\partial^2}{\partial t^2} h_0 = 0, \\ &h_{0,t}|_{t=0} = 0,\ g_{0,tt}|_{t=0} = 2(\hat{z}_{\rho\rho} - \mu\kappa\hat{z}_\rho)|_{\rho=0} \\ &h_0, g_0, h_{0,t}, g_{0,t} \to 0 \quad \text{as } t \to \infty. \end{aligned}$$

It is easily seen that the solutions of (1.12) are uniquely determined by (1.11), $i = 0$, and after an elementary calculation we find

$$(1.13) \qquad h_0 = ke^{-\lambda t} \sin(\lambda t + \pi/4), \ g_0 = ke^{-\lambda t} \sin(\lambda t - \pi/4),$$

with

$$k = 2\sqrt{2}\left[\frac{\hat{z}_{\rho\rho}}{\kappa \hat{z}_\rho} - \mu\right]\Bigg|_\Gamma$$

and

$$\lambda = \left[\frac{\kappa \hat{z}_\rho}{2}\right]\Bigg|_\Gamma^{1/2};$$

$\kappa\hat{z}_\rho|_\Gamma > 0$, from (1.3), (1.4).

The functions h_t, g_i, $i > 0$, are solutions of inhomogeneous systems, whose homogeneous part coincides with (1.12) because of (1.9). Their right hand sides are sums of terms, each consisting of functions h_j, g_j, $j < i$, already known and their derivatives times functions solely dependent on s. Since h_0, g_0 together with their derivatives decay exponentially as $t \to \infty$, we conclude inductively that the same is true for the functions h_i, g_i, $1 \leqq i \leqq n$. So the boundary layer functions $\hat{\hat{F}}_{ou}$, $\hat{\hat{W}}_{ou}$ decay exponentially as $t \to \infty$. (For a formal proof, cf. Lemma A.2 in [5]).

Next we define the boundary functions a_i, b_i, $(1.7)_i$, $2 \leqq i \leqq n + 2$, in such a way that F_n and W_n satisfy the boundary conditions $B_1[F_n] = 0$ and $B_3[W_n] = 0$ exactly:

$$(1.14) \qquad \begin{aligned} B_1[F_n] &= \Big(\sum_{i=0}^{n+2} f_i\varepsilon^i + \varepsilon^2 \sum_{i=0}^{n} h_i\varepsilon^i\Big)\Big|_\Gamma = 0, \\ B_3[W_n] &= \Big(\sum_{i=0}^{n+2} w_i\varepsilon^i + \varepsilon^2 \sum_{i=0}^{n} g_i\varepsilon^i\Big)\Big|_\Gamma = 0. \end{aligned}$$

Hence we set

$$a_i = -h_{i-2}|_{t=0},$$
$$b_i = -g_{i-2}|_{t=0},$$

noticing that the former choice of the functions f_0, f_1 and w_0, w_1 is compatible with (1.14). Using (1.13) we get for $i = 2$,

$$a_2 = b_2 = -2\left[\frac{\hat{z}_{\rho\rho}}{\kappa\hat{z}_\rho} - \mu\right]\Big|_\Gamma. \tag{1.15}$$

To complete the construction of the approximate solution (F_n, W_n) we should note that the boundary layer functions $\hat{\hat{F}}_{ou}$, $\hat{\hat{W}}_{ou}$ are defined for all $t \geqq 0$, whereas the coordinate system (t, s) makes sense only in $\Omega_d = \{(t, s) \mid 0 \leqq t < d/\varepsilon\}$. Therefore we multiply $\hat{F}_{ou}(\rho, s, \varepsilon) = \hat{\hat{F}}_{ou}(\rho/\varepsilon, s, \varepsilon)$ and $\hat{W}_{ou}(\rho, s, \varepsilon) = \hat{\hat{W}}_{ou}(\rho/\varepsilon, s, \varepsilon)$ by a cut-off function $\omega(\rho) \in C^\infty[0, \infty)$ which is defined as follows:

$$\omega(\rho) = \begin{cases} 1 \text{ for } 0 \leqq \rho \leqq d/2. \\ 0 \text{ for } \rho \geqq d \end{cases}$$

Now the formal approximations

$$F_n = F_{in} + \varepsilon^2\hat{F}_{ou}\omega, \; W_n = W_{in} + \varepsilon^2\hat{W}_{ou}\omega$$

are well defined in the whole of Ω.

In summary, the solutions $w_0 = -2z$ and $f_0 = 0$ of the reduced system $(1.7)_1$ are chosen as the basic inner approximations. The system $(1.7)_2$ is solved by $f_1 = w_1 = 0$, whereupon h_0, g_0 are calculated using (1.12). With the help of (1.14) one obtains a_2, b_2 and determines the solutions f_2, w_2 of $(1.7)_2$, etc., the last step of this iterative construction being the determination of the functions f_{n+2}, w_{n+2}. Matching the inner and outer approximations by means of ω yields the final representation of the formally approximate solution (F_n, W_n) of (1.1), (1.2).

The following theorem shows that (F_n, W_n) is indeed a formally approximate solution in the sense of (1.5), (1.6).

THEOREM 1. *The functions* F_n, $W_n \in C^\infty(\bar{\Omega})$ *satisfy the conditions* (1.5), (1.6).

PROOF. (1.6) is verified immediately using (1.10) and (1.14). To demonstrate the validity of (1.5), Ω is partitioned into three parts. In $\Omega\backslash\Omega_d$, $\Omega_{d/2}$, (1.5) follows from $(1.5)_{in}$, $(1.5)_{ou}$ accordingly by means of Taylors formula, since F_{in}, W_{in} and $\hat{\hat{F}}_{ou}$, $\hat{\hat{W}}_{ou}$ and their derivatives are bounded uniformly in ε. In $\Omega_d\backslash\Omega_{d/2}$, we have $|\hat{F}_{ou}\omega| < ce^{-\gamma/\varepsilon}$ and $|\hat{W}_{ou}\omega| < ce^{-\gamma/\varepsilon}$, $c, \gamma > 0$, because of the exponential decay of the boundary layer functions. So again (1.5) follows from $(1.5)_{in}$. (Cf. Theorem 3.6 in **[6]**.)

We note that in the case of a rotationally symmetric shell it is easy to calculate the functions F_n, W_n explicitly. Upon introducing polar coordinates one writes $z(x, y) = \hat{z}(r, \phi) = \hat{z}(r)$ and transforms the equations $(1.7)_i$ by means of the formulas

$$\triangle\triangle u = \frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\hat{u},$$

$$[u, v] = \frac{1}{r}\frac{d}{dr}(\hat{u}_r\hat{v}_r),$$

$$\hat{u} = \hat{u}(r), \hat{v} = \hat{v}(r).$$

This leads to algebraic equations for f_i, w_i which possess rotationally symmetric solutions.

Assuming without loss of generality that Ω is the unit ball, h_0, g_0 are obtained substituting $t = (1 - r)/\varepsilon$ in (1.13) where now $\kappa = 1$. To find h_i, g_i, $i > 0$, one has to solve the corresponding ordinary differential equations.

2. **Justification of the formal approximations.** It is our next goal to prove the existence of a nontrivial solution of (1.1), (1.2) with the asymptotic expansion (F_n, W_n).

The following lemma provides some functional analytic machinery for the subsequent discussion.

