ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 2, Number 1, Winter 1972

A CLOSURE PROPERTY OF REGRESSIVE ISOLS

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0. **Introduction.** Let ϵ^*, ϵ, Λ, Λ_R and Λ^* denote the collections of all integers, nonnegative integers, isols, regressive isols, and isolic integers respectively. Let $f(x_1, \cdots, x_n)$ be a recursive function, and let f_Λ denote the canonical extension of f to a mapping from Λ^n into Λ^*. Let Δ be any subcollection of Λ. We say that Δ *is closed under* f if $f_\Lambda(\Delta^n) \subseteq \Delta$. A. Nerode proved in [12] that Λ is closed under f if and only if f is almost recursive combinatorial. In [2], J. Barback showed that if f is a recursive function of one variable, Λ_R is closed under f if and only if f is eventually increasing. The purpose of this paper is to characterize the class of recursive functions of two variables mapping $\Lambda_R{}^2$ into Λ_R. The class obtained is surprisingly limited; it consists primarily of functions of the form $\min(f(x), g(y))$ where $\min(x, y)$ is the usual minimum function and $f(x)$ and $g(y)$ are eventually increasing and recursive. A precise statement of the main result requires the following two definitions. $f(x, y)$ will be called *flat* if there is a (recursive) function $g(x, y)$ such that $g(x, y) = 0$ for all but finitely many pairs $(x, y) \in \epsilon^2$ and $f(x, y) = \sum_{i=0}^{x} \sum_{j=0}^{y} g(i, j)$ for all $(x, y) \in \epsilon^2$. $f(x, y)$ will be called *reducible to the case of a single variable* if (i) there exist eventually increasing recursive functions $f_i(y)$, $i = 0, \cdots, m$, such that $f(x, y) = f_x(y)$ for $x \leqq m$ and $f(x, y) = f_m(y)$ for $x > m$, or (ii) condition (i) holds with the roles of x and y interchanged. The main result is the following:

Λ_R is closed under a recursive function $f(x, y)$ if and only if there is an $n \in \epsilon$ such that:

(1) For $i \leqq n$, $f(i, y)$ is an eventually increasing function of y and $f(x, i)$ is an eventually increasing function of x,

(2) $f(x + n, y + n) = m(x, y) + c_1(x, y) - c_2(x, y)$ for $x, y \in \epsilon$, where c_1 and c_2 are flat recursive functions and $m(x, y)$ is either (i) reducible to the case of a single variable or (ii) of the form $\min(g(x), h(y))$, where $g(x)$ and $h(y)$ are eventually increasing recursive functions of one variable.

Functions mapping $\Lambda_R{}^2$ into Λ_R have a natural use as Skolem func-

Received by the editors August 4, 1969.

AMS 1970 *subject classifications.* Primary 02F40; Secondary 02F20.

[1]The main results of this paper are part of a doctoral thesis written under the direction of Professor J. C. E. Dekker at Rutgers — The State University.

tions for first order sentences of the arithmetic of Λ_R. Thus the results of this paper show that the class of functions of two variables readily available for use as Skolem functions in Λ_R is rather limited. Similar negative results can be derived for functions of more than two variables.

1. **Preliminaries.** We shall assume familiarity with the concepts and main results of [4], [5], [6], [7], [8] and [10]. The definitions and theorems stated in this section are less widely known and play an essential part in the proofs of our main results.

NOTATION. For any set $\alpha \subseteq \epsilon$, Req (α) will denote the recursive equivalence type of α. We shall write $\alpha \mid \beta$ if α and β can be separated by disjoint r.e. sets.

We will often make use of the recursive pairing function $j(x, y)$ and the projection functions $k(x)$ and $l(x)$ defined by

$$j(x, y) = \frac{(x + y)(x + y + 1)}{2} + x,$$

$$kj(x, y) = x, \qquad lj(x, y) = y.$$

We will use ν_n or $\nu(n)$ as a notation for the initial segment $\{0, \cdots, n - 1\}$ of ϵ.

DEFINITION. Let $T \in \Lambda_R - \epsilon$ and let $f(x)$ be a strictly increasing function. Then

$$\phi_f(T) = \text{Req } \{\text{range } t_{f(n)}\}$$

where t_n is any regressive function ranging over a member of T.

PROPOSITION PR 1 (SANSONE [15]). *Let $f(x)$ be a strictly increasing recursive function and $T \in \Lambda_R - \epsilon$. Then $\phi_f(f_\Lambda(T)) = T$.*

DEFINITION. (a) Let a_n and b_n be any two one-to-one functions mapping ϵ into ϵ. We write $a_n \overset{*}{\vee} b_n$ if there is a partial recursive function $p(x)$ such that for all n, $p(a_n) = b_n$ or $p(b_n) = a_n$.

(b) Let A and B be any two infinite regressive isols. Then $A \overset{*}{\vee} B$ if $a_n \overset{*}{\vee} b_n$ for every pair of regressive functions a_n and b_n such that range $a_n \in A$, range $b_n \in B$ and range $a_n \mid$range b_n.

PROPOSITION PR 2 (BARBACK [3]). *For all infinite regressive isols A and B,*

$$A + B \in \Lambda_R \Rightarrow A \overset{*}{\vee} B.$$

PROPOSITION PR 3 (DEKKER [6]). *There exist A, $B \in \Lambda_R$ such that $A + B \notin \Lambda_R$.*

By a *number-theoretic function* of n variables we shall mean a function mapping ϵ^n into ϵ^*. Every number-theoretic function f can be written as the difference of two combinatorial functions f^+ and f^-, called the positive and negative parts of f; f is called recursive if f^+ and f^- are recursive. For a recursive number-theoretic function $f(x_1, \cdots, x_n)$, we can employ the usual canonical extension procedure to define f_Λ, i.e., for any n-tuple of isols $(x_1, \cdots, x_n)$,

$$f_\Lambda(x_1, \cdots, x_n) = f^+{}_\Lambda(x_1, \cdots, x_n) - f^-{}_\Lambda(x_1, \cdots, x_n).$$

Let $f(x, y)$ be recursive and number theoretic. For T, $U \in \Lambda_R$ we define

$$\sum_{(T,U)}^* f(x, y) = \sum_{(T,U)} f^+(x, y) - \sum_{(T,U)} f^-(x, y).$$

For any recursive function $f(x, y)$ we define

$$\begin{aligned} \hat{f}(x, y) &= 0, && \text{if } x = 0 \text{ or } y = 0,\\ &= f(x-1, y-1), && \text{otherwise,}\\ \Delta_x f(x, y) &= f(x+1, y) - f(x, y), &&\\ \Delta_y f(x, y) &= f(x, y+1) - f(x, y), &&\\ Df(x, y) &= \Delta_x \Delta_y \hat{f}(x, y), &&\\ Df^+(x, y) &= Df(x, y), && Df(x, y) \geqq 0,\\ &= 0, && \text{otherwise,}\\ Df^-(x, y) &= -Df(x, y), && Df(x, y) \leqq 0,\\ &= 0, && \text{otherwise.} \end{aligned}$$

The following theorem does not appear in the literature, but it is the natural generalization for functions of two variables of Theorem 2 of [**15**] and can be proved readily using the methods of [**15**].

PROPOSITION PR 4. *Let $f(x, y)$ be recursive. Then for T, $U \in \Lambda_R$*

$$f_\Lambda(T, U) = \sum_{(T+1,U+1)} Df^+ - \sum_{(T+1,U+1)} Df^-.$$

We shall often make use of recursive functions $j(x, y)$ and $j_3(x, y, z)$ and their associated projection functions, as defined in [5]. We will also make use of the function $x \dot{-} y$ defined by

$$\begin{aligned} x \dot{-} y &= x - y, && x > 0,\\ &= 0, && x \leqq y. \end{aligned}$$

If $\alpha \subseteq \epsilon$, we define the principal function of α to be the (unique) strictly increasing function whose range is α. The domain of this function is ϵ only if α is infinite. If α is finite, the principal function has a proper initial segment of ϵ as its domain.

2. Two mapping theorems.

NOTATION. Let (x_1, y_1) and (x_2, y_2) be members of ϵ^2. We write $(x_1, y_1) \leqq (x_2, y_2)$ if $x_1 \leqq x_2$ and $y_1 \leqq y_2$. We write $(x_1, y_1) < (x_2, y_2)$ if $(x_1, y_1) \leqq (x_2, y_2)$ and $(x_1, y_1) \neq (x_2, y_2)$.

DEFINITION 1. Let $g(n)$ and $h(n)$ be total functions. The pair $(g(n), h(n))$ is said to be a *strictly increasing pair* if for every $n \in \epsilon$, $(g(n+1), h(n+1)) > (g(n), h(n))$.

