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A NOTE ON INFINITE SERIES OF ISOLS

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1. **Introduction.** Let E denote the collection of all nonnegative integers (numbers), Λ the collection of all isols, and Λ_R the collection of all regressive isols. In [3], J. C. E. Dekker introduced the definition of an infinite series of isols, $\sum_T a_n$, where T denotes a regressive isol and a_n denotes a function from E into E. We recall the definition: When T is a finite regressive isol,

$$\sum_T a_n = a_0 + \cdots + a_{T-1} \qquad (0 \text{ if } T = 0).$$

When T is an infinite regressive isol,

$$\sum_T a_n = \mathrm{Req} \sum_0^\infty j(t_n, \nu(a_n)),$$

where $j(x, y)$ is a recursive function mapping E^2 one-to-one onto E, t_n is any regressive function ranging over a set in T, and for any number n, $\nu(n) = \{x \mid x < n\}$. Some properties of infinite series of isols were studied in [1] and [3]. By results in [3], $\sum_T a_n$ is an isol and is independent of the choice of the regressive function whose range is in T. In the particular case that a_n is a recursive function, then $\sum_T a_n$ is a regressive isol [1, Theorem 1]. In [2], J. Barback studied some properties of regressive isols that can be represented as infinite series $\sum_T a_n$ where the function a_n is not necessarily recursive. These regressive isols were of the form $\sum_T a_n$ where $T \leqq^* a_{n-1}$. The definition of the relation $T \leqq^* a_{n-1}$ is as follows: If T is finite, then $T \leqq^* a_{n-1}$; if T is infinite, $T \leqq^* a_{n-1}$ means that there is a regressive function t_n ranging over a set in T such that $t_n \leqq^* a_{n-1}$, that is, the mapping $t_n \to a_{n-1}$ has a partial recursive extension. The following properties are known about the $\leqq^*$ relation:

(1) If $T \leqq^* a_n$, then $t_n \leqq^* a_n$ for every regressive function t_n ranging over a set in T;

(2) if $T \leqq^* a_n$, then $T \leqq^* a_{n-1}$;

(3) if $T \leqq^* a_{n-1}$, then $\sum_T a_n$ is a regressive isol [2, Proposition 5]; and

(4) $T \leqq^* a_n$ for every recursive function a_n.

Our first two theorems below deal with the following question: Let

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T be an infinite regressive isol and a_n a function from E into E such that $T \leqq^* a_n$. By combining (2) and (3) above, we know that $\sum_T a_n$ is a regressive isol. Let S be an isol such that $S \leqq \sum_T a_n$. We know from [4] that S will be a regressive isol. Can S also be represented as an infinite sum over T? Theorem 1 shows that this is the case; Theorem 2, however, indicates that even if a_n is a recursive function, S need not have a representation as $\sum_T b_n$ for any recursive function b_n with $b_n \leqq a_n$. Theorem 3 shows that the converse of (3) above is false, i.e., $T \leqq^* a_{n-1}$ is not a necessary condition for $\sum_T a_n$ to be a regressive isol.

2. Three theorems.

THEOREM 1. *Let T be an infinite regressive isol and a_n a function from E into E such that $T \leqq^* a_n$. Let S be a (regressive) isol such that $S \leqq \sum_T a_n$. Then there exists a function b_n from E into E such that*

(a) $$T \leqq^* b_n,$$

(b) $$b_n \leqq a_n \qquad \textit{for all } n \in E,$$

(c) $$S = \sum_T b_n.$$

PROOF. Let t_n be any regressive function ranging over a set in T. Because $T \leqq {}^* a_n$, it follows from results in [2] that $\sum_T a_n$ is a regressive isol and that

(1) $$j(t_0, 0), \cdots, j(t_0, a_0 - 1), j(t_1, 0), \cdots, j(t_1, a_1 - 1), \cdots$$

represents a regressive enumeration of a set $\tau \in \sum_T a_n$. By the inequality $S \leqq \sum_T a_n$, there exists a set $\sigma \in S$ such that σ is a separated subset of τ. Let s_n denote the function ranging over σ which preserves the ordering established in (1). It follows that s_n is a regressive function. We now wish to define a function b_n in such a manner that $t_n \leqq^* b_n$. Let the number t_n be given for any $n \in E$. Because of the relation $t_n \leqq^* a_n$, the set

$$(j(t_n, 0), \cdots, j(t_n, a_n - 1)) \subset \tau$$

can be effectively generated. (Note that this set is empty in the event that $a_n = 0$.) Since σ is a separated subset of τ, the number of elements of σ which appear in this set, say k, can be effectively determined. We define $b_n = k$.