LEMMA 1. *Let $\tilde{\Lambda}$ be a positive definite selfadjoint operator in a Hilbert space X_0 equipped with the scalar product $(\cdot, \cdot)$, and X_1 be the closure of the domain of $\tilde{\Lambda}$, $D(\tilde{\Lambda})$, with respect to the energy norm $\|\cdot\|_1 = (\tilde{\Lambda}\cdot, \cdot)^{1/2}$.*

i) *The dual X_1' of X_1 is isometrically isomorphic to the closure X_{-1} of $D(\tilde{\Lambda})$ with respect to the "negative" energy norm $\|\cdot\|_{-1} = (\tilde{\Lambda}^{-1}\cdot, \cdot)^{1/2}$, i.e. for any $f \in X_1'$ there exists one and only one $x' \in X_{-1}$ such that $f(x) = (x', x)$ with $\|f\| = \|x'\|_{-1}$ for all $x \in X_1$.*

ii) *There exists a positive definite selfadjoint operator Λ in the Hilbert space X_{-1} equipped with the scalar product $(\cdot, \cdot)_{-1} = (\Lambda^{-1}\cdot, \cdot)$ which has the following properties:*

$D(\Lambda) = X_1$,

$\Lambda|_{D(\tilde{\Lambda})} = \tilde{\Lambda}$,

$\Lambda: X_1 \to X_{-1}$ *is an isometric isomorphism,*

$(\Lambda^{-1}x, y) = (x, \Lambda^{-1}y)$ *for all* $x, y \in X_{-1}$.

For the proof see [**11**].

We recall the well known fact that every positive definite symmetric densely defined operator $\tilde{\tilde{\Lambda}}$ has a unique positive definite selfadjoint extension $\tilde{\Lambda}$, namely Friedrich's extension.

Furthermore let $H^m(\Omega)$ denote the usual Sobolev spaces with the scalar products $\sum_{|\alpha|<m}(D^\alpha\cdot, D^\alpha\cdot)$ and the respective norms $\|\cdot\|_m$.

(ϕ, ψ) denotes throughout the scalar product $\int_\Omega \phi\psi dx$ in $H^0(\Omega) = L_2(\Omega)$.

We define the following subspaces of $H^2(\Omega)$:

i) Let $H_0^2(\Omega)$ be the closure of all functions $\phi \in C^\infty(\bar{\Omega})$ in $H^2(\Omega)$ satisfying the boundary conditions $B_1[\phi] = \phi|_\Gamma = 0$ and $B_2[\phi] = \phi_\rho|_\Gamma = 0$;

ii) let $H_1^2(\Omega)$ be the closure of all functions $\psi \in C^\infty(\bar{\Omega})$ in $H^2(\Omega)$ satisfying the boundary condition $B_3[\psi] = \psi|_\Gamma = 0$.

After these preliminaries we turn to the functional analytic formulation of the equation

$$N_1[F, W] \equiv \varepsilon^2 \triangle\triangle F + \frac{1}{2}[W, W] + [z, W] = 0, \qquad (2.1)$$
$$B_1[F] = B_2[F] = 0.$$

It is well known that $\tilde{A}_1 = \triangle\triangle$ with $D(\tilde{\tilde{A}}_1) = \{\phi | \phi \in C^\infty(\bar{\Omega}),\ B_1[\phi] = B_2[\phi] = 0\}$ is a positively definite symmetric operator in $L_2(\Omega)$. Let its Friedrich's extension $\tilde{A}_1$ play the role of $\tilde{A}$, $L_2(\Omega)$ that of X_0 of Lemma 1. In this setting X_1 of Lemma 1 is the space $H_0^2(\Omega)$, since the Sobolev norm $\|\cdot\|_2$ and the energy norm $(\tilde{A}_1 \cdot, \ \cdot)^{1/2} = (\triangle\cdot, \triangle\cdot)^{1/2}$ are equivalent on $D(\tilde{\tilde{A}}_1)$. We denote X_{-1} of Lemma 1 i) by $H_0^{-2}(\Omega)$, the respective norm by $\|\cdot\|_{-2}$, and the operator A of Lemma 1 ii) by A_1. An immediate consequence of Lemma 1 is the following lemma.

LEMMA 2. *A_1 is a continuous bijective operator mapping $H_0^2(\Omega)$ onto $H_0^{-2}(\Omega)$, and $A_1\phi = \triangle\triangle\phi$ holds for all $\phi \in C^\infty(\bar{\Omega})$.*

LEMMA 3. *The bilinear mapping $(\phi_1, \phi_2) \to [\phi_1, \phi_2]$ is continuous from $H^2(\Omega)$ into $H_0^{-2}(\Omega)$.*

PROOF. The definition of $[\cdot, \cdot]$ implies that the mapping is continuous from $H^2(\Omega)$ into $L_1(\Omega)$. It remains to show that $L_1(\Omega)$ is continuously imbedded in $H_0^{-2}(\Omega)$. This follows by means of Lemma 1 i): Each $f \in L_1(\Omega)$ defines a continuous linear functional $\ell(\phi) = (f, \phi)$ on $H^2(\Omega)$, since

$$|(f, \phi)| \leqq \int_\Omega |f| d\mathbf{x} \cdot |\phi|_0^\Omega \leqq c\|f\|_{L_1}\|\phi\|_2.$$

This inequality follows from Sobolev's imbedding theorem implying $\|\ell\| = \|f\|_{-2} \leqq c\|f\|_{L_1}$.

Based on Lemmas 2 and 3 we interpret $(F, W) \to N_1[F, W]$ as being a mapping from $H_0^2(\Omega) \times H_1^2(\Omega)$ into $H_0^{-2}(\Omega)$ and apply A_1^{-1} to equation (2.1). This gives

$$F = -\frac{1}{2\varepsilon^2} A_1^{-1}[W + 2z, W], \quad W \in H_1^2(\Omega), \quad F \in H_0^2(\Omega). \qquad (2.2)$$

Next we introduce the energy functional

$$p(W) = \frac{1}{2}\varepsilon^4 \langle W, W\rangle + \frac{1}{8}\|[W, W] + 2[z, W]\|_{-2}^2, \quad W \in H_1^2(\Omega), \qquad (2.3)$$

with

$$\langle u, v\rangle = \int_\Omega \left[u_{xx}v_{xx} + u_{yy}v_{yy} + 2(1-\mu)u_{xy}v_{xy} + \mu(u_{xx}v_{yy} + u_{yy}v_{xx})\right]d\mathbf{x}.$$

Denoting the Frechet derivative of p at the point ψ by $\delta^1(\psi)$ we have the following theorem.

THEOREM 2. *Let W^* be a stationary point of p, i.e., $\delta^1 p(W^*) = 0$. Then the pair (F^*, W^*) is classical solution of* (1.1), (1.2), *where F^* is defined by* (2.2) *with W^* instead of W.*

The proof will be accomplished with the aid of several lemmas.

LEMMA 4. *For all $u, v, w \in H_1^2(\Omega)$ we have $([u, v], w) = ([u, w], v)$.*

PROOF. For the time being let u, v, w be smooth with $u|_\Gamma = v|_\Gamma = w|_\Gamma = 0$.

$$\begin{aligned}([u, v], w) &= \int_\Omega (u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy})w d\mathbf{x} \\ &= -\int_\Omega (u_{xx}v_y w_y + u_{yy}v_x w_x - u_{xy}v_x w_y - u_{xy}v_y w_x)d\mathbf{x} + \int_\Gamma \kappa u_\rho v_\rho\, w ds \\ &= ([u, w], v),\end{aligned}$$

since the boundary integral vanishes. The same arguments as in the proof of Lemma 3 show that $([\cdot, \cdot], \cdot)$ is a continuous trilinear form on $H_1^2(\Omega)$, so (2.4) extends by continuity to all of $H_1^2(\Omega)$.

