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DIRECT SUMS OF COUNTABLE GROUPS

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1. If $\{G_\alpha \mid \alpha \in A\}$ is a collection of groups, then $\sum_{\alpha \in A} G_\alpha$ and $\prod_{\alpha \in A} G_\alpha$ will denote the direct sum and the unrestricted direct product of the G_α for α in A, respectively. If $G_\alpha = G$, for all α in A, then G^A will be used to denote $\prod_{\alpha \in A} G_\alpha$.

It was recently shown by T. Head [2] that if G is a group, then G^A is a direct sum of countable groups if and only if G is an Abelian group whose reduced part is of bounded exponent. The purpose of this note is to answer the following question: "If $\{G_\alpha \mid \alpha \in A\}$ is a collection of groups such that $\prod_{\alpha \in A} G_\alpha$ is a direct sum of countable groups, then what can one say about the G_α for α in A?" By modifying Head's proof we obtain a result similar to his.

In §3 we note that "except for possibly a finite subset of A", $\prod_{\alpha \in A} G_\alpha$ being a direct sum of countable groups is equivalent to $\sum_{\alpha \in A} G_\alpha$ being complemented in $\prod_{\alpha \in A} G_\alpha$.

2. In order to prove Theorem 1 we make use of the following fact which may be easily verified.

(*) If G is a reduced Abelian group which is a direct sum of countable groups and A is a countable subgroup of G, then the maximal divisible subgroup of G/A is countable.

THEOREM 1. *If $\mathcal{C} = \{G_\alpha \mid \alpha \in A\}$ is a collection of groups, then $\prod_{\alpha \in A} G_\alpha$ is the direct sum of countable groups if and only if there exist a positive integer k and a finite subset A_0 of A such that:*

(a) *If $\alpha \in A/A_0$, then G_α is Abelian and has reduced part of exponent less than k, and*

(b) *If $\alpha \in A_0$, then G_α is a direct sum of countable groups.*

PROOF. Assume that $G = \prod_{\alpha \in A} G_\alpha$ is a direct sum of countable groups. Suppose there exists an infinite subcollection $\{G_\beta \mid \beta \in B\}$ of $\mathcal{C}$ consisting of non-Abelian groups. Let $x_\beta \in G_\beta \setminus Z(G_\beta)$ for all β in B and define $(g_\alpha) \in G$ by

$$g_\alpha = \begin{cases} e, & \alpha \notin B, \\ x_\alpha, & \alpha \in B. \end{cases}$$

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Let C be the centralizer of (g_α) in G. Then $[\prod G_\alpha : C] \geqq 2^{\aleph_0}$ since $x_\alpha \notin Z(G_\alpha)$ for all α in B. However, considering (g_α) as an element in a direct sum of countable groups, it is easy to see that $[G : C] \leqq \aleph_0$. Hence, there exist at most a finite number of non-Abelian groups in $\mathcal{C}$.

Since a direct factor of a direct sum of countable groups is itself a direct sum of countable groups, (see [3, Corollary 3.2, p. 240]), there is no loss of generality in assuming now that all groups are Abelian. Let $G_\alpha = D_\alpha \oplus R_\alpha$, where D_α and R_α are respectively the maximal divisible subgroup and a reduced subgroup of G_α for all α in A. Let $G \cong \sum_{\gamma \in D} H_\gamma$, where each H_γ is countable. Then the reduced part of G is isomorphic to the direct sum of the reduced parts of H_γ for γ in D. Hence, $\prod_{\alpha \in A} R_\alpha$ is a direct sum of countable groups.

Now suppose the conclusion of the theorem is false. Then three possibilities exist.

(1) There is an infinite sequence $\delta_1, \delta_2, \cdots$ of indices in A such that R_{δ_i} contains an element g_i of infinite order for $i = 1, 2, \cdots$.

(2) There is an infinite sequence $\delta_1, \delta_2, \cdots$ of indices in A and a prime p such that R_{δ_i} contains an element g_i of order p^i for $i = 1, 2, \cdots$.

(3) There is an infinite sequence $\delta_1, \delta_2, \cdots$ of indices in A and distinct primes p_i such that G_{δ_i} contains an element g_i of order p_i for $i = 1, 2, \cdots$.

In each case, let $H = \prod_{i=1}^{\infty} \langle g_i \rangle$ and $K = \sum_{i=1}^{\infty} \langle g_i \rangle$. Then the maximal divisible subgroup of H/K in each case is uncountable (see [1, Proof of Theorem 1, p. 405]). Hence, the maximal divisible subgroup of $(\prod_{\alpha \in A} R_\alpha)/K$ is uncountable. However, since $\prod_{\alpha \in A} R_\alpha$ is a direct sum of countable groups, (*) implies that the maximal divisible subgroup of $(\prod_{\alpha \in A} R_\alpha)/K$ must be countable and hence we obtain a contradiction. It follows that all but a finite number of the reduced parts of the G_α, for α in A, are of exponent less than some fixed integer.

Since an Abelian group of finite exponent is a direct sum of cyclic groups, it is easy to see that the converse is true.

The following corollary is immediate.

COROLLARY 1. *If* $\prod_{\alpha \in A} G_\alpha$ *is a direct sum of countable groups, then*: (1) $\sum_{\alpha \in A} G_\alpha$ *is a direct summand of* $\prod_{\alpha \in A} G_\alpha$, (2) $\prod_{\alpha \in A} G_\alpha = (\sum_{\alpha \in A} G_\alpha) \oplus B \oplus D$ *where B is a bounded Abelian group and D is a divisible Abelian group, and* (3) *there exists a finite subset* A_0 *of* A *such that* $\prod_{\alpha \in A} G_\alpha = (\sum_{\alpha \in A_0} G_\alpha) \oplus B' \oplus D'$ *where B' is a bounded Abelian group and D' is a divisible Abelian group.*

3. B. H. Neumann [4] proved that if $\{G_\alpha \mid \alpha \in A\}$ is a collection of groups such that $\sum_{\alpha \in A} G_\alpha$ is complemented in $\prod_{\alpha \in A} G_\alpha$, then all but a

finite number of the G_α, for α in A, are Abelian. In addition, G. Baumslag and N. Blackburn [1] showed that if $\{G_\alpha \mid \alpha \in A\}$ is a collection of Abelian groups, then $\sum_{\alpha\in A} G_\alpha$ is complemented in $\prod_{\alpha\in A} G_\alpha$ if and only if all but a finite number of the G_α have reduced parts of exponent less than some fixed integer. Theorem 2 follows directly from these remarks. For the sake of completeness we include Theorem 3, which is merely a restatement of part (1) of Corollary 1.

THEOREM 2. *If $\sum_{\alpha\in A} G_\alpha$ is complemented in $\prod_{\alpha\in A} G_\alpha$, then there exists a finite subset $A_0 \subseteq A$ such that $\prod_{\alpha\in A\setminus A_0} G_\alpha$ is a direct sum of countable groups.*

THEOREM 3. *If $\prod_{\alpha\in A} G_\alpha$ is a direct sum of countable groups, then $\sum_{\alpha\in A} G_\alpha$ is a direct summand of $\prod_{\alpha\in A} G_\alpha$.*

The following corollary follows immediately from Theorem 1 and Theorem 2.

COROLLARY 2. *If $\sum_{\alpha\in A} G_\alpha$ is complemented in $\prod_{\alpha\in A} G_\alpha$, then there exists a finite subset $A_0 \subseteq A$ such that $\prod_{\alpha\in A\setminus A_0} G_\alpha$ is the direct sum of a bounded Abelian group and a divisible group.*

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