Now b_n is defined for all n, and clearly $t_n \leqq^* b_n$. Thus we have $T \leqq^* b_n$, and (a) is satisfied. Also it is clear that $b_n = k \leqq a_n$, so that

(b) also holds. Again by the results in [2], $\sum_T b_n$ is regressive and

$$j(t_0, 0), \cdots, j(t_0, b_0 - 1), j(t_1, 0), \cdots, j(t_1, b_1 - 1), \cdots$$

represents a regressive enumeration of a set $\gamma \in \sum_T b_n$. Let r_n be the regressive function determined by this enumeration of γ. We now have $S = \text{Req}\,\sigma$ and $\sum_T b_n = \text{Req}\,\gamma$, with $\sigma = \rho s_n$ and $\gamma = \rho r_n$. To complete the proof, it is necessary to show that $\gamma \simeq \sigma$. We will construct a one-to-one partial recursive function $g(x)$ such that $\gamma \subset \delta g$ and $g(\gamma) = \sigma$.

Making use of the assumption $T \leqq^* a_n$, let $f(x)$ denote the partial recursive function such that $\rho t_n \subset \delta f$ and $f(t_n) = a_n$. Also, since σ is a separated subset of τ, let σ^* and τ^* denote the disjoint recursively enumerable sets containing σ and $\tau - \sigma$, respectively. Let $k(x)$ and $l(x)$ denote the recursive functions such that $j(k(x), l(x)) = x$. Define the functions v and w by

$$\begin{aligned} v(x, i) &= j(k(x), i), \\ w(x) &= 1 \quad \text{for } x \in \sigma^*, \\ &= 0 \quad \text{for } x \in \tau^*. \end{aligned}$$

Clearly v and w are partial recursive functions. Finally, let

$$g(x) = v\left(x, (\mu y \leqq fk(x) - 1)\left[\sum_{i=0}^{y} wv(x, i) = l(x) + 1\right]\right).$$

Then $g(x)$ is a partial recursive function. Also, if $g(x) = g(z)$ for some x and z belonging to δg, then

$$\begin{aligned} g(x) = g(z) &\Rightarrow j(k(x), y_1) = j(k(z), y_2) \\ &\Rightarrow k(x) = k(z) \quad \text{and} \quad y_1 = y_2 \\ &\Rightarrow (\forall i)(v(x, i) = v(z, i)) \quad \text{and} \quad y_1 = y_2 \\ &\Rightarrow \sum_{i=0}^{y_1} wv(x, i) = \sum_{i=0}^{y_2} wv(z, i) \\ &\Rightarrow l(x) + 1 = l(z) + 1 \\ &\Rightarrow l(x) = l(z). \end{aligned}$$

Since $k(x) = k(z)$ and $l(x) = l(z)$, it follows that $x = z$ and g is one-to-one.

For $x \in \gamma$, say $x = r_n = j(t_p, q)$ with $0 \leqq q < b_p$, then $fk(x) - 1 = a_p - 1$ and $\{v(x, i) \mid 0 \leqq i \leqq a_p - 1\}$ is

$$\{j(t_p, 0), \cdots, j(t_p, a_p - 1)\} \subset \tau.$$

There are b_p elements of σ in this set and $q + 1 = l(x) + 1 \leqq b_p$. Hence $g(x)$ is defined, namely $g(x)$ is the $(q + 1)$st member from left to right in this set, as ordered above, which belongs to σ, and this is exactly s_n. Hence $g(r_n) = s_n$, that is, $\gamma \subset \delta g$ and $g(\gamma) = \sigma$. This completes the proof.

REMARK. The next theorem shows that if $T \leqq^* a_n$ is replaced in the hypothesis of Theorem 1 by the stronger condition that a_n is a recursive function, this recursive property need not carry through to the b_n function. Some preliminary ideas are needed. Let f be a one-to-one function from E into E and let T be an infinite regressive isol. Let t_n be any regressive function ranging over a set in T. We then let $\phi_f(T)$ be defined as the recursive equivalence type of the range of the function $t_{f(n)}$. M. Hassett proved in [**6**] that there exist infinite regressive isols S and T such that $S \leqq T$ while $S \neq \phi_f(T)$ for any strictly increasing recursive function f.

THEOREM 2. *There is an infinite regressive isol T, a recursive function a_n, and a regressive isol S such that $S \leqq \sum_T a_n$ and for no recursive function b_n, with $b_n \leqq a_n$ for all n, is it the case that $S = \sum_T b_n$.*

PROOF. Let S and T be infinite regressive isols such that $S \leqq T$ and $S \neq \phi_f(T)$ for any strictly increasing recursive function f. Let a_n denote the recursive function identically equal to 1. Then $T = \sum_T a_n$, and hence $S \leqq \sum_T a_n$. Let us assume that there is a recursive function b_n having the properties

$$b_n \leqq a_n \quad \text{for all } n \in E, \tag{1}$$

and

$$S = \sum_T b_n. \tag{2}$$

It follows from (1) and the definition of a_n that $b_n \leqq 1$ for all $n \in E$. However since S is infinite, (2) implies that $\{n \mid b_n = 1\}$ is an infinite set. Let $f(x)$ be the strictly increasing recursive function ranging over this set. Then

$$S = \sum_T b_n = \operatorname{Req}\,(j(t_{f(0)}, 0), j(t_{f(1)}, 0), \cdots) = \phi_f(T).$$

Therefore $S = \phi_f(T)$ for a strictly increasing recursive function f, and this is a contradiction. This completes the proof.