LEMMA 5. *For all $u \in C^\infty(\bar{\Omega})$, $\phi \in C^\infty(\bar{\Omega})$ with $u|_\Gamma = \phi|_\Gamma = 0$,*

$$\langle u, \phi\rangle = (\triangle\triangle u, \phi) - \int_\Gamma \phi_\rho(u_{\rho\rho} - \mu\kappa u_\rho)ds. \tag{2.5}$$

PROOF. We have

$$\langle u, \phi\rangle = \int_\Omega \triangle u \triangle\phi d\mathbf{x} - (1-\mu)\int_\Omega [u, \phi]d\mathbf{x}.$$

Furthermore

$$[u, \phi] = \operatorname{div}\begin{bmatrix} u_x\phi_{yy} - u_y\phi_{xy} \\ u_y\phi_{xx} - u_x\phi_{xy}\end{bmatrix},$$

and therefore, using the Gaussian integral theorem,

$$\begin{aligned}&\int_\Omega [u, \phi]d\mathbf{x} \\ &= -\int_\Gamma [u_y\phi_{xy}y_0'(s) - u_x\phi_{yy}y_0'(s) - u_x\phi_{xy}x_0'(s) + u_y\phi_{xx}x_0'(s)]ds \\ &= -\int_\Gamma [u_x(\phi_{yy}y_0'(s) + \phi_{xy}x_0'(s)) - u_y(\phi_{xy}y_0'(s) + \phi_{xx}x_0'(s))]ds,\end{aligned}$$

where $x_0 = x_0(s)$, $y_0 = y_0(s)$ is a parametrization of Γ by s being the arc length. Now $u|_\Gamma = 0$ implies $u_x|_\Gamma = -u_\rho y_0'(s)$ and $u_y|_\Gamma = u_\rho x_0'(s)$; using this and noting

$$\phi_x x_0''(s) + \phi_y y_0''(s) = \kappa\phi_\rho,$$

we obtain

$$\begin{aligned}\int_\Omega [u, \phi]d\mathbf{x} &= -\int_\Gamma u_\rho(\phi_{yy}y_0'(s)^2 + 2\phi_{xy}y_0'(s)x_0'(s) + \phi_{xx}x_0'(s)^2)ds \\ &= -\int_\Gamma u_\rho(\phi_{ss} - \kappa\phi_\rho)ds \\ &= \int_\Gamma \kappa u_\rho\phi_\rho ds,\end{aligned} \tag{2.6}$$

the latter because of $\phi|_\Gamma = 0$.
Integration by parts yields

$$\begin{aligned}\int_\Omega \triangle u \triangle\phi d\mathbf{x} &= \int_\Omega \triangle\triangle u\phi d\mathbf{x} - \int_\Gamma \triangle u\phi_\rho ds \\ &= \int_\Omega \triangle\triangle u\phi d\mathbf{x} - \int_\Gamma \phi_\rho(u_{\rho\rho} - \kappa u_\rho)ds,\end{aligned}$$

since $\triangle u|_\Gamma = u_{\rho\rho} - \kappa u_\rho + u_{ss}$. Therefore

$$\langle u, \phi\rangle = \int_\Omega \triangle\triangle u\phi d\mathbf{x} - \int_\Gamma \phi_\rho(u_{\rho\rho} - \kappa u_\rho)ds - (1-\mu)\int_\Gamma \kappa\phi_\rho u_\rho ds,$$

which proves Lemma 5.

LEMMA 6. *Let $w, \phi \in H_1^2(\Omega)$. Then*

$$\begin{aligned}\delta^1 p(W; \phi) &= \frac{\partial}{\partial\alpha} p(W + \alpha\phi)|_{\alpha=0} \\ &= \varepsilon^4\langle W, \phi\rangle + \left(\left[W + z, \frac{1}{2}A_1^{-1}[W + 2z, W]\right], \phi\right)\end{aligned} \tag{2.7}$$

PROOF. This follows by calculation, utilizing Lemmas 2, 3, and 4.

Formulas (2.2), (2.7) immediately imply the following lemma.

LEMMA 7. *$\delta^1 p(W^*) = 0$ is equivalent to*

$$\begin{aligned}&\varepsilon^2(\triangle F^*, \triangle\psi) + \frac{1}{2}([W^*, W^*], \psi) + ([z, W^*], \psi) = 0, \\ &\varepsilon^2\langle W^*, \phi\rangle - ([F^*, W^*], \phi) - ([z, F^*], \phi) = 0\end{aligned} \tag{2.8}$$

for all $(\psi, \phi) \in H_0^2(\Omega) \times H_1^2(\Omega)$.

We call every pair $(F^*, W^*) \in H_0^2(\Omega) \times H_1^2(\Omega)$ which satisfies the equations (2.8) a "generalized solution" of (1.1), (1.2).

To complete the proof of Theorem 2, we have to show that every generalized solution is also a classical solution, i.e., staisfies (1.1), (1.2) pointwise.

LEMMA 8. *For every generalized solution* (F^*, W^*), $(F^*, W^*) \in C^4(\bar{\Omega}) \times C^4(\bar{\Omega})$.

PROOF. This may be demonstrated in the same way as the proof of Lemma 1 in [3], using Agmon's L_p-regularity theory.

Now let (F^*, W^*) be a generalized solution. $F^* \in C^4(\bar{\Omega})$ as well as $F^* \in H_0^2(\Omega)$, whence F^* satisfies the boundary conditions $B_1[F^*] = B_2[F^*] = 0$, and, as partial integration shows, the first of the equations (1.1) in the classical sense. In the same way W^* satisfies the boundary condition $B_3[W^*] = 0$. Since $W^* \in C^4(\bar{\Omega})$, we may integrate by parts the second of the equations (2.8), and, with the aid of formula (2.5), obtain

$$\varepsilon^2(\triangle\triangle W^*, \phi) - \int_\Gamma \phi_\rho(W^*_{\rho\rho} - \mu\kappa W^*_\rho)ds \\ - ([F^*, W^*], \phi) - ([z, F^*], \phi) = 0$$

for all $\phi \in C^\infty(\bar{\Omega})$ with $\phi|_\Gamma = 0$. But this implies

$$\varepsilon^2\triangle\triangle W^* - [F^*, W^*] - [Z, F^*] = 0$$

and

$$B_4[W^*] = W^*_{\rho\rho} - \mu\kappa W^*_\rho = 0$$

which proves theorem 2.

Our object is to prove that the functional p indeed possesses a nontrivial minimum close to the formally approximate solution W_n which was constructed in §1.

LEMMA 9. *The functional p is weakly lower semicontinuous on* $H_1^2(\Omega)$.

PROOF. We write $p = p_1 + p_2$ with $p_1(\phi) = 1/2\ \varepsilon^2\langle\phi, \phi\rangle$ and $p_2(\phi) = 1/8\|[\phi, \phi] + 2[z, \phi]\|^2_{-2}$.

For the present, let $\phi \in C^\infty(\bar{\Omega})$; we have

$$\langle\phi, \phi\rangle = \int_\Omega \phi^2_{xx} + 2\phi^2_{xy} + \phi^2_{yy}\, d\mathbf{x} + \mu\int_\Omega [\phi, \phi]ds \\ = \sum_{|\alpha|=2} \int_\Omega |D^\alpha\phi|^2 d\mathbf{x} + \mu\int_\Gamma \kappa\phi^2_\rho\, ds,$$

where we used (2.6). Hence, since $\kappa > 0$,

$$\langle\phi, \phi\rangle \geqq \sum_{|\alpha|=2} \int_\Omega |D^\alpha\phi|^2\, d\mathbf{x} \geqq c_\beta\|\phi\|_2^2, \tag{2.9}$$

the last inequality being proved in the usual way by the fact that $H_1^2(\Omega)$

is compactly imbedded in $H^1(\Omega)$. Now a density argument shows the validity of (2.9) for all $\phi \in H_1^2(\Omega)$. Thus $\langle \cdot, \cdot \rangle$ is a positive definite continuous bilinear form on $H_1^2(\Omega)$, whence p_1 is weakly lower semicontinuous.

