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APPLICATIONS OF GLOBAL ANALYSIS TO SPECIFIC NONLINEAR EIGENVALUE PROBLEMS[1]

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Introduction. The aim of the following four lectures is threefold in that we wish to point out:

First, just how nonlinear eigenvalue problems arise in certain specific mathematical areas and to what extent the existing theory of nonlinear eigenvalue problems is successful in treating these specific applications.

Secondly, new results in the theory of nonlinear eigenvalue problems can be obtained by means of a careful study of specific mathematical disciplines.

Finally, we shall indicate basic research areas in the theory of nonlinear eigenvalue problems which are suggested by the above special problems.

At this point some general remarks and definitions are in order. Indeed, what is meant by the term "*nonlinear eigenvalue problem*"? In these lectures this term will mean the problem of studying all the solutions of certain operator equations $F(x, \lambda) = 0$, particularly in their dependence on the parameter λ. Here $F(x, \lambda)$ is an operator (generally nonlinear) defined on an open subset U of the Banach space $X \times Z$ with range in the Banach space Y. The parameter space Z will generally be R^1 or possibly R^N. Thus if $F(x, \lambda) = \lambda I - L$, and L is a bounded linear operator of a Banach space X into itself, we are concerned with the spectral theory of L. It is the merit of the nonlinear eigenvalue problems considered here that the *totality* of solutions of a given operator equation is given prime consideration. This is crucial when the operator F depends nonlinearly on x. It will be convenient to divide nonlinear eigenvalue problems into 4 parts: (i) bifurcation theory (the study of the solutions of the equation $F(x, \lambda) = 0$ near a point (x_0, λ_0) at which $F(x_0, \lambda_0) = 0$ and the Fréchet derivative of $F(x, \lambda)$ with respect to x at (x_0, λ_0), $F'(x_0, \lambda_0)$, has a nontrivial kernel); (ii) global theory (the study of solutions (x, λ) of $F(x, \lambda) = 0$ without regard to the norm of (x, λ) or the existence of nearby approximations (x_0, λ_0) for (x, λ)); (iii) singular perturbation theory (the study of the behavior of solutions of $F(x, \lambda) = 0$ as $\lambda \to \infty$); (iv) continuation theory (the study of the relations

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between (i)—(iii) mentioned above). We shall not consider (iii) here, as we are primarily concerned with the application of global methods.

In each of the following lectures one main area of mathematical investigation is considered. We begin each topic by stating some of the problems, known results, and conjectures of the area. We also show in which sense these results are related to the theory of nonlinear eigenvalue problems. Next we attempt to prove some of them by the methods of nonlinear eigenvalue problems. Finally we summarize our discussion in accordance with the threefold aim of the above paragraph.

Perhaps a word concerning the term "global analysis" is apropos at this stage. Because of their importance and relative simplicity, we have limited ourselves to those parts of global analysis centering around the well-known topics of critical point theory and the Leray-Schauder degree. Thus such topics as the theory of nonlinear Fredholm mappings, and the generalized degree for such mappings have been omitted. For the same reasons, we have omitted those applications requiring a general discussion of Hilbert and Banach manifolds.

Before proceeding, let us briefly review some of the important results of nonlinear functional analysis that will be used.

(i) *Gradient operators.* An operator f defined on an open set U of a Banach space X with range in X^* (the dual space) is called a gradient mapping if there is a C^1 real-valued function $F(x)$ whose Fréchet derivative $F'(x)$ is equal to $f(x)$ for every $x \in U$ (see [18]). In this case the solutions of $f(x) = 0$ coincide with the critical points of $F(x)$. Now the critical points of a large class of functionals can be associated with topological invariants in various ways (e.g. type numbers of M. Morse, minimax characterization by Ljusternik-Schnirelmann), and because these invariants are topological in nature, they have important *stability* properties under suitably restricted perturbations. If the functional $F(x)$ is defined on a smooth manifold $\mathcal{M}$, these topological invariants are intimately connected with the topology of $\mathcal{M}$. In fact, if the topological structure of $\mathcal{M}$ is nontrivial, any suitably restricted functional $F(x)$ defined on $\mathcal{M}$, must have a certain number of critical points. These remarks are crucial in understanding the following results.