THEOREM 1. *Let $f(x, y)$ be a recursive function such that $Df(x, y) \geqq 0$ for $(x, y) \in \epsilon^2$. Then Λ_R is closed under $f(x, y)$ if and only if one of the following conditions holds*:

(i) $(\exists n)(\forall x)(\forall y)[x > n \Rightarrow Df(x, y) = 0]$,

(ii) $(\exists n)(\forall x)(\forall y)[y > n \Rightarrow Df(x, y) = 0]$,

(iii) *there exists a strictly increasing pair* $(g(n), h(n))$ *of recursive functions such that* (a) $Df(g(n), h(n)) > 0$ *for all n, and* (b) *for all but finitely many pairs (x, y) which are not of the form* $(g(n), h(n))$ *for any n, $Df(x, y) = 0$.*

PROOF. Let $f(x, y)$ be a recursive function such that $Df(x, y) \geqq 0$. We begin by showing that if f satisfies one of the conditions (i), (ii), (iii), then f_Λ maps $\Lambda_R{}^2$ into Λ_R.

Case A. Condition (i) holds. Clearly we can restrict our attention to proving

$$(T, U) \in \Lambda_R{}^2 - \epsilon^2 \Rightarrow f_\Lambda(T, U) \in \Lambda_R.$$

Subcase 1. $T \geqq n$. Let $r(j) = \sum_{i=0}^{n} Df(i, j)$. Then by hypothesis of Case A and PR 4, $f(x+n, y) = \sum_{j<y+1} r(j)$ for x, $y \in \epsilon$. Hence $f_\Lambda(T, U) = \sum_{U+1} r(j) \in \Lambda_R$.

Subcase 2. $T \leqq n - 1$. Put $r(j) = \sum_{i<T+1} Df(i, j)$. Then, as before $f_\Lambda(T, U) = \sum_{U+1} r(j) \in \Lambda_R$.

Case B. Condition (ii) holds. This is similar to Case A.

Case C. Condition (iii) holds. If T or U is finite, the techniques of Cases A and B may be applied to obtain $f_\Lambda(T, U) \in \Lambda_R$. We assume, then, that T and U are infinite. Let t_n and u_n be regressive functions such that range $t \in T + 1$ and range $u \in U + 1$. Let k be the sum of all nonzero values of Df which are not of the form $Df(g(n), h(n))$. Put $s(n) = Df(g(n), h(n))$. Then

$$f_\Lambda(T, U) = k + \left\{ \operatorname{Req} \bigcup_{n=0}^{\infty} j_3[t_{g(n)}, u_{h(n)}, \nu_{s(n)}] \right\} .$$

We need only describe an effective procedure for regressing the set $\sigma = \bigcup_{n=0}^{\infty} j_3(t_{g(n)}, u_{h(n)}, \nu_{s(n)})$. We arrange the elements of σ in the following order:

$$\begin{array}{l} j_3(t_{g(0)}, u_{h(0)}, 0), \cdots, j_3(t_{g(0)}, u_{h(0)}, s(0)-1) \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ j_3(t_{g(n)}, u_{h(n)}, 0), \cdots, j_3(t_{g(n)}, u_{h(n)}, s(n)-1). \end{array}$$

Since t_n and u_n are regressive and $g(n)$, $h(n)$ and $s(n)$ are recursive, it is clear that we can regress through this array by proceeding from right to left in each row and from the left most element of any row to the right most element of the row above.

This completes the proof of the sufficiency of conditions (i), (ii), (iii). We shall now prove their necessity. Assume that f satisfies none of (i), (ii), (iii). Our ultimate aim is to show that f does not map $\Lambda_R{}^2$ into Λ_R. We distinguish two cases.

Case A. Df is eventually zero, i.e., there exists a number n such that

$$(\forall x)(\forall y)[Df(x+n, y+n)] = 0.$$

Since (i) does not hold, there exist infinitely many values of x, such that $(\exists y < n)[Df(x, y) > 0]$. Similarly there exist infinitely many values of y such that $(\exists x < n)[Df(x, y) > 0]$. Define

$$a(x) = \sum_{y<n} Df(x, y), \qquad b(y) = \sum_{x<n} Df(x, y).$$

Then both $\{x \mid a(x) > 0\}$ and $\{y \mid b(y) > 0\}$ are infinite sets. Let $x, y \in \epsilon$. Then

$$\begin{aligned} f(x+n, y+n) &= \sum_{i<x+n+1} \sum_{j<y+n+1} Df(i, j) \\ &= \sum_{i<n} \sum_{j<n} Df(i, j) + \sum_{i<x+1} a(i+n) + \sum_{j<y+1} b(j+n). \end{aligned}$$

Let $c = \sum_{i<n} \sum_{j<n} Df(i, j)$. Then the preceding identity can be extended to Λ_R, to obtain for $T, U \in \Lambda_R$,

$$(**) \qquad f_\Lambda(T+n, U+n) = c + \sum_{T+1} a(i+n) + \sum_{U+1} b(j+n).$$

We now define two strictly increasing recursive functions $v(x)$ and $w(x)$ by:

$$v(0) = (\mu y)[a(y+n) > 0],$$

$$v(x+1) = (\mu y)[y > v(x) \,\&\, a(y+n) > 0],$$

$$w(0) = (\mu y)[b(y+n) > 0],$$

$$w(x+1) = (\mu y)[y > w(x) \,\&\, b(y+n) > 0].$$

Clearly, for $x \in \epsilon$, $a(v(x)+n) > 0$ and $b(w(x)+n) > 0$. Furthermore, for $T \in \Lambda_R$, $\sum_T a(i+n) = \sum_{\phi_v(T)} a[v(i)+n]$ and $\sum_T b(i+n) = \sum_{\phi_w(T)} b[w(i)+n]$.

By PR 3, there exist two infinite regressive isols A and B such that $A + B \notin \Lambda_R$. Since v and w are strictly increasing recursive functions $v_\Lambda(A)$ and $w_\Lambda(B) \in \Lambda_R$. By our previous observations,

$$\begin{aligned} f_\Lambda(v_\Lambda(A) + n - 1, w_\Lambda(B) + n - 1) &= c + \sum_{v_\Lambda(A)} a(i+n) + \sum_{w_\Lambda(B)} b(j+n) \\ &= c + \sum_{\phi_v[v_\Lambda(A)]} a[v(i)+n] + \sum_{\phi_w[w_\Lambda(B)]} b[w(i)+n] \\ &= c + \sum_A a[v(i)+n] + \sum_B b[w(j)+n] \\ &\geqq A + B. \end{aligned}$$

Thus $f_\Lambda(v_\Lambda(A) + n - 1,\ w_\Lambda(B) + n - 1) \notin \Lambda_R$, and f does not map $\Lambda_R{}^2$ into Λ_R.

Case B. Df is not eventually zero.

We begin by defining four increasing recursive functions $c(x)$, $d(x)$, $p(x)$ and $q(x)$ with the following properties:

(1) The pairs $(p(x), q(x))$ and $(c(x), d(x))$ are strictly increasing pairs.

(2) $(\forall x)[Df(p(x), q(x)) > 0$ and $Df(c(x), d(x)) > 0]$.

(3) $(\forall x)[c(x) > p(x) \,\&\, q(x) > d(x)]$.

(4) The functions $c(x)$ and $q(x)$ are strictly increasing. There are three subcases in our defining procedure.

Subcase α. For some number m there are infinitely many numbers y such that $Df(m, y) > 0$. Since Df is not eventually zero by assumption, there exist pairs of numbers $(\bar{x}, \bar{y})$ such that $Df(\bar{x}, \bar{y}) > 0$ and $\bar{x} > m$. Let

$$a_0 = (\mu y)[Df(k(y), l(y)) > 0 \,\&\, k(y) > m].$$

Define $c(0) = k(a_0)$, $d(0) = l(a_0)$. Let $p(0) = m$ and define

$$q(0) = (\mu y)[y > d(0) \,\&\, Df(m, y) > 0] .$$

Clearly, $c(0)$, $d(0)$, $p(0)$, $q(0)$ satisfy (1)–(4). Suppose that for $n \leqq i$, $c(n)$, $d(n)$, $p(n)$ and $q(n)$ are defined and satisfy (1)–(4). Since Df is not eventually zero, there exist pairs $(\bar{x}, \bar{y})$ such that $\bar{x} > c(i)$ & $\bar{y} > q(i)$ and $Df(\bar{x}, \bar{y}) > 0$. Let

$$a_{i+1} = (\mu y)[Df(k(y), l(y)) > 0 \;\&\; k(y) > c(i) \;\&\; l(y) > q(i)].$$

Define $c(i+1) = k(a_{i+1})$ and $d(i+1) = l(a_{i+1})$. Note that $c(i+1) > c(i)$ and $d(i+1) > q(i) > d(i)$. Define $p(i+1) = m$ and $q(i+1) = (\mu y)[y > d(i+1) \;\&\; Df(m, y) > 0]$. Note that $q(i+1) > d(i+1) > q(i)$ and $p(i+1) = m < c(0) < c(i+1)$. This completes our inductive definition of c, d, p and q. Each function was effectively defined and is total. Hence each function is recursive. Each function was constructed as to satisfy (1)–(3). This completes Subcase α.