To prove the weak continuity of p_2 it is sufficient to show that $\delta^1 p_2$ is a compact mapping from $H_1^2(\Omega)$ into $H_1^2(\Omega)'$ (see [17], Theorem 8.2). From (2 7),

$$\delta^1 p_2(\phi) = \frac{1}{2}[\phi + z, A_1^{-1}[\phi + 2z, \phi]],$$

so, based on the lemmas 2 and 3, we only have to show that the mapping $(\phi, \psi) \to [\phi, \psi]$, $\phi, \psi \in H_1^2(\Omega)$, has this property. We shall use the interpolation theory from [7] as well as the notation there. Theorem 5.1, [7], implies that the embedding

$$[H_1^2(\Omega), L_2(\Omega)]_\theta \subset H^s(\Omega), \theta = 1 - S/2,$$

is continuous. It follows from a version of Sobolev's imbedding theorem for the spaces $H^s(\Omega)$, theorem 9.8, [7], that

$$[H_1^2(\Omega), L_2(\Omega)]_{3/8} \subset C^0(\bar{\Omega}),$$

and as in the proof of lemma 3 one sees that $(\phi, \psi) \to [\phi, \psi]$ is a continuous mapping from $H_1^2(\Omega)$ into $[H_1^2(\Omega), L_2(\Omega)]_{3/8}'$. The compactness of the embedding

$$H_1^2(\Omega) \subset [H_1^2(\Omega), L_2(\Omega)]_{3/8}$$

follows from theorem 16.2, [7], so transposition shows that the embedding

$$[H_1^2(\Omega), L_2(\Omega)]_{3/8}' \subset H_1^2(\Omega)'$$

is compact as well. Hence p, being the sum of p_1 and p_2, is weakly lower semicontinuous.

Next we expand p about the approximation W_n; of course, $W_n \in H_1^2(\Omega)$, since W_n satisfies the boundary condition $B_3[W_n] = 0$ exactly. An elementary calculation yields

$$\begin{aligned} &p(W_n + \phi) - p(W_n) = \\ &\varepsilon^4\langle W_n, \phi\rangle + ([W_n + z, \frac{1}{2} A_1^{-1}[W_n + 2z, W_n]], \phi) \\ (2.10)\quad &+ \frac{1}{2}\Big[\varepsilon^4\langle\phi, \phi\rangle + \Big(\frac{1}{2}A_1^{-1}[W_n + 2z, W], [\phi, \phi]\Big) + \|[W_n + z, \phi]\|_{-2}^2\Big] \\ &+ \frac{1}{2}([A_1^{-1}[\phi, \phi], \phi], W_n + z) + \frac{1}{8}\|[\phi, \phi]\|_{-2}^2 \end{aligned}$$

for $\phi \in H_1^2(\Omega)$.

Next we estimate the terms on the right hand side of (2.10) consecutively. We set

$$\ell_1(\phi) \equiv \delta^1 p(W_n; \phi) =$$
$$\varepsilon^4\langle W_n, \phi\rangle + \left(\left[W_n + z, \frac{1}{2}A_1^{-1}[W_n + 2z, W_n]\right], \phi\right).$$

LEMMA 10. *If* $\varepsilon < 1$, *then*

$$|\ell_1(\phi)| \leqq c_\gamma \varepsilon^{n+1}\|\phi\|_2, \quad \phi \in H_1^2(\Omega). \tag{2.11}$$

PROOF. For the present, let ϕ be smooth. (2.5) implies

$$\langle W_n, \phi\rangle = (\triangle\triangle W_n, \phi) - \int_\Gamma \phi_\rho(W_{n,\rho\rho} - \mu\kappa W_{n,\rho})ds. \tag{2.12}$$

Recall (1.5):

$$N_1[F_n, W_n] = \varepsilon^2\triangle\triangle F_n + \frac{1}{2}[W_n + 2z, W_n] = R_n$$

with $|R_n|_0^\Omega < c_\alpha\varepsilon^{n+1}$. Applying A_1^{-1} to this equation yields

$$\varepsilon^2 A_1^{-1}\triangle\triangle F_n + \frac{1}{2}A_1^{-1}[W_n + 2z, W_n] = A_1^{-1}R_n, \tag{2.13}$$

where we set

$$A_1^{-1}\triangle\triangle F_n = F_n + \Upsilon. \tag{2.14}$$

According to the definition of A_1 we have $F_n + \Upsilon \in H_0^2(\Omega)$, whence $(F_n+\Upsilon)|_\Gamma = 0$ and $(F_{n,\rho} + \Upsilon_\rho)|_\Gamma = 0$. Now

$$\Upsilon|_\Gamma = 0, \ |\Upsilon_\rho|_0^\Gamma < c_\alpha\varepsilon^{n+2}, \tag{2.15}$$

because of (1.6). We apply $\triangle\triangle$ to (2.14) and obtain

$$\triangle\triangle\Upsilon = 0. \tag{2.16}$$

Using a maximum principle for biharmonic functions (Teorema II, [**8**]), we get from (2.15), (2.16) $|\Upsilon|_0^\Omega \leqq c_1\varepsilon^{n+2}$. Trivially $\|R_n\|_{-2} \leqq c_2|R_n|_0^\Omega$, and therefore

$$|A_1^{-1}R_n|_0^\Omega \leqq c_3\|A_1^{-1}R_n\|_2 \leqq c_4\|R_n\|_{-2} \leqq c_\alpha c_4\varepsilon^{n+1}.$$

(2.13) implies

$$\frac{1}{2}A_1^{-1}[W_n + 2z, W_n] = -\varepsilon^2 F_n - \varepsilon^2\Upsilon + A_1^{-1}R_n, \tag{2.17}$$

and consequently

$$\ell_1(\phi) = \varepsilon^2(\varepsilon^2\triangle\triangle W_n - [W_n + z, F_n], \phi)$$
$$+([W_n + z, \phi], A_1^{-1}R_n - \varepsilon^2\Upsilon)$$

$$-\varepsilon^4 \int_\Gamma \phi_\rho B_4[W_n] ds,$$

using (2.12), (2.4).

We estimate the first term by means of $(1.5)_2$:

$$\varepsilon^2 |(N_2[F_n, W_n], \phi)| \leqq c_\alpha c_5 \varepsilon^{n+3} \|\phi\|_2,$$

for the second, we have:

$$\begin{aligned} &\left| \int_\Omega [W_n + z, \phi](A_1^{-1} R_n - \varepsilon^2 \Upsilon) d\mathbf{x} \right| \\ &\leqq |A_1^{-1} R_n - \varepsilon^2 \Upsilon|_0^\Omega \int_\Omega |[W_n + z, \phi]| d\mathbf{x} \\ &\leqq c_6 \varepsilon^{n+1} \| W_n + z\|_2 \|\phi\|_2 \leqq c_7 \varepsilon^{n+1} \|\phi\|_2. \end{aligned}$$

The validity of the last inequality follows from the fact that the leading term of the boundary layer expansion $\varepsilon^2 W_{ou}\omega$ is of order $O(\varepsilon^2)$ implying that the second derivatives of $W_n + z$ are uniformly bounded with respect to ε.

The modulus of the boundary integral satisfies

$$\left| \int_\Gamma \phi_\rho B_4[W_n] ds \right|_0^\Gamma \leqq c_8 |B_4[W_n]|_0^\Gamma \left(\int_\Gamma \phi_\rho^2 ds \right)^{1/2} \leqq c_\alpha c_9 \varepsilon^{n+1} \|\phi\|_2,$$

where $(1.6)_2$, and the trace theorem 8.3 in [5] have been used.

From these estimates (2.11) follows for smooth ϕ which proves lemma 10 by the usual limiting process.

