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POINTWISE COMPLETENESS OF DIFFERENTIAL-DIFFERENCE EQUATIONS

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1. **Introduction.** Let A_i, $i = 0, 1, \cdots, m$, be complex $n \times n$ matrices and let x be a complex n-dimensional column vector. Further, let $0 < \tau_1 < \tau_2 < \cdots < \tau_m$ be given real numbers. We consider the system of differential-difference equations

$$x'(t) = A_0x(t) + A_1x(t-\tau_1) + \cdots + A_mx(t-\tau_m), \qquad t \geqq 0. \tag{1}$$

Let $\mathbf{C}^n$ denote n-dimensional complex Euclidean space and let $\mathcal{B}$ denote the set of all continuous functions from $[-\tau_m, 0]$ into $\mathbf{C}^n$. If $\varphi \in \mathcal{B}$, we denote by $x(t; \varphi)$ the unique solution of (1) satisfying the initial condition

$$x(t; \varphi) = \varphi(t), \qquad -\tau_m \leqq t \leqq 0. \tag{2}$$

The system (1) is called *pointwise complete* if for any $t \geqq 0$, the set $\{x(t; \varphi) : \varphi \in \mathcal{B}\}$ equals $\mathbf{C}^n$, and *pointwise degenerate* otherwise.

In 1967, Weiss [5] posed the question whether the system

$$x'(t) = Ax(t) + Bx(t-1) \tag{3}$$

is pointwise complete for any pair of $n \times n$ matrices A and B. Since then, several people have worked on this question and several sufficient conditions for the pointwise completeness of (3) have been established. In the case $n \leqq 2$, (3) is pointwise complete for any choice of A and B (see Halanay and Yorke [3]); however, for dimension $n > 2$, pointwise degenerate systems exist as Popov [4] has recently demonstrated by showing that any solution $x(t)$ of (3), where

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix},$$

is orthogonal to the vector $(1, -2, -1)$ for $t \geqq 2$. In the same paper [4], Popov shows that (3) is pointwise complete whenever B is of the

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form $B = bc^T$, where b and c are constant column vectors (c^T is the transpose of c).

In this paper, we prove pointwise completeness of (1) in another very general situation, namely whenever the matrices A_i, $i = 0, 1, \cdots, m$, commute. We approach the problem by constructing a certain transcendental matrix equation whose solvability provides a sufficient condition for pointwise completeness. We then use Gelfand transform methods to show that this matrix equation has a solution whenever the matrices A_i commute.

Our methods have the advantage that we are also able to obtain global existence results for solutions of autonomous differential-difference equations of advanced and neutral type and further show that a concept similar to pointwise completeness holds for such equations.

2. **An auxiliary equation.** Together with (1), we consider the following matrix equation

$$X'(t) = A_0X(t) + A_1X(t - \tau_1) + \cdots + A_mX(t - \tau_m), \tag{4}$$

where $X(t)$ is an $n \times n$ matrix. Observe that $X(t)c$, c a constant vector, is a solution of (1) whenever $X(t)$ is a solution of (4).

Let M_n denote the algebra of all complex $n \times n$ matrices equipped with the operator norm. For $Y \in M_n$, we denote by e^Y the element of M_n given by

$$e^Y = \sum_{j=0}^{\infty} Y^j/j!\,.$$

If $Y \in M_n$, then $X(t) = e^{tY}$ is a solution of (4) (for all t) if and only if

$$Y = A_0 + A_1e^{-\tau_1 Y} + \cdots + A_me^{-\tau_m Y}. \tag{5}$$

If (5) has a solution Y, then, as observed above, $x(t) = e^{tY}c$ is a solution of (1) for any constant vector c, and since e^{tY} is nonsingular, we conclude that (1) is pointwise complete whenever (5) has a solution.

3. **Solution of the auxiliary equation.** In this section, we study equation (5) in case $A_iA_j = A_jA_i$, $i, j = 0, 1, \cdots, m$. For the sake of brevity, we adopt much of the notation and terminology of Browder [2].

THEOREM. *Let $A_iA_j = A_jA_i$, $i, j = 0, 1, \cdots, m$. Then there exists a solution Y of (5) and (1) is pointwise complete.*

PROOF. We verify the theorem in case $A_2 = \cdots = A_m = 0$. The general case may be proved in much the same way. Further there is no loss in generality in assuming that $\tau_1 = 1$. Equation (5) then takes the form

$$Y = A + Be^{-Y}, \tag{6}$$

where A and B commute.