Subcase β. For some number m there are infinitely many x such that $Df(x, m) > 0$. This is similar to Subcase α.

Subcase γ. For each number m there are only finitely many x such that $Df(x, m) > 0$ and only finitely many y such that $Df(m, y) > 0$. We shall need the following lemma, whose proof is left to the reader.

LEMMA 1. *Let $f(x, y)$ be a recursive function such that $Df \geqq 0$, Df is not eventually zero, and*

$$(\gamma') \quad \begin{aligned} &(\forall x_1)(\forall x_2)(\forall y_1)(\forall y_2)[[Df(x_1, y_1) > 0 \;\&\; Df(x_2, y_2) > 0] \\ &\qquad \Rightarrow [(x_1, y_1) \geqq (x_2, y_2) \vee (x_2, y_2) \leqq (x_1, y_1)]]. \end{aligned}$$

Then there exists a strictly increasing pair $(g(n), h(n))$ of recursive functions such that for $x, y \in \epsilon$,

$$Df(x, y) > 0 \Longleftrightarrow (\exists n)[x = h(n) \;\&\; y = g(n)].$$

As before, we will define c, d, p and q by induction in such a manner that at each stage the conditions (1), (2), (3) and (4) are satisfied.

Since condition (iii) of the theorem cannot hold, (γ') cannot hold by Lemma 1. Thus there must be numbers x_1, y_1, x_2, y_2 such that $Df(x_1, y_1) > 0$ and $Df(x_2, y_2) > 0$ while $x_1 > x_2$ and $y_2 > y_1$. Define

$$\begin{aligned} a_0 = (\mu y)[&kk(y) > kl(y) \;\&\; ll(y) > lk(y) \\ &\& \; Df(kk(y), lk(y)) > 0 \;\&\; Df(kl(y), ll(y)) > 0], \end{aligned}$$

$$c(0) = kk(a_0), \qquad d(0) = lk(a_0),$$

$$p(0) = kl(a_0), \qquad q(0) = ll(a_0).$$

Suppose that $c(n)$, $d(n)$, $p(n)$ and $q(n)$ are defined for $n \leqq i$ and satisfy (1)–(4). Let $G = \{(x, y) \mid x > c(i) \;\&\; y > q(i)\}$. By the assumption of Subcase γ, $Df(x, y) = 0$ for all but finitely many pairs (x, y) in $\epsilon^2 - G$. Thus there cannot be a strictly increasing pair $(g(n), h(n))$ of recursive functions such that for $(x, y) \in G$,

$$Df(x, y) > 0 \Longleftrightarrow (\exists n)[x = g(n) \;\&\; y = h(n)].$$

By Lemma 1 there must be number pairs (x_1, y_1) and (x_2, y_2) in G such that $[x_1 > x_2 \;\&\; y_1 < y_2]$ and $Df(x_1, y_1) > 0$ and $Df(x_2, y_2) > 0$. Clearly, $x_1 > c(i)$, $x_2 > p(i)$, $y_1 > d(i)$ and $y_2 > q(i)$, since (x_1, y_1) and (x_2, y_2) are members of G. Define

$$\begin{aligned} a_{i+1} = (\mu y)[&Df(kk(y), lk(y)) > 0 \;\&\; Df(kl(y), ll(y)) > 0 \\ &\& \; kk(y) > kl(y) \;\&\; lk(y) < ll(y) \\ &\& \; kk(y) > c(i) \;\&\; lk(y) > d(i) \\ &\qquad\qquad \& \; kl(y) > c(i) \;\&\; ll(y) > d(i)]. \end{aligned}$$

Define

$$c(i+1) = kk(a_{i+1}), \qquad d(i+1) = lk(a_{i+1}),$$
$$p(i+1) = kl(a_{i+1}), \qquad q(i+1) = ll(a_{i+1}).$$

This completes the definition of c, d, p, q. As before, it is immediate from the definition that c, d, p, and q are recursive and that (1)–(4) are satisfied. We have now shown that in any case we can define the four functions $c(x)$, $d(x)$, $p(x)$ and $q(x)$ with the stated properties.

Let t_n and u_k be two retraceable functions with immune ranges. Let range $t_n \in T + 1$ and range $u_n \in U + 1$ and represent $f_\Lambda(T, U)$ as the RET of

$$\sigma = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{\infty} j_3[t_n, u_k, \nu(Df(n, k))].$$

Let

$$\alpha(t, u) = \{j_3(t_{c(n)}, u_{d(n)}, 0) \mid n \in \epsilon\},$$
$$\beta(t, u) = \{j_3(t_{p(n)}, u_{q(n)}, 0) \mid n \in \epsilon\}.$$

It is clear that
(i) $\alpha(t, u) \mid \beta(t, u)$,
(ii) $\alpha(t, u) \cup \beta(t, u) \subset \sigma$,
(iii) $\alpha(t, u) \cup \beta(t, u) \mid \sigma - (\alpha(t, u) \cup \beta(t, u))$.

Hence, $\operatorname{Req}[\alpha(t u) \cap \beta(t, u)] = \operatorname{Req}[\alpha(t, u)] + \operatorname{Req}[\beta(t, u)] \leqq f_\Lambda(T, U)$. In order to show that $f_\Lambda(T, U)$ need not be regressive, we need only produce retraceable sets t_n and u_k such that

$$(*) \qquad \mathrm{Req}\,[\alpha(t,u) \cup \beta(t,u)] \notin \Lambda_R.$$

We note that both $\alpha(t,u)$ and $\beta(t,u)$ are regressive if t_n and u_k are regressive. Hence we may appeal to PR 2 and prove the existence of t_n and u_k satisfying $(*)$ by producing retraceable functions t_n and u_k such that $\alpha(t,u) \overset{*}{\vee} \beta(t,u)$ fails. We do this as follows.

Let $\overline{p_i}(x)$ be a function of two variables such that a function of one variable is partial recursive if and only if it appears in the sequence $\overline{p_0}(x), \overline{p_1}(x), \cdots$. Let θ_n be an infinite sequence of sets such that a set is infinite and recursive if and only if it occurs in the infinite sequence $\theta_0, \theta_1, \cdots$. For every number n, let the principal function of θ_n be denoted by $e_n(x)$. We will simultaneously define the functions t and u by induction in the stages indicated below. (The reader will note that we make use of the fact that c and q are strictly increasing in this display.)

(0) (a)	(0) (b)	(1) (a)	(1) (b)	
$t_0, \cdots, t_{c(0)-1}$	$t_{c(0)}$	$t_{c(0)+1}, \cdots, t_{c(1)-1}$	$t_{c(1)}$	$\cdots$
$u_0, \cdots, u_{q(0)-1}$	$u_{q(0)}$	$u_{q(0)+1}, \cdots, u_{q(1)-1}$	$u_{q(1)}$	

We first observe that $c(0) > p(0) \geqq 0$ and $q(0) > d(0) \geqq 0$. Hence Part (a) of Stage (0) cannot be empty, and $u_{d(0)}$ and $t_{p(0)}$ are defined during Part (a) of Stage (0).

Stage (0). *Part* (a). Let $t_0 = u_0 = 1$. Define

$$t_{i+1} = j(t_i, 0), \qquad 0 \leqq i < c(0) - 1,$$
$$u_{k+1} = j(u_k, 0), \qquad 0 \leqq k < q(0) - 1.$$

Part (b). Let

$$\alpha = \epsilon - \{le_0(c(0))\},$$
$$\beta = \epsilon - \{le_0(q(0))\}.$$

We define

$$t_{c(0)} = j(t_{c(0)-1}, s_0),$$
$$u_{q(0)} = j(u_{q(0)-1}, v_0),$$

where $s_0 \in \alpha$ and $v_0 \in \beta$ are chosen so that

$$\overline{p_0} j_3(t_{c(0)}, u_{d(0)}, 0) \neq j_3(t_{p(0)}, u_{q(0)}, 0)$$

and

$$\overline{p_0}j_3(t_{p(0)}, u_{q(0)}, 0) \neq j_3(t_{c(0)}, u_{q(0)}, 0).$$

We can show that such an s_0 and v_0 can be chosen as follows:

NOTATION. For $s \in \alpha$, $j(t_{c(0)-1}, s) = t^s_{c(0)}$; for $v \in \beta$, $j(u_{q(0)-1}, v) = u^v_{q(0)}$.