We define

$$\begin{aligned} \ell_2(\phi) &\equiv \delta^2 p(W_n; \phi, \phi) \\ &= \varepsilon^4 \langle \phi, \phi \rangle + \left(\frac{1}{2} A_1^{-1}([W_n, W_n + 2z], [\phi, \phi] \right) + \|[W_n + z, \phi\|_{-2}^2 \end{aligned}$$

for $\phi \in H_1^2(\Omega)$.

To proceed further, we make the following assumption:

$$\text{(H)} \qquad -\frac{z_{\rho\rho}}{\kappa z_\rho}\bigg|_\Gamma < \frac{1}{2(1 + e^{-\pi})} - \mu.$$

The left hand side of this inequality being always positive, it obviously imposes a mutual restriction on both the boundary behaviour of z and μ. However, it is met in relevant applications.

Now (H) allows us to state and prove the crucial lemma.

LEMMA 11. *Let* (H) *be satisfied,* $n \geqq 4$, *and* ε *sufficiently small. Then* $\ell_2(\phi) \geqq \varepsilon^4 c_\delta \|\phi\|_2^2$ for all $\phi \in H_1^2(\Omega)$.

PROOF. Formula (2.17) and the above estimates imply

$$\begin{aligned}
\ell_2(\phi) &\geqq \varepsilon^4\langle\phi,\phi\rangle - \varepsilon^2(F_n,[\phi,\phi]) + \|[W_n + z,\phi]\|_{-2}^2 - c_9\varepsilon^{n+1}\langle\phi,\phi\rangle \\
&= \varepsilon^4\langle\phi,\phi\rangle - \varepsilon^4(f_2 + h_0,[\phi,\phi]) + \varepsilon^2(F_n - F_2,[\phi,\phi]) \\
&\qquad + \|[W_n + z,\phi]\|_{-2}^2 - c_9\varepsilon^{n+1}\langle\phi,\phi\rangle \\
&\geqq \varepsilon^4\langle\phi,\phi\rangle - \varepsilon^4(f_2 + h_0,[\varphi,\varphi]) \\
&\qquad + \|[W_n + z,\phi]\|_{-2}^2 - c_{10}\varepsilon^{n+1}\langle\phi,\phi\rangle,
\end{aligned}$$

where we have used $\int_\Omega|[\phi,\phi]|d\mathbf{x} \leqq \langle\phi,\phi\rangle$, $F_2 = f_2 + h_0$, and $|F_n - F_2|_0^\Omega = O(\varepsilon^3)$.

Integration by parts as we employed in (2.4) yields

$$\begin{aligned}
\ell_2(\phi) &\geqq \varepsilon^4\langle\phi,\phi\rangle - \varepsilon^4(\phi,[f_2,\phi]) - \varepsilon^4\int_\Gamma f_2\kappa\phi_\rho^2 ds \\
&\quad - \varepsilon^4(h_0,[\phi,\phi]) + \|[W_n + z,\phi]\|_{-2}^2 - c_{10}\varepsilon^5\langle\phi,\phi\rangle.
\end{aligned}$$

Use (2.6) to obtain

$$\begin{aligned}
\ell_2(\phi) &\geqq \varepsilon^4\langle\phi,\phi\rangle - \varepsilon^4|f_2|_0^\Gamma\int_\Omega[\phi,\phi]d\mathbf{x} - \varepsilon^4(h_0,[\phi,\phi]) \\
&\quad - \varepsilon^4(\phi,[f_2,\phi]) + \|[W_n + z,\phi]\|_{-2}^2 - c_{10}\varepsilon^5\langle\phi,\phi\rangle \\
&\geqq \varepsilon^4\langle\phi,\phi\rangle - (|f_2|_0^\Gamma + |h_0|_0^\Omega)\varepsilon^4\int_\Omega|[\phi,\phi]|d\mathbf{x} \\
&\quad - \varepsilon_4(\phi,[f_2,\phi]) + \|[W_n + z,\phi]\|_{-2}^2 - c_{10}\varepsilon^5\langle\phi,\phi\rangle \\
&\geqq \varepsilon^4\langle\phi,\phi\rangle - 2\left|\left[\mu - \frac{z_{\rho\rho}}{\kappa z_\rho}\right]\right|_0^\Gamma(1 + e^{-\pi})\varepsilon^4\langle\phi,\phi\rangle \\
&\quad - \varepsilon^4(\phi,[f_2,\phi]) + \|[W_n + z,\phi]\|_{-2}^2 - c_{10}\varepsilon^5\langle\phi,\phi\rangle.
\end{aligned}$$

This last inequality follows easily from (1.13), (1.15), and, together with (H), implies

$$\begin{aligned}
\ell_2(\phi) &\geqq \varepsilon^4(1 - \eta)\langle\phi,\phi\rangle - \varepsilon^4(\phi,[f_2,\varphi]) \\
&\qquad + \|[W_n + z,\phi]\|_{-2}^2 - c_{10}\varepsilon^5\langle\phi,\phi\rangle, \quad 0 < \eta < 1.
\end{aligned}$$

We now want to show that there exists $\varepsilon_0 > 0$ such that

$$(1 - \eta)/2 - \ell_1(\phi) + \varepsilon^{-4}\ell_2(\varepsilon,\phi) \geqq 0 \tag{2.18}$$

for all $\phi \in H_1^2(\Omega)$ with $\langle\phi,\phi\rangle = 1$ and all $\varepsilon < \varepsilon_0$, where $\ell_1(\phi) = (\phi,[f_2,\phi])$ and $\ell_2(\varepsilon,\phi) = \|[W_n + z,\phi]\|_{-2}^2$. ℓ_1 is a weakly continuous functional on $H_1^2(\Omega)$; to see this, differentiate ℓ_1 to obtain

$$\delta^1\ell_1(\phi;\psi) = (\psi,[f_2,\phi]) + (\phi,[f_2,\psi]) = 2(\psi,[f_2,\phi]).$$

Thus $\delta^1\ell_1\colon \phi \to 2[f_2,\phi]$, and as in the proof of lemma 9 it is seen that this is a compact mapping from $H_1^2(\Omega)$ into $H_1^2(\Omega)$. Hence ℓ_1 is weakly continuous. Likewise $\tilde{\ell}_2\colon \phi \to \|[z,\phi]\|_{-2}^2$ is weakly continuous.

We argue by contradiction: Assume (2.18) is not valid. Then there exists a seuqence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$, and a sequence $\{\phi_n\}$, $\langle\phi_n,\phi_n\rangle = 1$, such that

$$(1 - \eta)/2 - \ell_1(\phi_n) + \varepsilon_n^{-4}\ell_2(\varepsilon_n, \phi_n) < 0$$

for all n.

We may assume that ϕ_n converges weakly in $H_1^2(\Omega)$ to $\hat{\phi}$. We distinguish between two cases:

1. $\hat{\phi} = 0$: Then $\lim_{n\to\infty}(\ell_1\phi_n) = 0$, but this leads to a contradiction, since ℓ_2 is positive.

