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A FINITE p-GROUP $P = AB$ WITH $\mathrm{Core}(A) = \mathrm{Core}(B) = \{1\}$

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Let A and B be subgroups of the finite p-group P, and let $P = AB$. We give an example of such a p-group with $\mathrm{Core}(A) = \mathrm{Core}(B) = \{1\}$. A few conditions are given which do insure the existence of a proper normal subgroup of P contained in A or B. The notations and terminology are standard.

Theorem 1. *Let $P = AB$ be a finite p-group.*

(i) *If A has subnormal defect less than or equal to* 2, *then* $\mathrm{Core}(A) \neq \{1\}$ *or* $\mathrm{Core}(B) \neq \{1\}$.

(ii) *If the order of P is less than or equal to p^5, then* $\mathrm{Core}(A) \neq \{1\}$ *or* $\mathrm{Core}(B) \neq \{1\}$.

Proof. (i) Let H be a normal subgroup of P minimal with respect to $H \neq \{1\}$ and $H = (H \cap A)(H \cap B)$. Let $A_1 = H \cap A$ and $B_1 = H \cap B$. Suppose H is contained in neither A nor B. Then by the minimality of H, $A_1{}^P = H = B_1{}^P$; that is, $H = A_1[B, A_1] = B_1[A, B_1]$. Hence $[H, A] = [A_1[B, A_1], A] = [B_1[A, B_1], A] = [B_1, A]$. Therefore $[A_1, A][B, A_1, A] = [B_1, A]$. Since A has subnormal defect less than or equal to 2, $[B, A, A] \leqq A$. Hence $[B_1, A] \leqq A$. Therefore $[H, P] = ([H, P] \cap B_1)[B_1, A] = ([H, P] \cap B) \cdot ([H, P] \cap A)$. Since $[H, P] < H$, $[H, P] = \{1\}$. Hence $H = A_1{}^P = A_1 \leqq A$.

(ii) This follows immediately from (i).

Theorem 2. *There exists a p-group $P = AB$ of order p^6 with* $\mathrm{Core}(A) = \mathrm{Core}(B) = \{1\}$.

Proof. Let V be an elementary abelian p-group with minimal generating set $\{a_1, a_2, b_1, b_2\}$. Let a_3 and b_3 be the automorphisms of V whose matrices with respect to (a_1, a_2, b_1, b_2) are

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

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respectively. Let $H = \langle a_3, b_3 \rangle$. Then direct computation shows that H is abelian of type (p, p). Let P be the relative holomorph of V by H. Then the order of P is p^6. Identify the elements of V and H with the elements of P in the usual way. Let $A = \langle a_1, a_2, a_3 \rangle$ and $B = \langle b_1, b_2, b_3 \rangle$. Then $A \cap B = \{1\}$ and $|A| = |B| = p^3$. Hence $P = AB$. Since $C_P(V) = V$, $Z(P) \leqq V$. Hence $Z(P) \cap A = Z(P) \cap A \cap V = Z(P) \cap \langle a_1, a_2 \rangle = \{1\}$. Therefore Core$(A) = \{1\}$. Similarly Core$(B) = \{1\}$.

THEOREM 3. *Let $P = AB$ be a finite p-group. The following are equivalent:*

(i) *If $N \lhd P$, then there exists a proper normal subgroup of P/N contained in AN/N or BN/N.*

(ii) *There exists a chief series $\{1\} = H_0 \lhd H_1 \lhd \cdots \lhd H_r = P$ with $H_i = (H_i \cap A)(H_i \cap B)$ for $0 \leqq i \leqq r$.*

(iii) *There exists a chief series of B, $\{1\} = B_0 \lhd B_1 \lhd \cdots \lhd B_s = B$, such that AB_i is a subgroup of P for $0 \leqq i \leqq s$.*

(iv) *There exists a chief series of A, $\{1\} = A_0 \lhd A_1 \lhd \cdots \lhd A_t = A$, such that A_iB is a subgroup of P for $0 \leqq i \leqq t$.*

PROOF. (i) implies (ii): Let $H_1 \lhd P$ be of order p and contained in A or B. The hypothesis of (i) is clearly valid for $P/H_1 = (AH_1/H_1)(BH_1/H_1)$. By induction, there exists a chief series $\{1\} = H_0 \lhd H_1 \lhd \cdots \lhd H_r = P$ with

$$\begin{aligned} H_i &= (H_i \cap AH_1)(H_i \cap BH_1) \\ &= (H_i \cap A)(H_i \cap B)H_1 \\ &= (H_i \cap A)(H_i \cap B) \quad \text{for } 0 \leqq i \leqq t. \end{aligned}$$

(ii) implies (iii): Let $\{1\} = H_0 \lhd H_1 \lhd \cdots \lhd H_t = P$ be a chief series of P with

$$H_i = (H_i \cap A)(H_i \cap B) \quad \text{for } 1 \leqq i \leqq t.$$

Let $B_i = H_i \cap B$ for all i. Then the set $\{B_i : 0 \leqq i \leqq t\}$ is the collection of terms of a chief series of B, and $AB_i = AH_i$ is a subgroup of P for $0 \leqq i \leqq t$.

(iii) implies (i): Let $N \lhd P$. Then the hypothesis of (iii) is true for $P/N = (AN/N)(BN/N)$. Hence we may suppose $N = \{1\}$. Induct on the index of A in P. Let $\{1\} \neq B_1 \leqq Z(B)$ be such that AB_1 is a subgroup of P. If $B_1 \leqq A$, then $B_1{}^P = B_1{}^A \leqq A$ and Core$(A) \neq \{1\}$. Suppose B_1 is not contained in A. Then $P = (AB_1)B$, AB_1 and B satisfy (iii), and the index of AB_1 in P is smaller than the index of A in P. We may suppose Core$(B) = \{1\}$. Hence by induction there exists $1 \neq ab \in Z(P) \cap AB_1$ where $a \in A$ and $b \in B_1$. Since $\mathrm{Core}(B) = \{1\}$, $a \neq 1$. Then $a \in C_P(B)$, $\langle a \rangle^P = \langle a \rangle^A \leqq A$, and $\mathrm{Core}(A) \neq \{1\}$.

The equivalence of (i), (ii), and (iv) results from the symmetry of the situation.

COROLLARY 3.1. *Let P be a finite p-group and A a subgroup of P. Then A is quasinormal in P if and only if for every subgroup H of P there exists a chief series of A all of whose terms permute with H.*

COROLLARY 3.2. *Let P be a finite p-group, A a quasinormal subgroup of P, and A_0 a subgroup of A which is a term in every chief series of A. Then A_0 is quasinormal in P. In particular, every subgroup of a cyclic quasinormal subgroup of P is quasinormal in P.*

REFERENCE

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