(1) (Ljusternik). Suppose $F(x)$ is a smooth, even, sequentially weakly continuous functional defined on a Hilbert space H. Then the equation $x = \lambda F'(x)$ has a countably infinite number of distinct solutions $(x_N(R), \lambda_N(R))$ on each sphere $\|x\| = R > 0$. The elements x_N can be characterized by the Ljusternik-Schnirelmann minimax principle, and $\lambda_N \to \infty$ as $N \to \infty$.

(2) (Berger). If in (1) above $F(x) = \frac{1}{2}(Lx, x) + o(\|x\|^2)$ as $\|x\| \to 0$, then $(x_N(R), \lambda_N(R)) \to (0, \lambda_N)$ as $R \to 0$, where λ_N is the Nth eigenvalue of the linear eigenvalue problem $x = \lambda Lx$ (ordered by magnitude and counted by multiplicity). [Note that this result implies that if $\lambda_N = \lambda_{N+1} = \cdots = \lambda_{N+p-1}$, then there are at least p distinct families $(x_{N+i}(R), \lambda_{N+i}(R)) \to (0, \lambda_N)$ as $R \to 0$ $(i = 0, 1, \cdots, p-1)$.]

(3) (Krasnoselskiĭ). If L is a compact selfadjoint linear operator defined on a Hilbert space H, then the eigenvalues $\{\lambda_N\}$ of the operator equation $x = \lambda Lx$ are stable in the sense that if $\mathscr{N}(x)$ is a smooth weakly continuous functional, every sufficiently small neighborhood of $(0, \lambda_N)$ contains a solution of $x = \lambda\{Lx + \mathscr{N}'(x)\}$ distinct from $(0, \lambda_N)$. [In this case we say $(0, \lambda_N)$ is a point of bifurcation for the equation $x = \lambda\{Lx + \mathscr{N}'(x)\}$.]

The proofs of these basic results will be found in [16], [17], [18]. Since the known proofs of (3) are long and involved, we include a sketch of a short proof of a result, more general than (3) in several respects, in Appendix A at the end of the paper.

(ii) *Compact operators.* An operator C defined on an open set U of a Banach space X with range in a Banach space Y is compact if the closure of $C(B)$ in Y is a compact set for each bounded set B in U. Now if C is a compact operator (in this sense) and $X = Y$, then the Leray-Schauder degree $d(f, p, D)$ [31] of the mapping $f = I + C$ at a point p is defined relative to any open bounded set $D \subset U$ provided that $f(x) \neq p$ on ∂D. Now the Leray-Schauder degree $d(f, p, D)$ is an *algebraic* count of the number of solutions of $f(x) = p$ in D and is a topological invariant in the sense that it is a homotopy invariant, i.e. if $f(\lambda) = I + C(\lambda)$ depends (uniformly) continuously on $\lambda \in [0, 1]$ and $f(\lambda)(x) \neq p$ on ∂D, then $d(f(\lambda), p, D) = \text{const}$ independent of $\lambda \in [0, 1]$. Now we shall make use of the following results.

(4) (Leray-Schauder). If $d(\lambda) = d(f(\lambda), p, D)$ is defined for $\lambda = 0$ and $\lambda = 1$ and $d(0) \neq d(1)$, then $f(\lambda_0)(x) = p$ has a solution on ∂D for some $\lambda_0 \in (0, 1)$.

(5) (Schwartz) [19] Suppose X is a complex Banach space. An operator f is a complex analytic map of $U \to X$ if, for all $x \in U$, $y \in X$, $f(x + ty)$ is an analytic function of the complex variable t for $|t|$ sufficiently small. Then if $d(f(\lambda), p, D)$ is defined in a small neighborhood D of a point x_0, at which $f(\lambda)(x_0) = p$ and $\operatorname{Ker} f'(\lambda)(x_0) \neq \{0\}$, then $d(f(\lambda), p, D) \geqq 2$.