Case 1. Suppose there is a $v \in \beta$ such that for all but finitely many $s \in \alpha$ we have

$$\overline{p_0}j_3(t^s_{c(0)}, u_{d(0)}, 0) = j_3(t_{p(0)}, u^v_{q(0)}, 0).$$

Then for all but finitely many $s \in \alpha$, if $\bar{v} \neq v$, $\bar{v} \in \beta$, we have

$$(*) \qquad \overline{p_0}j_3(t_{c(0)}, u_{d(0)}, 0) \neq j_3(t_{p(0)}, u^{\bar{v}}_{q(0)}, 0).$$

Let v_0 be the smallest such $\bar{v}$. If $\overline{p_0}j_3(t_{p(0)}, u^{v_0}_{q(0)}, 0)$ is undefined, any one of the infinitely many s in α which satisfy $(*)$ will serve as s_0. If $\overline{p_0}j_3(t_{p(0)}, u^{v_0}_{q(0)}, 0)$ is defined, there are still infinitely many numbers s belonging to α and satisfying $(*)$ from which we can choose s_0 so that

$$j_3(t^{s_0}_{c(0)}, u_{d(0)}, 0) \neq \overline{p_0}j_3(t_{p(0)}, u^{v_0}_{q(0)}, 0).$$

Case 2. Suppose Case 1 does not hold. Let v_0 be the least element of β. Then for infinitely many $s \in \alpha$ we have

$$(**) \qquad \overline{p_0}j_3(t^s_{c(0)}, u_{d(0)}, 0) \neq j_3(t_{p(0)}, u^{v_0}_{q(0)}, 0).$$

Then we have infinitely many $s \in \alpha$, s satisfying $(**)$, from which to choose s_0 so that

$$j_3(t^{s_0}_{c(0)}, u_{d(0)}, 0) \neq \overline{p_0}j_3(t_{p(0)}, u^{v_0}_{q(0)}, 0).$$

We note that t and u are strictly increasing. Consequently since $s_0 \in \alpha$ and $v_0 \in \beta$, $e_0c(0) \neq t_{c(0)}$ and $e_0q(0) \neq u_{q(0)}$.

Stage $(i+1)$. Suppose that $t_0, \cdots, t_{c(i)}$ and $u_0, \cdots, u_{q(i)}$ have all been defined so that

$$(\mathrm{A}_i) \qquad \begin{aligned} &t_0 < t_1 < \cdots < t_{c(i)}, \\ &u_0 < u_1 < \cdots < u_{q(i)}; \end{aligned}$$

$$(\mathrm{B}_i) \qquad \begin{aligned} &t_{c(n)} \neq e_n c(n), \qquad n \leqq i, \\ &u_{q(n)} \neq e_n q(n), \qquad n \leqq i; \end{aligned}$$

$$(\mathrm{C}_i) \qquad \begin{aligned} &\overline{p_n}j_3(t_{c(n)}, u_{d(n)}, 0) \neq j_3(t_{p(n)}, u_{q(n)}, 0), \qquad n \leqq i, \\ &\overline{p_n}j_3(t_{p(n)}, u_{q(n)}, 0) \neq j_3(t_{c(n)}, u_{d(n)}, 0), \qquad n \leqq i. \end{aligned}$$

Part (a). Define

$$t_{n+1} = j(t_n, 0), \qquad c(i) \leqq n < c(i+1) - 1,$$
$$u_{k+1} = j(u_k, 0), \qquad q(i) \leqq k < q(i+1) - 1.$$

Part (b). Since $c(i+1) > p(i+1)$ and $d(i+1) < q(i+1)$, we see that $t_{p(i+1)}$ and $u_{d(i+1)}$ have already been defined. Let

$$\alpha_{i+1} = \epsilon - \{le_{i+1}c(i+1)\},$$
$$\beta_{i+1} = \epsilon - \{le_{i+1}q(i+1)\}.$$

We then define

$$t_{c(i+1)} = j(t_{c(i+1)-1}, s_{i+1}),$$
$$u_{q(i+1)} = j(u_{q(i+1)-1}, v_{i+1}),$$

where $s_{i+1} \in \alpha_{i+1}$ and $v_{i+1} \in \beta_{i+1}$ are chosen so that

$$\bar{p}_{i+1}j_3(t_{c(i+1)}, u_{d(i+1)}, 0) \neq j_3(t_{p(i+1)}, u_{q(i+1)}, 0),$$
$$\bar{p}_{i+1}j_3(t_{p(i+1)}, u_{q(i+1)}, 0) \neq j_3(t_{c(i+1)}, u_{d(i+1)}, 0).$$

The existence of s_{i+1} can be proved in a manner similar to that used for s_0 and v_0. This completes the definition of t_n and u_k. It follows directly from the definition that the properties (A_i), (B_i), (C_i) hold for all i.

Clearly the functions t_n and u_k are retraceable by the function $k(x)$. Since $t_{c(n)} \neq e_n c(n)$ for any $n \in \epsilon$, t_n cannot range over any recursive set θ_n. Hence the range of t_n is immune. Similarly the range of u_n is immune.

Finally we assert that it is not true that $\alpha(t, u) \overset{*}{\nabla} \beta(t, u)$. For suppose this were true. Since the two sets in question can be regressed in the respective orders

(a) $j_3(t_{c(0)}, u_{d(0)}, 0), j_3(t_{c(1)}, u_{d(1)}, 0), \cdots,$

(b) $j_3(t_{p(0)}, u_{q(0)}, 0), j_3(t_{p(1)}, u_{q(1)}, 0), \cdots,$

there must be some partial recursive function $\bar{p}_n(x)$ such that for each number:

$$\bar{p}_n j_3(t_{c(i)}, u_{d(i)}, 0) = j_3(t_{p(i)}, u_{q(i)}, 0),$$

or

$$\bar{p}_n j_3(t_{p(i)}, u_{q(i)}, 0) = j_3(t_{c(i)}, u_{d(i)}, 0).$$

The above identities must hold for $i = n$; this contradicts property (C_i) of t and u. Hence $\alpha(t, u) \overset{*}{\nabla} \beta(t, u)$ is false and $f_\Lambda(T, U) \notin \Lambda_R$ for $T + 1 = \mathrm{Req}\,(t_n)$ and $U + 1 = \mathrm{Req}\,(u_n)$. This completes the proof of Case B, and of Theorem 1.

NOTATION. By $p(x) \notin \alpha$ we mean either (a) $p(x)$ is undefined, or (b) $p(x)$ is defined and not a member of α.

By $p(\alpha) \not\subset \beta$ we mean that for some $x \in \alpha$, $p(x) \not\in \beta$.

Lemma 2. *Let $\{a_0, \cdots, a_n\}$ and $\{b_0, \cdots, b_k\}$ be finite sequences of numbers. Let $\{t_0, \cdots, t_n\}$ and $\{u_0, \cdots, u_k\}$ be finite sequences of distinct numbers. Let α and β be any two infinite sets, and suppose that for $s \in \alpha$, $v \in \beta$ we write*

$$t_{n+1}^s = j(t_n, s), \qquad u_{k+1}^v = j(u_k, v).$$

Let $p(x)$ be any partial recursive one-to-one function. Then there exist infinitely many distinct ordered pairs $(s, v) \in \alpha \times \beta$ such that

$$pj_3(t_{n+1}^s, u_{k+1}^v, 0) \not\in \bigcup_{i=0}^{k} j_3(t_{n+1}^s, u_i, \nu_{b(i)}) \cup \bigcup_{i=0}^{n} j_3(t_i, u_{k+1}^v, \nu_{a(i)}).$$

Proof. If $pj_3(t_{n+1}^s, u_{k+1}^v, 0)$ is undefined for infinitely many pairs $(s, v) \in \alpha \times \beta$, we are finished. Suppose then that $pj_3(t_{n+1}^s, u_{k+1}^v, 0)$ is defined for all but finitely many members of $\alpha \times \beta$. Assume that the lemma does not hold. Then for all but finitely many pairs $(s, v) \in \alpha \times \beta$

$$pj_3(t_{n+1}^s, u_{k+1}^v, 0) \in \bigcup_{i=0}^{k} j_3(t_{n+1}^s, u_i, \nu_{b(i)}) \cup \bigcup_{i=0}^{n} j_3(t_i, u_{k+1}^v, \nu_{a(i)}).$$