2. $\hat{\phi} \neq 0$: Write ℓ_2 in the following form:

$$\ell_2(\varepsilon, \phi) = \bar{\ell}_2(\phi) + \hat{\ell}_2(\varepsilon, \phi),$$

with $\bar{\ell}_2$ as above, and

$$\hat{\ell}_2(\varepsilon, \phi) = ([W_n^{(2)}, \phi], A_1^{-1}[W_n^{(2)} - 2z, \phi]),$$

where

$$W_n^{(2)} = W_n^{(2)}(\varepsilon) = \sum_{i=2}^{n+2} (w_i + g_{i-2}\omega)\varepsilon^i.$$

Now (1.4) implies $\lim_{n\to\infty}\bar{\ell}_2(\phi_n) = \bar{\ell}_2(\hat{\phi}) > 0$, furthermore

$$\begin{aligned}|\hat{\ell}_2(\varepsilon_n, \phi_n)| &\leqq c_{11}\int_\Omega |[W_n^{(2)}, \phi_n]|d\mathbf{x} \leqq c_{12}\|W_n^{(2)}(\varepsilon_n)\|_2\|\phi_n\|_2 \\ &\leqq c_{13}\left[\sum_{|\alpha|=2}\int_\Omega |D^\alpha W_n^{(2)}|^2 d\mathbf{x}\right]^{1/2} \\ &= c_{13}\left[\sum_{|\alpha|=2}\int_\Omega \varepsilon_n^4|D^\alpha g_0(\varepsilon_n, \rho, s)|^2 d\mathbf{x}\right]^{1/2} + O(\varepsilon_n),\end{aligned}$$

since the derivatives $D^\alpha W_n^{(2)}$, $|\alpha| = 2$, are bounded uniformly with respect to ε. From (1.13),

$$\varepsilon_n^4|D^\alpha g_0(\varepsilon_n, \rho, s)|^2 < c_{14}e^{-\frac{\gamma\rho}{\varepsilon_n}},$$

$\gamma > 0$, whence $\lim_{n\to\infty} \hat{\ell}_2(\varepsilon_n, \phi_n) = 0$.

Altogether, $\lim_{n\to\infty}\varepsilon^{-4}(\varepsilon_n, \phi_n) = \infty$ which again yields a contradiction, since ℓ_1 is bounded. This proves (2.18), and consequently

$$\ell_2(\phi) \geqq \frac{1}{2}\varepsilon^4(1 - \eta)\langle\phi, \phi\rangle - c_{10}\varepsilon^5\langle\phi, \phi\rangle$$

for all $\phi \in H_1^2(\Omega)$, completing the proof of lemma 11.

We should like to insert the following remark. In **[15]**, Theorem 4.3, a condition analogous to (H) is stated and used—without proof—to obtain $|h_0 + f_2|_0^\Omega \leqq 1 - \eta$. This would greatly facilitate the estimation of ℓ_2:

$$\begin{aligned}\ell_2(\phi) &\geqq \varepsilon^4\langle\phi,\phi\rangle - \varepsilon^4(f_2 + h_0, [\phi,\phi])\\ &\geqq \varepsilon^4\langle\phi,\phi\rangle - \varepsilon^4|f_2 + h_0|_0^\Omega\int_\Omega |2\phi_{xx}\phi_{yy} - 2\phi_{xy}^2|d\mathbf{x}\\ &\geqq \varepsilon^4\langle\phi,\phi\rangle - \varepsilon^4(1-\eta)\sum_{|\alpha|=2}|D^\alpha\phi|^2 d\mathbf{x}\\ &\geqq \varepsilon^4\langle\phi,\phi\rangle - \varepsilon^4(1-\eta)\langle\phi,\phi\rangle\\ &\geqq \varepsilon^4\eta\langle\phi,\phi\rangle,\end{aligned}$$

where we have omitted the higher order terms.

However, set $\phi = z$. Then

$$\begin{aligned}&\langle z, z\rangle - (f_2 + h_0, [z, z])\\ &= \langle z, z\rangle - (f_2, [z, z]) + O(\varepsilon\ln\varepsilon)\\ &= \langle z, z\rangle - (z, [f_2, z]) - \int_\Gamma f_2\kappa z_\rho^2 ds + O(\varepsilon\ln\varepsilon)\\ &= \langle z, z\rangle - 2(z, \triangle\triangle z) - \int_\Gamma f_2\kappa z_\rho^2 ds + O(\varepsilon\ln\varepsilon)\\ &= \langle z, z\rangle - 2\langle z, z\rangle + 2\int_\Gamma z_\rho z_{\rho\rho} - \mu\kappa z_\rho ds - 2\int_\Gamma z_\rho z_{\rho\rho} - \mu\kappa z_\rho ds + O(\varepsilon\ln\varepsilon)\\ &= -\langle z, z\rangle + O(\varepsilon\ln\varepsilon) < 0\end{aligned}$$

for sufficiently small ε. So the estimate $|h_0 + f_2|_0^\Omega \leqq 1 - \eta$ does not hold for any z.

We resume our discussion by setting

$$\begin{aligned}\ell_3(\phi) &\equiv \frac{1}{6}\delta^3 p(W_n; \phi, \phi, \phi) + \frac{1}{24}\delta^4(W_n; \phi, \phi, \phi, \phi)\\ &= \frac{1}{2}([A_1^{-1}[\phi,\phi],\phi], W_n + z) + \frac{1}{8}\|[\phi,\phi]\|_{-2}^2.\end{aligned}$$

LEMMA 12. *For all $\phi \in H_1^2(\Omega)$ with $\|\phi\|_2 < 1$, $|\ell_3(\phi)| \leqq c_\eta\|\phi\|_2^3$.*

PROOF. Lemma 12 follows from the continuity of $\delta^3 p(W_n)$ and $\delta^4 p(W_n)$.

THEOREM 3. *Let (H) be satisfied, $n > 8$, and ε sufficiently small. Then p has an isolated relative minimum $W^* \in H_1^2(\Omega)$, and the estimate $\|W^* - W_n\|_2 < \varepsilon^{n-4}$ holds.*

PROOF. Lemmas 10, 11, and 12 show the validity of the inequality

$$p(W_n + \phi) - p(W_n) \geqq -c_\gamma\varepsilon^{n+1}\|\phi\|_2 + \frac{1}{2}c_\delta\varepsilon^4\|\varphi\|_2^2 - c_\eta\|\varphi\|_2^3,$$

for sufficiently small ε and $\|\phi\|_2$. Thus, for $n > 8$, on the sphere $S = \{W_n + \phi \mid \|\phi\|_2 = \varepsilon^{n-4}\}$ we have

$$
(2.19)\qquad \begin{aligned} p(W_n + \phi) - p(W_n) &\geqq -c_\gamma \varepsilon^{2n-3} + \frac{1}{2}c_\delta \varepsilon^{2n-4} - c_\eta \varepsilon^{3n-12} \\ &= \varepsilon^{2n-4}\Big(-c_\gamma \varepsilon + \frac{1}{2}c_\delta - c_\eta \varepsilon^{n-8}\Big) > 0. \end{aligned}
$$

According to lemma 9, p is weakly lower semi continuous, and therefore attains its minimum W^* on the weakly compact ball $B = \{W_n + \phi \mid \|\phi\|_2 \leqq \varepsilon^{n-4}\}$. Because of (2.19) this minimum does not lie on S. So it is a stationary point of p. By continuity it follows from lemma 11 that the second variation of p at the point W^* is positively definite. This implies that W^* is isolated.

Theorems 2 and 3 yield the existence of a nontrivial solution (F^*, W^*) of (1.1), (1.2); so we have gained the objective of this paragraph. We remark, that the Lyapunov stability of the stationary solution W^* of the time dependent shell equation in a suitable function space may be proved with the above results as in **[13]**.

3. **Uniqueness for big** ε**.** In this paragraph we shall show that for "thick" shells the trivial solution $(F_0, W_0) = (0, 0)$ of (1.1), (1.2) is unique. First we define another subspace of $H^2(\Omega)$. Let $H_2^2(\Omega)$ be the closure of the set of all functions $\phi \in C^\infty(\bar{\Omega})$ in $H^2(\Omega)$ which satisfy the boundary conditions $B_3[\phi] = \phi|_\Gamma = 0$, and $B_4[\phi] = \phi_{\rho\rho} - \mu\kappa_\rho|_\Gamma = 0$.

It follows from formulas (2.5), (2.9) that the operator $\tilde{\tilde{A}}_2 = \triangle\triangle$ with $D(\tilde{\tilde{A}}_2) = \{\phi \mid \phi \in C^\infty(\bar{\Omega}),\ B_3[\phi] = B_4[\phi] = 0\}$ is positive definite and symmetric in $L_2(\Omega)$. As Lemma 1 shows, these properties imply, in analogy with Lemma 2, the following lemma.