Many important applications of nonlinear eigenvalue problems will not be considered here. Chief among these is the body of work centering around the Navier-Stokes equations, since other lecturers will discuss such results here.

Lastly, a word is in order concerning the formulation of nonlinear eigenvalue problems. In a large number of problems, implicit parameters and constraints are present. A basic tenet in our work is to bring these items to light *explicitly* by introducing them formally into the relevant equations defining the problem. By so doing, many diverse mathematical situations fall into the category of nonlinear eigenvalue problems.

Lecture 1. Periodicity for autonomous dynamical systems. The importance of periodic phenomena in both mathematics and nature is well known. Yet many basic questions concerning periodicity are unanswered. We mention two fundamental classes of problems for some classical dynamical systems that seem typical of this field of study:

(A) *Global problems* (i.e. problems in which there is no obvious first approximation for the desired periodic solution). For example:

If $f(0,0) = 0$, determine the structure of all periodic solutions of the system of N ordinary differential equations

$$x_{tt} + f(x, x_t) = 0, \tag{1.1}$$

including those which are not close to the stationary point $x \equiv 0$.

A variant of this problem is obtained by replacing the term x_{tt} with the second covariant derivative of x with respect to some metric structure on a manifold $\mathcal{M}$, so that one obtains

$$\frac{D}{dt}\left(\frac{dx}{dt}\right) + f(x, x_t) = 0. \tag{1.2}$$

If $f(x, x_t) \equiv 0$, this problem is equivalent to finding all closed geodesics on $\mathcal{M}$.

(B) *Local problems* (in which there is an obvious first approximation for the desired periodic solutions). For example:

Compare the periodic solutions of

$$x_{tt} + Ax + f(x, x_t) = 0 \tag{1.3}$$

with the periodic solutions of the linear system $x_{tt} + Ax = 0$. Here A is an $(N \times N)$ matrix and $f(x, y) = o(|x| + |y|)$, so that $x = 0$ is a stationary point of (1.3).

Two classes of deep and unsolved problems of type (A) and (B) are:

(C) *Problems of celestial mechanics* (in which one uses Newton's laws of gravitation to determine periodic motions of the planets and satellites). For example:

Again as in (A) and (B), determine periodic solutions of systems of the form

$$x_{tt} + \nabla U(x) = 0, \tag{1.4}$$

$$x_{tt} + B(x)x_t + \nabla U(x) = 0. \tag{1.5}$$

Here, because of Newton's laws of gravitation, $U(x)$ possesses "singularities" and one often resorts to a process called "regularization" to remove these singularities [20].

(D) *Problems for "continuous" systems.* Determine the periodic solutions of the mixed boundary value problem

$$\begin{aligned} u_{tt} &= \Delta u - f(x, u), \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{1.6}$$

Here Ω is a bounded domain in $\boldsymbol{R}^N$ and Δ denotes the Laplacian. Note that if $f(x, u) \equiv 0$, then the desired periodic solutions coincide with the so-called "normal modes" of vibration for (1.6).

REMARKS. (i) Poincaré has conjectured that the periodic solutions are dense in the set of all solutions for problems of the type (1.4) or (1.5). This conjecture emphasizes the importance of periodicity in dynamical systems. "Dense" in this context means that given any solution $x(t)$, then there is a periodic solution differing only slightly from $x(t)$ for a given length of time [24, p. 82].

(ii) Another conjecture in this direction is that in the neighborhood of a given periodic solution, there exists a countably infinite number of other distinct periodic solutions (cf. Poincaré's last geometric theorem).

All the problems (A)–(D) can be regarded as nonlinear eigenvalue problems. One observes that in each case the period of the desired periodic solutions is unknown a priori. Hence one might set $t = \lambda s$ where $2\pi\lambda$ is the unknown period, in any of the equations (1.1) to (1.6), and seek 2π-periodic solutions in s. Thus nonlinear eigenvalue problems are formed.