Let $\bar{v}$ be any member of β. By the above, for all but finitely many $s \in \alpha$,

$$pj_3(t_{n+1}^s, u_{k+1}^{\bar{v}}, 0) \in \bigcup_{i=0}^{k} j_3(t_{n+1}^s, u_i, \nu_{b(i)}) \cup \bigcup_{i=0}^{n} j_3(t_i, u_{k+1}^{\bar{v}}, \nu_{a(i)}).$$

Since the second set above is finite and $p(x)$ is one-to-one, for all but finitely many $s \in \alpha$,

$$pj_3(t_{n+1}^s, u_{k+1}^{\bar{v}}, 0) \in \bigcup_{i=0}^{k} j_3(t_{n+1}^s, u_i, \nu_{b(i)}).$$

From this it follows directly that if $v_0, \cdots, v_m$ are m distinct members of β, for all but finitely many $s \in \alpha$,

$$(\forall i \leqq m)\left[j_3(t^s_{n+1}, u^{v_i}_{k+1}, 0) \in \bigcup_{i=0}^{k} j_3(t^s_{n+1}, u_i, \nu_{b(i)})\right].$$

Let $m = \sum_{j=0}^k b_j$. Then by the above we see that there are infinitely many $\bar{s} \in \alpha$ such that $p(x)$ is everywhere defined on the $m+1$ element set $\bigcup_{i=0}^m j_3(t^{\bar{s}}_{n+1}, u^{v_i}_{k+1}, 0)$ and maps it one-to-one onto the m element set $\bigcup_{i=0}^k j_3(t^{\bar{s}}_{n+1}, u_i, \nu_{b(i)})$. This contradiction completes the proof.

The following theorem is the two variable analogue of the main theorem of [2]. The technique used in its proof is similar to that employed in Theorem 95 of [7].

Theorem 2. *Let $f(x, y)$ be a recursive function such that Df^- is not eventually zero. Then $f_\Lambda(X, Y)$ does not map $\Lambda_R{}^2$ into Λ_R.*

Proof. Let $f(x, y)$ be recursive and suppose that $Df^-(x, y)$ is not eventually zero. Let $p_i(x)$ be a function of two variables such that a function of one variable is partial recursive and one-to-one if and only if it occurs in the sequence $p_0(x), p_1(x), \cdots$. We shall define two retraceable functions t_i and u_k with immune ranges such that for $y \in \epsilon$,

$$(1)\qquad p_y \bigcup_{i=0}^{\infty} \bigcup_{k=0}^{\infty} j_3[t_i, u_k, \nu Df^-(i, k)] \not\subseteq \bigcup_{i=0}^{\infty} \bigcup_{k=0}^{\infty} j_3[t_i, u_k, \nu Df^+(i, k)].$$

Putting $T + 1 = \mathrm{Req}\,(t_i)$ and $U + 1 = \mathrm{Req}\,(u_k)$ we see that $\sum_{T+1, U+1} Df^+ - \sum_{T+1, U+1} Df^- \notin \Lambda$. Thus $f_\Lambda(T, U) \notin \Lambda_R$ if (1) holds. We complete the proof by defining t_i and u_k satisfying (1).

We first observe that since Df^- is not eventually zero, we can find two strictly increasing recursive functions $g(x)$ and $h(x)$ such that $g(0) > 0$, and $h(0) > 0$ and $(\forall n)[Df^-(g(n), h(n)) > 0]$. For example, let

$$a_0 = (\mu y)[k(y) > 0 \;\&\; l(y) > 0 \;\&\; Df^-(k(y), l(y)) > 0],$$

$$g(0) = k(a_0), \qquad h(0) = l(a_0).$$

Suppose that $g(0), \cdots, g(n)$ and $h(0), \cdots, h(n)$ have all been defined as desired. Let

$$a_{n+1} = (\mu y)[k(y) > g(n) \,\&\, l(y) > h(n) \,\&\, Df^{-}(k(y), l(y)) > 0],$$
$$g_{n+1} = k(a_{n+1}), \qquad h_{n+1} = l(a_{n+1}).$$

Let θ_n be an infinite sequence of sets such that a set is infinite and recursive if and only if it occurs in the sequence $\theta_0, \theta_1, \cdots$. Let $e_n(x)$ be the principal function of θ_n. We will define the functions t and u in the following stages

(0) (a)	(0) (b)	(1) (a)	(1) (b)
$t_0, \cdots, t_{g(0)-1}$	$t_{g(0)}$	$t_{g(0)+1}, \cdots, t_{g(1)-1}$	$t_{g(1)}$
$u_0, \cdots, u_{h(0)-1}$	$u_{h(0)}$	$u_{h(0)+1}, \cdots, u_{h(1)-1}$	$u_{h(1)}$

.

We will perform this construction so that at the completion of stage n the following conditions hold.

$(\mathrm{I})_n$ $t_0 < t_1 < \cdots < t_{g(n)} \,\&\, u_0 < u_1 < \cdots < u_{h(n)}$.

$(\mathrm{II})_n$ $(\forall z \leqq n)[t_{g(z)} \neq e_z g(z) \,\&\, u_{h(z)} \neq e_z h(z)]$.

$(\mathrm{III})_n$ For $z \leqq n$,

$$p_z j_3(t_{g(z)}, u_{h(z)}, 0) \notin \bigcup_{x=0}^{g(n)} \bigcup_{y=0}^{h(n)} j_3[t_x, u_y, \nu Df^{+}(x, y)].$$

Stage 0. (a) Define

$$t_0 = u_0 = 1.$$

Define

$$t_{i+1} = j(t_i, 0), \qquad 0 \leqq i < g(0) - 1,$$
$$u_{k+1} = j(u_k, 0), \qquad 0 \leqq k < h(0) - 1.$$

(b) Let $\alpha_0 = \epsilon - \{le_0 g(0)\}$, $\beta_0 = \epsilon - \{le_0 h(0)\}$.
Then by Lemma 2, there exist infinitely many pairs (s, v) in $\alpha_0 \times \beta_0$ such that

$$(2) \quad \begin{aligned} p_0 j_3(t^{s}_{g(0)}, u^{v}_{h(0)}, 0) \notin & \bigcup_{i=0}^{g(0)-1} j_3[t_i, u^{v}_{h(0)}, \nu Df^{+}(i, h(0))] \\ & \cup \bigcup_{k=0}^{h(0)-1} j_3[t^{s}_{g(0)}, u_k, \nu Df^{+}(g(0), k)]. \end{aligned}$$

Since $p_0(x)$ is one-to-one, there are infinitely many pairs (s, v) in $\alpha_0 \times \beta_0$ which satisfy (2) and also have the property

$$(3) \qquad p_0 j_3(t^s_{g(0)},\ u^v_{h(0)},\ 0) \notin \bigcup_{k=0}^{h(0)-1} \bigcup_{i=0}^{g(0)-1} j_3[t_i, u_k, \nu Df^+(i,k)].$$

Let (s_0, v_0) be that member (s, v) of $\alpha_0 \times \beta_0$ which satisfies (2) and (3), and for which $j(s, v)$ is minimal. Define $t_{g(0)} = t^{s_0}_{g(0)}$, $u_{h(0)} = u^{v_0}_{h(0)}$. It is clear that $(\mathrm{I})_0$ holds. Since (s_0, v_0) was chosen from $\alpha_0 \times \beta_0$, $(\mathrm{II})_0$ must hold. Finally, combining the facts that (s_0, v_0) satisfies (2) and (3), and that $Df^+(g(0), h(0)) = 0$, we see that $(\mathrm{III})_0$ holds.