Let $\mathscr{M}$ denote the closure in M_n of the algebra $\{p(A, B) : p$ is a polynomial in two indeterminates over $\mathbf{C}^1\}$. Then $\mathscr{M}$ is a commutative Banach algebra with identity, and is, moreover, generated (polynomially) by A and B. Denote by $S(A)$ and $S(B)$ the spectra of A and B, respectively, considered as elements of $\mathscr{M}$, and by spec $\mathscr{M}$ the spectrum of the algebra $\mathscr{M}$ (the set of all multiplicative linear functionals on $\mathscr{M}$). Then the mapping

$$T : \operatorname{spec} \mathscr{M} \to S(A) \times S(B) \subseteq \mathbf{C}^2,$$

defined by $T(\varphi) = (\varphi(A), \varphi(B))$, is a homeomorphism of spec $\mathscr{M}$ onto a subset R of $S(A) \times S(B)$ (see Browder [2, pp. 36–37]).

Now $S(A) = \sigma(A)$, since $\sigma(A)$ (the operator spectrum) is a finite set in $\mathbf{C}^1$. We identify spec $\mathscr{M}$ with R. With this identification, we have the Gelfand transform $\mu \to \hat{\mu}$ mapping $\mathscr{M}$ into the continuous functions on R, $C(R)$, and $\hat{A}(\alpha) = \alpha_1$, $\hat{B}(\alpha) = \alpha_2$ for $\alpha = (\alpha_1, \alpha_2) \in R$.

If $\lambda \in R$, then $\{\lambda\}$ is open and closed in R, so there exists an element $E_\lambda \in \mathscr{M}$ such that $\hat{E}_\lambda = \chi_{\{\lambda\}}$ (the characteristic function of $\{\lambda\}$), for if $\alpha = \lambda$, there exists $A_\alpha \in \mathscr{M}$ such that $\hat{A}_\alpha(\lambda) = 1$, $\hat{A}_\alpha(\alpha) = 0$; let $E_\lambda = \prod_{\alpha \neq \lambda} A_\alpha \in \mathscr{M}$. Hence, if $f \in C(R)$, we must have

$$f = \sum_{\lambda \in R} a_\lambda \chi_{\{\lambda\}} = \sum_{\lambda \in R} a_\lambda \hat{E}_\lambda = \Big(\sum_{\lambda \in R} a_\lambda E_\lambda \Big)^{\wedge}.$$

Hence $\hat{\mathscr{M}} = C(R)$.

The above now yield that we must find a function $f : R \to \mathbf{C}^1$ so that

$$f = \hat{A} + \hat{B}e^{-f} \quad \text{on } R,$$

i.e.,

$$f(\lambda) = \lambda_1 + \lambda_2 e^{-f(\lambda)}, \qquad \lambda \in R.$$

This, however, says only that for each $\lambda \in R$, we must find a $z \in \mathbf{C}^1$ such that

$$z = \lambda_1 + \lambda_2 e^{-z}, \tag{7}$$

thus reducing (6) to a quasipolynomial equation (7). Such quasipolynomial equations have been extensively studied and it is well known (see, e.g., Bellman and Cooke [1, Chapter 12]) that (7) has a solution for any choice of $(\lambda_1, \lambda_2) \in \mathbf{C}^2$.

REMARK. Our method may also be employed in the quest for global solutions of neutral type differential-difference equations.

For example, consider the neutral-type equation

$$x' = Ax + Bx(t-1) + Cx(t+1). \tag{8}$$

The transcendental matrix equation obtained in this case is

$$Y = A + Be^{-Y} + Ce^{Y}. \tag{9}$$

Again under the assumption that A, B, and C commute, we reduce (9) to the quasipolynomial scalar equation

$$z = \lambda_1 + \lambda_2 e^{-z} + \lambda_3 e^{z}. \tag{10}$$

Such equations again have been extensively studied (see [1]).

Knowing that (10) may be solved, we obtain a solution Y of (9) and hence for any constant vector c, $x(t) = e^{tY}c$ is a global solution of (8).

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