LEMMA 13. *There exists a continuous bijective operator A_2 from $H_2^2(\Omega)$ onto $H_2^{-2}(\Omega)$ with $A_2\phi = \triangle\triangle\phi$ for all $\phi \in C^\infty(\bar{\Omega})$, where $H_2^{-2}(\Omega)$ is isomorphic to $H_2^2(\Omega)'$ in the sense of Lemma* 1.

Exactly as in the proof of Lemma 3, one sees that $(\phi, \psi) \to [\phi, \psi]$ is a continuous mapping of $H_0^2(\Omega) \times H_2^2(\Omega)$ into $H_2^{-2}(\Omega)$. Therefore, as before, we may formulate the problem (1.1), (1.2) in $H_0^2(\Omega) \times H_2^2(\Omega)$:

$$
(3.1)\qquad \begin{gathered} \varepsilon^2 A_1 F + \frac{1}{2}[W, W] + [z, W] = 0, \\ \varepsilon^2 A_2 W - [F, W] - [z, F] = 0, \\ (F, W) \in H_0^2(\Omega) \times H_2^2(\Omega). \end{gathered}
$$

Again we solve the first equation for F, and substitute the expression obtained into the second equation.

$$
(3.2)\qquad \varepsilon^4 A_2 W + \frac{1}{2}[W + z,\ A_1^{-1}[W + 2z,\ W]] = 0, \quad W \in H_2^2(\Omega).
$$

We multiply (3.2) by W, and, by means of Lemma (ii), formula (2.5), as well as some elementary manipulations, arrive at

$$(3.3)\qquad \varepsilon^4\langle W, W\rangle + \frac{1}{2}\|[W, W] + \frac{2}{2}[z, W]\|_{-2}^2 - \frac{1}{8}\|[z, W]\|_{-2}^2 = 0.$$

Assume that (3.1) possesses a nontrivial solution $\tilde{W}$. For this we should have $\varepsilon^4\langle \tilde{W}, \tilde{W}\rangle \leqq 1/8\|[z, \tilde{W}]\|_{-2}^2$ which follows immediately from (3.3). Thus, if

$$(3.4)\qquad \varepsilon^4\langle W, W\rangle > \frac{1}{2}\|[z, W]\|_{-2}^2$$

holds for all $W \in H^2(\Omega)\backslash\{0\}$, then (3.1), and consequently (1.1), (1.2) have no nontrivial solution.

THEOREM 4. *If* $\varepsilon^4 > c_\xi\int_\Omega z_{xx}^2 + z_{yy}^2 dx$, *then problem* (1.1), (1.2) *has no nontrivial solution. Here* c_ξ *is a positive constant which depends only on* Ω.

PROOF. The continuity of $[\cdot, \cdot]$ and (2.9) imply

$$\|[z, W]\|_{-2}^2 \leqq c_1\|z\|_2^2\|W\|_2^2 \leqq c_\xi \langle z, z\rangle\langle W, W\rangle.$$

Furthermore

$$\begin{aligned}\langle z, z\rangle &= \int_\Omega(\triangle z)^2 d\mathbf{x} - (1-\mu)\int_\Omega [z, z]\, d\mathbf{x}\\ &= \int_\Omega(\triangle z)^2 d\mathbf{x} - (1-\mu)\int_\Gamma \kappa z_\rho\, ds\\ &\leqq \int_\Omega z_{xx}^2 + 2z_{xx}z_{yy} + z_{yy}^2\, d\mathbf{x}\\ &\leqq 2\int_\Omega z_{xx}^2 + z_{yy}^2\, d\mathbf{x}\end{aligned}$$

which follows from (2.6) and $\kappa > 0$. Combining both estimates, we obtain

$$\|[z, W]\|_{-2}^2 \leqq 2 \int_\Omega z_{xx}^2 + z_{yy}^2 d\mathbf{x}\langle W, W\rangle.$$

This yields (3.4).

4. **Existence of a third solution.** We assume that condition (H) is fulfilled. From Theorems 2 and 3 we know that for sufficiently small values of the parameter ε there exists a nontrivial solution of the equation (3.2) which at the same time is an isolated minimum of the energy functional (2.3). We are going to show that this implies the existence of a second nontrivial solution.

First we establish the apriori boundedness of all solutions of (3.2).

LEMMA 14. *Let* $\varepsilon > \varepsilon_0 > 0$. *Then there exists a positive constant* M, *depending on* ε_0 *only, such that all solutions* $W(\varepsilon)$ *of equation* (3.2) *lie inside the ball* $B_M = \{\phi|\phi \in H_2^2(\Omega), \|\phi\|_2 < M\}$.

PROOF. Assuming the contrary, there would exist a sequence of solutions

$\{W(\varepsilon_n)\}^{n\in\mathbf{N}}$, $\varepsilon_n > \varepsilon_0$, such that $\|W(\varepsilon_n)\|_2 \to \infty$ for $n \to \infty$. Recalling theorem 4, we may assume that the sequence $\{\varepsilon_n\}_{n\in\mathbf{N}}$ is bounded. (3.4) implies that

$$\|[z, W(\varepsilon_n)]\|_{-2}^2 > 8\varepsilon_n^4 \langle W(\varepsilon_n), W(\varepsilon_n)\rangle > c\varepsilon_0^4 \|W(\varepsilon_n)\|_2^2 \tag{4.1}$$

holds for all $W(\varepsilon_n)$. Without loss of generality, $W(\varepsilon_n) = \|W(\varepsilon_n)\|_2 v_n$, with $v_n \to_s v$, where $\to_s$ denotes weak convergence in $\mathring{H}_2^2(\Omega)$. As in the proof of Lemma 9 one sees that the functionals $p_3(\phi) = \|[\phi, \phi]\|_{-2}^2$ and $p_4(\phi) = \|[z, \phi]\|_{-2}^2$ are weakly continuous on $\mathring{H}_2^2(\Omega)$. From (3.3), using (4.1),

$$c\varepsilon_0^4\|W(\varepsilon_n)\|_2^2 + \|[z, W(\varepsilon_n)]\|_{-2}^2 + \frac{3}{2}([W(\varepsilon_n), W(\varepsilon_n)], [z, W(\varepsilon_n)])_{-2} + \frac{1}{2}\|[(W\varepsilon_n), W(\varepsilon_n)]\|_{-2}^2 < 0.$$

We divide this inequality by $\|W(\varepsilon_n)\|_2^4$ obtaining

$$\frac{c\varepsilon_0^4}{\|W(\varepsilon_n)\|_2^2} + \frac{\|[z, v_n]\|_{-2}^2}{\|W(\varepsilon_n)\|_2^2} + \frac{3}{2}\frac{([z, v_n], [v_n, v_n])_{-2}}{\|W(\varepsilon_n)\|_2} + \frac{1}{2}\|[v_n, v_n]\|_{-2}^2 < 0.$$

Letting n tend to infinity, we conclude $[v, v] = 0$. The lemma below shows that this implies $v = 0$. But. $\|[z, v_n]\|_{-2}^2 > c\varepsilon_0\|v_n\|_2^2$, because of (4.1), whence $\|v_n\|_2 \to 0$ by the weak continuity of p_4. This contradicts $\|v_n\|_2 = 1$ for all n.

LEMMA 15. *Let Ω be strictly convex, $\phi \in H^2(\Omega)$, and $\phi|_\Gamma = 0$. Then $[\phi, \phi] = 0$ implies $\phi = 0$.*

PROOF. This result is easily proved for $\phi \in C^2(\bar{\Omega})$, since $[\phi, \phi]$ is proportional to the Gaussian curvature of the surface $\phi(\mathbf{x})$. If $[\phi, \phi] = 0$, then $\phi(\mathbf{x})$ is a plane, and therefore vanishes identically, since $\phi|_\Gamma = 0$. For $\phi \in H^2(\Omega)$, the proof known to us uses deep topological results; we refer to **[10]**.