Stage $(i+1)$. Assume that $t_0, \cdots, t_{g(i)}, u_0, \cdots, u_{h(i)}$ have been defined and that $(\mathrm{I})_i, (\mathrm{II})_i, (\mathrm{III})_i$ hold. For $n \leqq i$, we let

$$\begin{aligned} m_n &= lk_1 p_n j_3(t_{g(n)}, u_{h(n)}, 0), && p_n j_3(t_{g(n)}, u_{h(n)}, 0) \text{ defined,} \\ &= 0, && p_n j_3(t_{g(n)}, u_{h(n)}, 0) \text{ undefined,} \\ w_n &= lk_2 p_n j_3(t_{g(n)}, u_{h(n)}, 0), && p_n j_3(t_{g(n)}, u_{h(n)}, 0) \text{ defined,} \\ &= 0, && p_n j_3(t_{g(n)}, u_{h(n)}, 0) \text{ undefined,} \end{aligned}$$

$$m_i^* = \max_{n \leqq i}(m_n), \qquad w_i^* = \max_{n \leqq i}(w_n).$$

(a) Define

$$\begin{aligned} t_{g(i)+k+1} &= j(t_{g(i)+k}, m_i^* + 1), && 0 \leqq k < g(i+1) - g(i) - 1, \\ u_{h(i)+k+1} &= j(u_{h(i)+k}, w_i^* + 1), && 0 \leqq k < h(i+1) - h(i) - 1. \end{aligned}$$

(b) Let

$$\alpha_{i+1} = \{x \mid x > \max[m_i^*, le_{i+1}g(i+1)]\},$$
$$\beta_{i+1} = \{x \mid x > \max[w_i^*, le_{i+1}h(i+1)]\}.$$

Proceeding as in (b) of Stage (0), we can use Lemma 1 and the one-to-one-ness of $p_{i+1}(x)$ to select a pair (s_{i+1}, v_{i+1}) in $\alpha_{i+1} \times \beta_{i+1}$ such that

$$(4) \qquad p_{i+1} j_3(t^{s_{i+1}}_{g(i+1)}, u^{v_{i+1}}_{h(i+1)}, 0) \notin \bigcup_{x=0}^{g(i+1)} \bigcup_{y=0}^{h(i+1)} j_3[t_x, u_y, \nu Df^+(x,y)].$$

Let $t_{g(i+1)} = t^{s_{i+1}}_{g(i+1)}$ and $u_{h(i+1)} = u^{v_{i+1}}_{h(i+1)}$. It is clear that $(\mathrm{I})_{i+1}$ holds. The choice of (s_{i+1}, v_{i+1}) from $\alpha_{i+1} \times \beta_{i+1}$ immediately yields that $t_{g(i+1)} \neq e_{i+1}g(i+1)$ and $u_{h(i+1)} \neq e_{i+1}h(i+1)$. Combining this result with our inductive assumption of $(\mathrm{II})_i$, we obtain $(\mathrm{II})_{i+1}$.

$(\mathrm{III})_{i+1}$: Let $0 < k \leqq g(i+1) - g(i)$, $0 \leqq y \leqq h(i+1)$. Suppose $\nu Df^+(g(i)+k, y) \neq 0$ and let $z \in Df^+(g(i)+k, y)$. By definition of $t_{g(i+k)}$, we have

$$lk_1 j_3(t_{g(i+k)}, u_y, z) \geqq m_i^* + 1 > m_i^* \\ \geqq lk_1 p_n j_3(t_{g(n)}, u_{h(n)}, 0)$$

if $n \leqq i$ and $p_n j_3(t_{g(n)}, u_{h(n)}, 0)$ is defined. Thus for $n \leqq i$,

$$(5)\quad p_n j_3(t_{g(n)},\ u_{h(n)}, 0) \notin \bigcup_{k=1}^{g(i+1)} \bigcup_{y=0}^{h(i+1)} j_3[t_{g(i)+k}, u_y, \nu Df^+(g(i)+k, y)].$$

Similarly, for $n \leqq i$,

$$(6)\quad p_n j_3(t_{g(n)}, u_{h(n)}, 0) \notin \bigcup_{k=1}^{h(i+1)-h(i)} \bigcup_{x=0}^{g(i+1)} j_3[t_x, u_{h(i)+k}, \nu Df^+(x, h(i)+k)].$$

Combining (4), (5), (6) and our inductive assumption of $(\mathrm{III})_i$, we obtain $(\mathrm{III})_{i+1}$. This completes the definition of t_i and u_k, satisfying $(\mathrm{I})_i$, $(\mathrm{II})_i$ and $(\mathrm{III})_i$ for all i.

It is clear that t_i and u_k are retraceable functions. However, neither can be recursive. For the assumption that range $(t_i) = \theta_m$ (for some index m in our enumeration of all infinite recursive sets) leads to the conclusion that $t_{g(m)} = e_m g(m)$, contrary to $(\mathrm{II})_m$. Hence range (t_i) is immune. Similarly range (u_k) is immune.

We now show that t_i and u_k satisfy (1). Suppose the contrary. Let $p_m(x)$ be a partial recursive function for which the inclusion denied by (1) holds. Since

$$\bigcup_{n=0}^{\infty} \{j_3(t_{g(n)}, u_{h(n)}, 0)\} \subset \bigcup_{i=0}^{\infty} \bigcup_{k=0}^{\infty} \{j_3[t_i, u_k, \nu Df^-(i, k)]\}$$

we obtain

$$p_m \left[\bigcup_{n=0} \{j_3(t_{g(n)}, u_{h(n)}, 0)\}\right] \subset \bigcup_{i=0}^{\infty} \bigcup_{k=0}^{\infty} \{j_3[t_i, u_k, \nu Df^+(i, k)]\}.$$

In particular, we would have $p_m j_3(t_{g(m)}, u_{h(m)}, 0) = j_3(t_{\bar{i}}, u_{\bar{k}}, z)$ where $(\bar{i}, \bar{k}) \in \epsilon^2$ and $0 \leqq z < Df^+(\bar{i}, \bar{k})$. Since $g(x)$ and $h(x)$ are strictly increasing functions, there is a number s, $s > m$, such that $g(s) > \bar{i}$ and $h(s) > \bar{k}$. But then using $(\mathrm{III})_s$ we obtain

$$p_m j_3(t_{g(m)}, u_{h(m)}, 0) \notin \bigcup_{i=0}^{g(s)} \bigcup_{k=0}^{h(s)} \{j_3[t_i, u_k, \nu Df^+(i, k)]\}$$

where $j_3(t_{\bar{i}}, u_{\bar{k}}, z)$ belongs to the second set above. This contradiction proves (1) and completes the proof of Theorem 2.

3. The main theorem.

PROPOSITION 1. *The function* $\min_\Lambda(x, y)$ *maps* Λ^2 *into* Λ_R.

PROOF. This is immediate from Theorem 1, since

$$\begin{aligned} \mathrm{Dmin}\,(x, y) &= 0; \quad && x = 0 \text{ or } y = 0, \\ &= 1; && x, y > 0 \text{ and } x = y, \\ &= 0; && x, y > 0 \text{ and } x \neq y. \end{aligned}$$

Let $\mathrm{Min}\,(T, U)$ denote the minimum of two regressive isols as defined by Dekker in [6]. It is immediate from the equation $\min_\Lambda(T, U) = \sum_{T+1, U+1} \mathrm{Dmin}\,(x, y)$ that $\min_\Lambda(T, U) = \mathrm{Min}\,(T, U)$ for $T, U \in \Lambda_R$. This fact and PR 1 were first observed by J. Barback.

We leave to the reader the proof of the following simple proposition.

PROPOSITION 2.

(a) $(\forall x)(\forall y)[x > m \Longrightarrow f(x, y) = f(m, y)] \Longleftrightarrow (\forall x)(\forall y)[x > m \Longrightarrow Df(x, y) = 0]$,

(b) $(\forall x)(\forall y)[y > m \Longrightarrow f(x, y) = f(x, m)] \Longleftrightarrow (\forall x)(\forall y)[y > m \Longrightarrow Df(x, y) = 0]$.

PROPOSITION 3. *Let* $f(x, y)$ *be a recursive function of two variables such that* $Df \geqq 0$. *Suppose that there is a strictly increasing pair of recursive functions* $(g(n), h(n))$ *such that for all* n, $Df(g(n), h(n)) > 0$ *and for all but finitely many pairs* (x, y) *which are not of the form* $(g(n), h(n))$, $Df(x, y) = 0$. *Then either*

(a) f *is reducible to the case of a single variable, or*

(b) *there exist eventually increasing functions* $a(x)$ *and* $b(y)$ *and a flat recursive function* $c(x, y)$ *such that for* $x, y \in \epsilon$, $f(x, y) = \min\,(a(x), b(y)) + c(x, y)$.

PROOF. *Case* A. $g(n)$ is a bounded function. Then

$$(\exists m)(\forall x)(\forall y)[x > m \Longrightarrow Df(x, y) = 0].$$

Thus by PR 2, for $x > m$, $f(x, y) = f(m, y)$. We recall that for $0 \leqq k \leqq m$, $f(k, y) = \sum_{i=0}^{k} \sum_{j=0}^{y} Df(i, j)$. Since $Df \geqq 0$, $f(k, y)$ is increasing, $0 \leqq k \leqq m$. Hence $f(x, y)$ is reducible to the case of a single variable.

Case B. $h(n)$ is bounded. This is similar to Case A.

