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EXISTENCE OF POSITIVE AND Φ-BOUNDED HARMONIC FUNCTIONS ON RIEMANNIAN MANIFOLDS

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1. By a *Riemannian manifold* R we mean a connected, orientable C^∞-manifold of dimension $n \geqq 2$ possessing a C^∞-metric tensor. Let $\Phi(t)$ be any nonnegative real-valued function defined on $[0, \infty)$. A harmonic function u on R is said to be Φ-bounded on R if the composite function $\Phi(|u|)$ possesses a harmonic majorant on R. We denote by $H\Phi(R)$, or simply $H\Phi$, the class of all Φ-bounded harmonic functions on R and by $O_{H\Phi}$ the null class consisting of all Riemannian manifolds R on which every Φ-bounded harmonic function reduces to a constant. The problem of classifying Riemann surfaces with respect to $O_{H\Phi}$ was first attempted by Parreau [4] for the special case where Φ was increasing and convex. Later Nakai [1] completely determined $O_{H\Phi}$ for general Φ in the 2-dimensional case. Recently Ow [3] extended the Φ-bounded notion to harmonic spaces and determined $O_{H\Phi}$ there. In his paper mentioned above Nakai also considered the classification of Riemann surfaces with regular boundaries. In this investigation he partially characterized the class $SO_{H\Phi}$, where a subsurface G with regular boundary is said to belong to $SO_{H\Phi}$ if every Φ-bounded harmonic function on G which vanishes continuously on ∂G is identically zero.

The purpose of this paper is to determine the class $SO_{H\Phi}$ completely for the 2-dimensional (i.e. Riemann surface) as well as for the higher dimensional cases. An important factor in this regard is a theorem of Parreau on the existence of positive harmonic functions. In higher dimensions the class of admissible subregions will consist of smooth subregions of Riemannian n-manifolds. Here a subregion G of R will be called *smooth* if its relative boundary $\partial G \neq \emptyset$ satisfies the following: Each point $p \in \partial G$ has a neighborhood N and a diffeomorphism h of N with a region in E^n such that $h(N \cap \partial G)$ is contained in a hyperplane.

2. Before giving a characterization of $SO_{H\Phi}$ we shall first give some necessary preliminary results. The following theorem of Parreau [4],

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proven for the 2-dimensional case, is also valid in the higher dimensional case:

THEOREM 1. *If G is any smooth subregion with noncompact closure $\bar{G}$ then there exists a nonconstant positive harmonic function u on G with continuous boundary value zero on ∂G.*

A simple proof of the above theorem for Riemann surfaces has also been given by Nakai [2]. The proof of the theorem for general Riemannian manifolds will be given in §4.

3. Denote by $HP(G)$ the class of nonnegative harmonic functions on a subregion $G \subset R$. Let $G \subset R$ be smooth and $\{u_n\} \subset HP(G)$ a sequence such that $u = \lim_n u_n$ exists in G. Then we have the following counterpart of Nakai's result (cf. [5]):

LEMMA 1. *If each u_n is continuous on $G \cup \partial G$ and $u_n \mid \partial G = 0$ then u is continuous on $G \cup \partial G$ and vanishes continuously on ∂G.*

PROOF. Let $p_0 \in \partial G$ and B be an open parametric ball about p_0 chosen so that $B \cap G$ is itself a relatively compact smooth ball about p_0 with boundary $\partial(B \cap G)$ consisting of $(\partial B) \cap G$ and $(\partial G) \cap \bar{B}$. Denote by $g(p, a)$ the Green's function for $B \cap G$ with pole $a \in B \cap G$. Define a regular Borel measure μ_n on $\partial(B \cap G)$ by $d\mu_n(p) = u_n(p)dS(p)$ where $dS(p)$ denotes the surface element on $\partial(B \cap G)$. Now

$$u_n(a) = - \int_{\partial(B \cap G)} u_n(p) \frac{\partial g(p, a)}{\partial n} dS(p) \tag{1}$$

where $\partial g/\partial n$ is the normal derivative defined in terms of the Hodge star operator $*$ by $(\partial g/\partial n)\, dS = * dg$. It is well known that $- \partial g(p, a)/\partial n$ is continuous and positive, and therefore

$$\inf_{\partial(B \cap G)} - \frac{\partial g(p, a)}{\partial n} = m > 0.$$

Hence

$$u_n(a) \geqq m \int_{\partial(B \cap G)} d\mu_n(p)$$

and consequently $\mu_n(\partial(B \cap G)) \leqq u_n(a)/m$. As a result the sequence $\{\mu_n(\partial(B \cap G))\}$ is bounded and, hence, by the selection theorem there exists a regular Borel measure $\mu(p)$ such that

$$\lim_n \int_{\partial(B \cap G)} \lambda(p)\, d\mu_n(p) = \int_{\partial(B \cap G)} \lambda(p)\, d\mu(p)$$

for any real-valued continuous function λ on $\partial(B \cap G)$. In particular if $\lambda(p) = -\partial g(p, a)/\partial n$ we have from (1) that

$$u(a) = -\int_{\partial(B \cap G)} \frac{\partial g(p, a)}{\partial n} \, d\mu(p).$$

Now $u_n \mid (\partial G) \cap \bar{B} = 0$, and so $\mu_n((\partial G) \cap \bar{B}) = 0$. Consequently $\mu((\partial G) \cap \bar{B}) = 0$ and hence

$$u(a) = -\int_{(\partial B) \cap G} \frac{\partial g(p, a)}{\partial n} \, d\mu(p).$$

Finally since $\lim_{a \to \partial G} \partial g(p, a)/\partial n = 0$ it follows that $\lim_{a \to \partial G} u(a) = 0$ as asserted.