We now let the operator A_2^{-1} act on equation (3.2), set $\lambda = \varepsilon^{-4}$, thus obtaining

$$W + \lambda\frac{1}{2}A_2^{-1}[W + z, A_1^{-1}[W + 2z, W]] = 0, \quad W \in \mathring{H}_2^2(\Omega.) \tag{4.2}$$

We denote the left hand side of this equation by $F_\lambda(W)$, and note that F_λ: $\mathring{H}_2^2(\Omega) \to \mathring{H}_2^2(\Omega)$ is a mapping of the form $F_\lambda = \mathrm{id} + \lambda K$, where id is the identity and K a compact mapping. The compactness of K follows from lemma 9.

LEMMA 16. *F_λ is a potential operator, whose potential is given by $\lambda\tilde{p}$, $\lambda\tilde{p}$ denoting the restriction of the energy functional p, (2.3), to $\mathring{H}_2^2(\Omega)$.*

PROOF. For the duality between $\mathring{H}_2^2(\Omega)$ and $\mathring{H}_2^2(\Omega)'$ one uses the inner product $\langle \cdot, \cdot \rangle = (A_1 \cdot, \cdot)$, identifying $\mathring{H}_2^2(\Omega)'$ and $\mathring{H}_2^2(\Omega)$ by means of the Riesz isomorphism. Then the result of lemma 16 follows by differentiating $\lambda \tilde{p}$.

We now apply a degree argument to prove the existence of a third solution of (1.1), (1.2) (cf. [9]). It is shown by lemma 14 that the Leray-Schauder degree of F_λ with respect to $B_M \subset \mathring{H}_2^2(\Omega)$ and $0 \in \mathring{H}_2^2(\Omega)$ for $\lambda \in (0, \varepsilon^{-4}]$ is well defined. Denote it by $DG(F_\lambda, B_M, 0)$. The index of an isolated solution of $F_\lambda(\phi) = 0$ will be denoted by $\mathrm{ID}(F_\lambda, \phi, 0)$.

LEMMA 17 *Let ϕ be an isolated local minimum of $\lambda\tilde{p}$. Then* $\mathrm{ID}(F_\lambda, \phi, 0) = 1$.

PROOF. See [9].

THEOREM 5. *Let* (H) *be satisfied. Then the problem* (1.1), (1.2) *has at least three solution, provided ε is small enough.*

PROOF. Let ε^* be so small that the equation $F_{\lambda^*}(W) = 0$, $\lambda^* = (\varepsilon^*)^{-4}$, possesses besides the trivial solution $W_0 = 0$ the solution W^* as guaranteed by theorems 2 and 3. Theorem 3 shows that this solution is an isolated local minimun of $\lambda\tilde{p}$. Furthermore, choose ε_* big enough so that $W_0 = 0$ is the only solution of $F_{\lambda_*}(W) = 0$, $\lambda_* = (\varepsilon_*)^{-4}$; this is possible by theorem 4. Since $\lambda\tilde{p}(W) > 0$ for $W \neq 0$, now W_0 is an isolated minimum as well. (One can show that the absolute minimum W_0 is isolated for any ε.) Lemma 14 implies that

$$\mathrm{DG}(F_{t\lambda_* + (1-t)\lambda^*}, B_M, 0), \quad 0 \leqq t \leqq 1, \; M = M(\varepsilon^*/2),$$

is well defined. Now it is a consequence of the homotopy invariance of the degree that

$$\mathrm{DG}(F_{\lambda_*}, B_M, 0) = \mathrm{DG}(F_{\lambda^*}, B_M, 0).$$

From Lemma 17,

$$\mathrm{DG}(F_{\lambda_*}, B_M, 0) = \mathrm{ID}(F_{\lambda_*}, W_0, 0) = 1,$$

whereas

$$\mathrm{DG}(F_{\lambda^*}, B_M, 0) = \mathrm{ID}(F_{\lambda^*}, W_0, 0) + \mathrm{ID}(F_{\lambda^*}, W^*, 0) = 2,$$

if W_0 and W^* are the only solutions of $F_{\lambda^*}(W) = 0$. This is a contradiction implying that there exists at least another solution W^{**}. Defining F^{**} by $(\varepsilon^*)^4 F^{**} = -\frac{1}{2} A_1^{-1}[W^{**} + 2z, W^{**}]$, one sees as in §2 that (F^{**}, W^{**}) is a classical solution of the problem (1.1), (1.2).

With regard to the example given in the introduction, one may be led to the conjecture that there are many more solutions than just these three exhibiting sharp transitions inside Ω.

References

1. M. S. Berger, *On von Karman's equations and the buckling of a thin elastic plate*, I, Comm. Pure Appl. Math. **20** (1967), 687–719.

2. ——— and P. C. Fife, *Von Karman's equations and the buckling of a thin elastic plate*, II, Comm. Pure Appl. Math. **21** (1968), 227–241.

3. ——— and L. E. Fraenkel, *On the asymptotic solution of a nonlinear Dirichlet problem*, J. Math. Mech. **19** (1969/70), 553–585.

4. G. Fichera, *Linear Elliptic Differential Systems and Eigenvalue Problems*, Springer, Berlin-Heidelberg-New York, 1965.

5. P. C. Fife, *Semilinear elliptic boundary value problems with small parameters*, Arch. Rational Mech. Anal. **52** (1973), 205–232.

6. ——— and W. M. Greenlee, *Transition layers for elliptic boundary value problems*, Russ. Math. Surveys, **29** (1974), 103–131.

7. J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, I, Springer, Berlin-Heidelberg-New York, 1972.

8. C. Miranda, *Formule di maggiorazione e teorema di esistenza per le funzioni biarmoniche di due variabili*, Giron. Mat. Battaglini, **78** (1948/49), 97–118.

9. P. H. Rabinowitz, *A note on topological degree for potential operators*, Preprint, 1974.

10. J. Rauch and B. A. Taylor, *The Dirichlet problem for the multidimensional Monge-Ampere-equation*, Preprint.

11. G. Rupprecht, *Operatoren in Hilbertraum-Tripeln, Diplomarbeit*, Stuttgart, 1972.

12. L. S. Srubshchik and V. I. Judovich, *The asymptotic integration of the system of equations for the large deflections of symmetrically loaded shells of revolution*, Journal of Applied Mathematics and Mechanics (translation of Prikladjana Matematika I Mechanika) **26** (1962), 1378–1392.

13. ———, *Note on the stability of membrane solutions in the nonlinear theory of plates and shells*, Journal of Applied Mathematics and Mechanics (translation of Prikladjana Matematika I Mechanika) **30** (1966), 137–147.

14. L.S. Srubshchik, *Nonstiffness of spherical shells*, Journal of Applied Mathematics and Mechanics (translation of Prinkladjana Matematika I Mechanika) **31** (1967), 737–743.

15. ———, *On the question of nonrigidity in the theory of gently sloping shells*, Math. USSR Izvestija **6** (1972), 883–903.

16. M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, Holden-Day, San Fransisco-London-Amsterdam, 1964.

17. V. Velte, *Direkte Methoden der Variationsrechnung*, Teubner, Stuttgart, 1976.

18. M. I. Visik and L. A. Lyusternik, *Regular degeneration and boundary layers for linear differential equations with small parameter*, Amer, Math. Soc. Trans. **20** (1962), 239–364.

19. H. J. Weinitschke, *Existenz-and Eindeutigkeitssätze in der nichtlinearen Theorie elastischer Schalen*, Technische Universität, Berlin, 1968.

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