Case C. Both $g(n)$ and $h(n)$ are unbounded. Let Γ be the finite collection of all ordered pairs (x, y) such that $Df(x, y) > 0$ and

$(x, y) \notin \{(g(n), h(n)) \mid n \in \epsilon\}$. Let $c(x, y) = \sum Df(p, q)$, where the summation is performed over all $(p, q) \in \Gamma$ for which $p \leqq x$ and $q \leqq y$. The function $c(x, y)$ is clearly flat. Furthermore, for $x \in \epsilon$,

$$f(x, y) = \left(\sum_{g(n) \leqq y \& h(n) \leqq y} Df(g(n), h(n)) \right) + c(x, y).$$

To complete Case C we need only define eventually increasing recursive functions $a(x)$ and $b(y)$ such that

$$(*) \qquad \sum_{g(n) \leqq x, h(n) \leqq y} Df(g(n), h(n)) = \min(a(x), b(y)).$$

Define increasing resursive functions $\bar{g}$, $\bar{h}$, a and b by

$$\begin{aligned}
\bar{g}(n) &= (\mu y)[g(y) \geqq n], \\
\bar{h}(n) &= (\mu y)[h(y) \geqq n], \\
a(x) &= 0, && \text{if } \bar{g}(x+1) = 0, \\
&= \sum_{n < \bar{g}(x+1)} Df(g(n), h(n)), && \text{if } \bar{g}(x+1) > 0, \\
b(y) &= 0, && \text{if } \bar{h}(y+1) = 0, \\
&= \sum_{n < \bar{h}(y+1)} Df(g(n), h(n)), && \text{if } \bar{h}(y+1) > 0.
\end{aligned}$$

(That the functions $\bar{g}$, $\bar{h}$, a, b are total follows from the hypothesis of Case C.) We now prove that $a(x)$ and $b(y)$ satisfy $(*)$. We note first that $g(n) \leqq x \Longleftrightarrow n < \bar{g}(x+1)$ and $n < \bar{h}(y+1) \Longleftrightarrow h(n) \leqq y$. Hence

$$\begin{aligned}
\sum_{g(n) \leqq x \& h(n) \leqq y} Df(g(n), h(n)) &= \sum_{n < \bar{g}(x+1) \& n < \bar{h}(y+1)} Df(g(n), h(n)) \\
&= \min\left(\sum_{n < \bar{g}(x+1)} Df(g(n), h(n)), \sum_{n < \bar{h}(y+1)} Df(g(n), h(n)) \right) \\
&= \min(a(x), b(y)).
\end{aligned}$$

This proves $(*)$, and completes the proof of PR 3.

We note that PR 3 tells us that if condition (iii) of Theorem 1 holds for a recursive function $f(x, y)$ with $Df \geqq 0$, then f satisfies either (a) or (b). We have already seen (in PR 2) that functions $f(x, y)$ satisfying (i) or (ii) of Theorem 1 with $Df \geqq 0$ are reducible to the case of a single variable. Thus we obtain

Proposition 4. *Let $f(x, y)$ be a recursive function such that $Df \geqq 0$. If Λ_R is closed under f, then either*

(a) *f is reducible to the case of a single variable, or*

(b) *there exist eventually increasing recursive functions* $a(x)$ *and* $b(y)$ *and a flat recursive function* $c(x, y)$ *such that for* $x, y \in \epsilon$, $f(x, y) = \min(a(x), b(y)) + c(x, y)$.

The next two propositions show that the converse of PR 4 holds even if the condition "$Df \geqq 0$" is removed.

PROPOSITION 5. *Let* $f(x, y)$ *be a recursive function which is reducible to the case of a single variable. Then* Λ_R *is closed under* f.

PROOF. Suppose that $f(x, y) = f(m, y)$ for $x > m$ and $f(i, y)$ is an eventually increasing function of y for $0 \leqq i \leqq m$. Let $T, U \in \Lambda_R$.

Case 1. $T \leqq m$. Let $f(T, y) = g(y)$. Then $f_\Lambda(T, U) = g_\Lambda(U) \in \Lambda_R$.

Case 2. $T > m$. Note that $f(x + m, y) = f(m, y)$ for all x, y. Thus $f_\Lambda(T, U) = f_\Lambda(T - m + m, U) = f_\Lambda(m, U) \in \Lambda_R$. A similar proof applies if the roles of x and y are reversed.

PROPOSITION 6. *Let* $a(x)$ *and* $b(y)$ *be eventually increasing recursive functions and* $c(x, y)$ *a flat recursive function. Then* Λ_R *is closed under* $\min(a(x), b(y)) + c(x, y)$.

PROOF. Let $T, U \in \Lambda_R$. Then $c_\Lambda(T, U)$ is finite and $\min_\Lambda(a_\Lambda(T), b_\Lambda(U)) \in \Lambda_R$ by Proposition 1.

We shall now proceed to state and prove our main theorem.

THEOREM 4. *Let* $f(x, y)$ *be a recursive function of two variables. Then* Λ_R *is closed under* f *if and only if there exists an integer* n *such that*:

(1) *for* $i \leqq n$, $f(i, y)$ *is an eventually increasing function of* y *and* $f(x, i)$ *is an eventually increasing function of* x, *and*

(2) $f(x + n, y + n) = m(x, y) + (c_1(x, y) - c_2(x, y))$ *where* c_1 *and* c_2 *are flat and recursive and* $m(x, y)$ *is either*:

(i) *reducible to the case of a single variable, or*

(ii) *of the form* $\min(g(x), h(y))$, *where* $g(x)$ *and* $h(y)$ *are eventually increasing functions of one variable.*

PROOF. *Sufficiency of* (1) *and* (2). We first prove

(*) Let $a(x, y) = m(x, y) + c_1(x, y) - c_2(x, y)$ be a recursive function with c_1 and c_2 flat and $m(x, y)$ satisfying (i) or (ii) above. Then Λ_R is closed under $a(x, y)$.

PROOF OF (*). *Case* A. $m(x, y)$ is reducible to the case of a single variable. Then $a(x, y)$ is also reducible to the case of a single variable, and Λ_R is closed under $a(x, y)$.

Case B. $m(x, y) = \min(g(x), h(y))$, $g(x)$ and $h(x)$ increasing and recursive. Let $T, U \in \Lambda_R$. We distinguish four subcases.

Subcase 1. T finite, U infinite. Then the function $a(T, y)$ is an eventually constant function of y. Thus $a_\Lambda(T, U) \in \Lambda_R$.

Subcase 2. T infinite, U finite. This is similar to Subcase 1.

Subcase 3. T and U are infinite and neither $g(x)$ nor $h(y)$ is eventually constant. Then $\min(g_\Lambda(T), h_\Lambda(U)) \in \Lambda_R - \epsilon$, while $c_{1\Lambda}(T, U), c_{2\Lambda}(T, U) \in \epsilon$. Thus $a_\Lambda(T, U) \in \Lambda_R - \epsilon$.

Subcase 4. T and U are infinite and at least one of g and h is eventually constant. Then $a(x, y)$ is eventually equal to some constant c, and $a_\Lambda(T, U) = c \in \epsilon$. This completes the proof of (*).

Let $f(x, y)$ satisfy (1) and (2) and $T, U \in \Lambda_R$. If $T \leqq n$, $f(T, y)$ is an eventually increasing function of y and $f_\Lambda(T, U) \in \Lambda_R$. Similarly, if $U \leqq n$, $f_\Lambda(T, U) \in \Lambda_R$. If $T \geqq n$ and $U \geqq n$, $f_\Lambda(T, U) = a_\Lambda(T - n, U - n)$ and by (*) $a_\Lambda(T - n, U - n) \in \Lambda_R$.

Necessity. Suppose that

$$T, U \in \Lambda_R \Longrightarrow f_\Lambda(T, U) \in \Lambda_R.$$

We consider the following three cases.

Case 1. There are only finitely many ordered pairs (x, y) such that $Df(x, y) \neq 0$. Then $f(x, y)$ is of the form $c_1(x, y) - c_2(x, y)$, c_1, c_2 flat recursive functions.

Case 2. $\{(x, y) \mid Df^+(x, y) > 0\}$ is infinite, but $\{(x, y) \mid Df^-(x, y) > 0\}$ is finite. For $x, y \in \epsilon$, define

$$c_2(x, y) = \sum_{i=0}^{x} \sum_{j=0}^{y} Df^-(i, j), \qquad m(x, y) = \sum_{i=0}^{x} \sum_{j=0}^{y} Df^+(i, j).$$

For x, $y \in \epsilon$, $f(x, y) = m(x, y) - c_2(x, y)$. Thus for T, $U \in \Lambda_R$, $f_\Lambda(T, U) + c_{2\Lambda}(T, U) = m_\Lambda(T, U)$. Since c_2 is flat, we see that $m_\Lambda(T, U) \in \Lambda_R$ for T, $U \in \Lambda_R$. By definition of $m(x, y)$, $Dm \geqq 0$. Thus $m(x, y)$ satisfies either (a) or (b) of Proposition 4. Combining (a) and (b) with the representation $f(x, y) = m(x, y) - c_2(x, y)$ the desired representation of f is obtained (with $n = 0$).