4. **Proof of Theorem 1.** Let $p_0 \in G$ be fixed and $\{p_n\}$ a sequence of points in G converging to the ideal boundary of G, i.e. the Alexandroff compactification point. Set

$$u_n(p) = g(p, p_n)/g(p_0, p_n),$$

where $p \in G - p_n$ and $g(p, p_n)$ is the Green's function for G. Since $u_n(p_0) = 1$ and $u_n > 0$ there exists a convergent subsequence, again denoted by $\{u_n\}$. Let $u = \lim_n u_n$. Then $u(p_0) = 1$ and $u > 0$ on G. By Lemma 1, u vanishes continuously on ∂G.

5. If $G \subset R$ is a smooth subregion we say that $G \in SO_{HP}$ provided that every $u \in HP(G)$ which vanishes continuously on ∂G is identically zero on G. For any smooth subregion $G \subset R$ we denote by $H_0\Phi = H_0\Phi(R, G)$ the class of harmonic functions u on G vanishing continuously on ∂G and such that $\Phi(|u|)$ possesses a harmonic majorant on G. Here Φ is an arbitrary nonnegative real-valued function defined on $[0, \infty)$. The corresponding null class $SO_{H\Phi}$ will consist of those smooth subregions G for which $H_0\Phi = \{0\}$. We observe that if Φ is not bounded in any neighborhood of $t = 0$ then $SO_{H\Phi}$ consists of all smooth subregions $G \subset R$. On the other hand suppose Φ is bounded in some neighborhood of $t = 0$. Denote by SO_{HB} the class of smooth subregions $G \subset R$ such that every bounded harmonic function which vanishes continuously on ∂G is identically zero on G. We then have

THEOREM 2. (a) *If Φ is not bounded in some finite neighborhood of $t = 0$ then $SO_{H\Phi} = SO_{HB}$.*

(b) *If Φ is bounded in any finite neighborhood of $t = 0$ then $SO_{H\Phi} = SO_{HP}$ ($SO_{H\Phi} = SO_{HB}$) provided that* $\limsup_{t \to \infty} \Phi(t)/t < \infty$ ($\limsup_{t \to \infty} \Phi(t)/t = \infty$).

Before proving Theorem 2 we state

Lemma 2. *The class SO_{HP} consists of all relatively compact smooth subregions of R.*

Proof. Clearly any relatively compact smooth subregion belongs to SO_{HP}. By Theorem 1, it follows that SO_{HP} consists precisely of relatively compact smooth subregions.

6. **Proof of Theorem 2.** We first prove part (a). By assumption $\Phi(t) \leqq c = \text{const}$ for $t \leqq t_0$. If u is a nonconstant bounded harmonic function on G vanishing continuously on ∂G, then, upon multiplication by a suitable constant c_0, we have $|c_0 u| \leqq t_0$ and hence $\Phi(|c_0 u|) \leqq c$. Therefore, $SO_{H\Phi} \subset SO_{HB}$. In order to obtain $SO_{HB} \subset SO_{H\Phi}$ we note that by assumption there exists a neighborhood $[0, t^*]$ such that $\sup_{0 \leqq t \leqq t^*} \Phi(t) = \infty$. Clearly $u < t^*$. This completes the proof of part (a).

We next prove part (b). First assume that $\limsup_{t\to\infty} \Phi(t)/t < \infty$ and Φ is bounded in any finite neighborhood of $t = 0$. It follows that there exist constants c_1 and c_2 such that $\Phi(t) \leqq c_1 + c_2 t$ on $[0, \infty)$. Therefore, every HP-function is an $H\Phi$-function. This remark together with Lemma 2 proves the first half of part (b). A proof of the remaining part can be found in [3]. This completes the proof of Theorem 2.

Summarizing these results diagramatically:

Φ unbounded in any neighborhood of $t = 0$: $SO_{H\Phi} = \{\text{all smooth subregions}\}$

general Φ

Φ unbounded in some finite neighborhood of $t = 0$: $SO_{H\Phi} = SO_{HB}$

Φ bounded in some neighborhood of $t = 0$

Φ bounded in any finite neighborhood of $t = 0$

$\limsup_{t\to\infty} \Phi(t)/t < \infty$: $SO_{H\Phi} = SO_{HP}$

$\limsup_{t\to\infty} \Phi(t)/t = \infty$: $SO_{H\Phi} = SO_{HB}$

Remark. Observe that the manifold structure of R is needed only in the proof of the inclusion $SO_{HP} \subset SO_{H\Phi}$ where Theorem 1 is used.

All other relations obtained in the diagram are therefore valid for harmonic spaces. It remains an open question whether the relation $SO_{H\Phi} = SO_{HP}$ is itself valid for harmonic spaces.

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