REMARK. The next result illustrates that for T an infinite regressive isol, the converse of the theorem

$$T \leqq^* a_{n-1} \Rightarrow \sum_T a_n \in \Lambda_R$$

does not hold.

THEOREM 3. *There exist an infinite regressive isol T and a function a_n from E into E such that $\sum_T a_n \in \Lambda_R$ but $T \nleqq^* a_{n-1}$.*

PROOF. Let $\tilde{t}_n$ be any regressive function ranging over an isolated set which does not include zero. We define a function t_n from E into E by the enumeration

$$3\tilde{t}_0, 3\tilde{t}_0 + 1, 3\tilde{t}_0 + 2, 3\tilde{t}_1, 3\tilde{t}_1 + 1, 3\tilde{t}_1 + 2, \cdots.$$

It is clear that t_n is a regressive function. Also, $\mathrm{Req}\, \rho t_n = 3\, \mathrm{Req}\, \rho \tilde{t}_n$, so that t_n ranges over an isolated set. Letting $T = \mathrm{Req}\, \rho t_n$, T is thus an infinite regressive isol. In addition, if $p(x)$ denotes the partial recursive function $p(x) = x + 1$, we see that $p(t_n) = t_{n+1}$ for $n = 3k$ or $n = 3k + 1$, $k \in E$.

We now define a function a_n from E into E by

$$\begin{aligned} a_n &= t_n && \text{if } n = 3k \text{ for some } k \in E, \\ &= t_{n+2} && \text{if } n = 3k + 1 \text{ for some } k \in E, \\ &= t_n && \text{if } n = 3k + 2 \text{ for some } k \in E. \end{aligned}$$

We know that $\sum_T a_n \in \Lambda$. We wish to show that $\sum_T a_n \in \Lambda_R$. Let the set σ be defined as $\sigma = \sum_0^\infty j(t_n, \nu(a_n))$; then $\sigma \in \sum_T a_n$. Consider the following enumeration of σ, where the ordering proceeds from left to right on the first line, then from left to right on the second line, etc.:

$$\begin{aligned} \sigma = {} & j(t_0, 0), \cdots, j(t_0, t_0 - 1), j(t_2, 0), \cdots j(t_2, t_2 - 1), j(t_1, 0), \cdots, \\ & j(t_1, t_3 - 1), j(t_3, 0), \cdots j(t_3, t_3 - 1), j(t_5, 0), \cdots, j(t_5, t_5 - 1), \\ & j(t_4, 0), \cdots, j(t_4, t_6 - 1), j(t_6, 0), \cdots. \end{aligned}$$

Let b_n be the function determined by this enumeration. Then σ will be a regressive set if we can establish that b_n is a regressive function. To see that b_n is in fact regressive, let $b_{n+1} = j(t_i, y)$ be given. If $y \neq 0$, then $b_n = j(t_i, y - 1)$. If $y = 0$, we consider the index i and determine whether $i = 3k$, $i = 3k + 1$, or $i = 3k + 2$ for some $k \in E$. If $i = 3k$, then $b_n = j(t_{i-2}, t_i - 1)$. If $i = 3k + 2$, then $b_n = j(t_{i-2}, t_{i-2} - 1)$. If $i = 3k + 1$, then $b_n = j(p(t_i), p(t_i) - 1)$. In any case, b_n can be effectively determined from b_{n+1}. Thus b_n is a regressive function and σ is a regressive set. Therefore $\sum_T a_n \in \Lambda_R$.

Suppose now that $T \leqq^* a_{n-1}$. Let $q(x)$ denote the partial recursive function such that $\rho t_n \subset \delta q$ and $q(t_n) = a_{n-1}$. We note that if $n = 3k + 2$ for some $k \in E$, then $q(t_n) = a_{n-1} = t_{n+1}$. Define a function $s(x)$ on ρt_n by

$$\begin{aligned} s(t_n) &= p(t_n) \quad \text{if } n = 3k \text{ for some } k \in E, \\ &= p(t_n) \quad \text{if } n = 3k + 1 \text{ for some } k \in E, \\ &= q(t_n) \quad \text{if } n = 3k + 2 \text{ for some } k \in E. \end{aligned}$$

Then the function $s(x)$ has a partial recursive extension mapping t_n into t_{n+1}. Hence $t_n \leqq {}^*t_{n+1}$, which is a contradiction. Thus $T \nleqq {}^* a_{n-1}$. This completes the proof.

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