Case 3. $\{(x, y) \mid Df^-(x, y) > 0\}$ is infinite. Since Λ_R is closed under f, Theorem 2 yields the existence of a number n such that for $x, y \in \epsilon$, $Df^-(x + n, y + n) = 0$. Define

$$\hat{a}(i) = \sum_{j=0}^{n-1} Df(i, j), \qquad \hat{b}(j) = \sum_{i=0}^{n-1} Df(i, j),$$

$$a(i) = \hat{a}(i + n), \qquad b(j) = \hat{b}(j + n),$$

$$c = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} Df(i, j).$$

Then for $x, y \in \epsilon$,

$$(1)\quad f(x+n, y+n) = c + \sum_{i=n}^{x+n} \hat{a}(i) + \sum_{j=n}^{x+n} \hat{b}(j) + \sum_{i=n}^{x+n} \sum_{j=n}^{y+n} Df(i,j)$$

$$(2)\quad = c + \sum_{i<x+1} a(i) + \sum_{j<y+1} b(j) + \sum_{i=0}^{x} \sum_{j=0}^{y} Df^{+}(i+n, j+n).$$

Thus for $T, U \in \Lambda_R$,

$$(3)\quad f_{\Lambda}(T+n, U+n) = c + \sum_{T+1}^{*} a(i) + \sum_{U+1}^{*} b(j) + \sum_{T+1, U+1} Df^{+}(i+n, j+n).$$

We shall now prove that $a(i)$ is eventually nonnegative. We first note that for $x \in \epsilon$, $f(x, n-1) = \sum_{i=0}^{x} \hat{a}(i)$. Since $\Lambda_R{}^2$ is closed under $f(x, y)$, Λ_R is closed under $f(x, n-1)$. Hence $f(x, n-1)$ is eventually increasing and $\hat{a}(i+n) = a(i)$ is eventually nonnegative. Similarly, $b(j)$ is eventually nonnegative.

Furthermore, at least one of the two functions $a(i)$ and $b(j)$ must be eventually zero. For suppose the contrary. Let $h(x)$ be the recursive function which enumerates $\{x \mid b(x) > 0\}$ and $g(x)$ the recursive function which enumerates $\{x \mid a(x) > 0\}$. Then for $T \in \Lambda_R$,

$$\sum_{T}^{*} a(i) = \Big(\sum_{\phi_g(T)} ag(i) \Big) - \tilde{a}(T),$$

$$\sum_{T}^{*} b(j) = \Big(\sum_{\phi_h(T)} bh(j) \Big) - \tilde{b}(T),$$

where $ag(x) > 0$ and $bh(x) > 0$ for all x and $\tilde{a}$ and $\tilde{b}$ are finite for all $T \in \Lambda_R$.

Let A and B be regressive isols such that $A + B \notin \Lambda_R$. Then $h_{\Lambda}(B) \in \Lambda_R$ and $g_{\Lambda}(A) \in \Lambda_R$. However, by (3) and PR 1

$$f_{\Lambda}(g_{\Lambda}(A) + n - 1, h_{\Lambda}(B) + n - 1) \geqq \sum_{g_{\Lambda}(A)}^{*} a(i) + \sum_{h_{\Lambda}(B)}^{*} b(j)$$

$$= \sum_{A} ag(i) + \sum_{B} bh(i) - (\tilde{a}(A) + \tilde{b}(B))$$

$$\geqq A + B - (\tilde{a}(A) + \tilde{b}(B)) \notin \Lambda_R.$$

Thus Λ_R is not closed under f. This is a contradiction.

Thus at least one of the two functions $a(i)$, $b(j)$ is eventually zero. We assume that $a(i)$ is eventually zero and note that the case in which $b(j)$ is eventually zero can be treated similarly. We define

$$\begin{aligned} a^+(x) &= a(x), \quad \text{if } a(x) > 0, & a^-(x) &= 0, \quad \text{if } a(x) > 0,\\ &= 0, \quad \text{if } a(x) < 0, & &= -a(x), \quad \text{if } a(x) < 0. \end{aligned}$$

Define $b^+(x)$ and $b^-(x)$ similarly. We recall that each of the functions a^+, a^- and b^- assumes nonzero values only finitely many times. Clearly each of the functions

$$c_1(x, y) = \left(\sum_{i=0}^{x} a^+(i) \right) + c,$$

$$c_2(x, y) = \left(\sum_{i=0}^{x} a^-(i) \right) + \sum_{j=0}^{y} b^-(j)$$

is a flat recursive function. With this notation, equation (2) becomes for $x, y \in \epsilon$,

$$\begin{aligned} f(x + n, y + n) = \; & c_1(x, y) - c_2(x, y) + \sum_{j=0}^{y} b^+(j) \\ & + \sum_{i=0}^{x} \sum_{j=0}^{y} Df^+(i + n, y + n). \end{aligned} \tag{4}$$

We define recursive functions $q(i, j)$ and $h(x, y)$ by

$$\begin{aligned} q(i, j) &= b^+(j) + Df^+(n, j + n), & i &= 0,\\ &= Df^+(i + n, j + n), & i &> 0, \end{aligned}$$

$$h(x, y) = \sum_{i=0}^{x} \sum_{j=0}^{y} q(i, j).$$

Then equation (4) becomes, for $x, y \in \epsilon$,

$$f(x + n, y + n) = c_1(x, y) - c_2(x, y) + h(x, y). \tag{5}$$

Clearly (5) holds for T, $U \in \Lambda_R$. Since c_1 and c_2 are flat and Λ_R is closed under f, Λ_R is closed under h. From the definition of h and the fact that $q(i, j) \geqq 0$, we see that $Dh(x, y) \geqq 0$ for x, $y \in \epsilon$. Hence $h(x, y)$ satisfies either (a) or (b) of Proposition 4. If $h(x, y)$ satisfies (a), then (5) is the desired representation for $f(x + n, y + n)$. If $h(x, y)$ satisfies (b), then (5) becomes

$$f(x + n, y + n) = c_1(x, y) - c_2(x, y) + \min(a(x), b(y)) + c(x, y).$$

Since $c_3(x, y) = c_1(x, y) + c(x, y)$ is flat, the desired representation of $f(x + n, y + n)$ is obtained. Let $i \leqq n$. We complete the proof of Case 3 by observing that since $f(x, y)$ maps $\Lambda_R{}^2$ into Λ_R, the functions $f(x, i)$ and $f(i, y)$ map Λ_R into Λ_R and are thus eventually increasing. This completes the proof of Theorem 4.

4. Applications.

PROPOSITION A1. *Λ_R is closed under the function* $\min_\Lambda(x \dot{-} 1, y) + \min_\Lambda(x, y)$.

PROOF. This follows directly from the identity $\min (x \dot{-} 1, y) + \min (x, y) = \min (2x \dot{-} 1, 2y)$ and Theorem 4.

PROPOSITION A2. *Λ_R is not closed under the function* $q(x, y) = \min (x, y) + \min (x \dot{-} 2, y)$.

PROOF. A simple computation shows that

$$\begin{aligned} Dq(x, y) &= 0, && \text{for } x = 0 \text{ or } y = 0, \\ &= 1, && \text{for } x, y > 0 \text{ and } x = y, \\ &= 1, && \text{for } x, y > 0 \text{ and } x = y + 2, \\ &= 0, && \text{for } x, y > 0 \text{ and } x \neq y,\ x \neq y + 2. \end{aligned}$$

Thus $Dq \geqq 0$ and Theorem 1 is applicable. Since Dq clearly fails to satisfy conditions (i), (ii), (iii) of that theorem, Λ_R is not closed under q.

PROPOSITION A3. *There exist infinite regressive isols T and U such that* $\min_\Lambda(T - 2, U) \nleqq \min_\Lambda(T - 1, U)$.

PROOF. Assume the contrary. Then for $T, U \in \Lambda_R$, $\min_\Lambda(T - 2, U) + \min_\Lambda(T, U) \leqq \min_\Lambda(T - 1, U) + \min_\Lambda(T, U)$. This would imply that $\min_\Lambda(T - 2, U) + \min_\Lambda(T, U) \in \Lambda_R$ for $T, U \in \Lambda_R$, contrary to Proposition A2.

We note that by Proposition A1 we also have $\min_\Lambda(T - 2, U) + \min_\Lambda(T - 1, U) \in \Lambda_R$ for the isols T, U of Proposition A3.

PROPOSITION A5. *The function* $\max_\Lambda(X, Y)$ *does not map* $\Lambda_R{}^2$ *into* Λ_R.

PROOF. $D\max = -1$ for $x, y > 0$ and $x = y$. Thus Λ_R is not closed under $\max_\Lambda(X, Y)$ by Theorem 2.

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