We shall discuss one aspect of problem (A), namely the case $f(x, x_t) = \nabla U(x)$ in equation (1.1). A classical existence theorem in this direction is due to Whittaker and Tonelli [1, pp. 387–389], but is restricted to the case $N = 2$. We shall show how the theory of nonlinear eigenvalue problems can be used to prove the following N-dimensional result.

THEOREM 1.1 [2]. *There exists a one-parameter family of distinct periodic solutions for the system*

$$\ddot{x} + \nabla U(x) = 0 \tag{1.7}$$

where $x \in \mathbf{R}^N$ *and* $U(x)$ *is a* $C^1(\mathbf{R}^N)$, *real-valued function such that*
(i) $0 = U(0) \leqq U(x)$ *for* $x \in \mathbf{R}^N$,
(ii) $U(x)$ *is convex and* $U(x) \to \infty$ *as* $|x| \to \infty$.
The family of periodic solutions is parametrized by the average of U *(the potential energy) over a period.*

The proof we give of Theorem 1.1 has the advantage that it can be easily extended to the analogous case defined by equation (1.2).

IDEA OF PROOF OF THEOREM 1.1. The obvious approach, by means of the theory of nonlinear eigenvalue problems, is to apply critical point theory by minimizing $\int_0^{2\pi} \dot{x}^2(s)\, ds$ subject to the constraint $\int_0^{2\pi} U(x(s))\, ds = \text{const} = R > 0$ (say). This approach fails however, since this minimum is zero. Now the equation (1.7) possesses "natural constraints" mentioned in the last paragraph of the introduction, and these must be introduced explicitly into the variational problem. Indeed, after reparametrization and integration over the period 2π, one finds that these constraints are $\int_0^{2\pi} \nabla U(x(s))\, ds = 0$. To show that this modified variational problem Π_R leads to a proof of Theorem 1.1, we show that if $U(x) \in C^2(\mathbf{R}^N)$ and strictly convex, then the critical points of Π_R are in (1-1) correspondence with the periodic solutions of (1.7) parametrized as indicated. For the other details we refer to [2]. The Euler-Lagrange equations of the modified problem Π_R are $\ddot{x} + \beta_0 \nabla U(x) = \nabla(\boldsymbol{\beta} \cdot \nabla U(x))$ where $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_N)$ is an N-vector of constants. Integrating over the period 2π, we find that if $H(U(x))$ denotes the Hessian matrix of $U(x)$, the quadratic form $\int_0^{2\pi} H(U(x))\boldsymbol{\beta} \cdot \boldsymbol{\beta}\, ds = 0$. Now by the hypothesis (i), $H(U(x))$ for fixed x is a nonnegative selfadjoint matrix. Thus $\nabla(\boldsymbol{\beta} \cdot \nabla U(x)) = 0$, as desired. In the same way one finds $\beta_0 > 0$. The desired parametrization of the family of periodic solutions follows by setting $t = \lambda s$ in the equation $\int_0^{2\pi} U(x(s))\, ds = R$.

A close scrutiny of the proof in [2] enables us to formulate *and prove* the following:

THEOREM 1.1′. *Suppose* L *is a selfadjoint, nonnegative operator with closed range and finite-dimensional kernel, defined on a Hilbert space* H. *Then, if* $\mathscr{N}(u)$ *is a* C^2 *weakly sequentially continuous, strictly convex functional defined on* H *such that* $\mathscr{N}(u + tv) \to \infty$ *as* $|t| \to \infty$ *(for any fixed* u, *and nonzero* $v \in H$*) and* $0 = \mathscr{N}(0) \leqq \mathscr{N}(u)$, *the operator equation* $Lu = \lambda \mathscr{N}'(u)$ *has a one-parameter family of nontrivial solutions* $(u(R), \lambda(R))$ *such that* $\mathscr{N}(u(R)) = R$.

IDEA OF PROOF OF THEOREM 1.1′. As in the idea of proof of Theorem 1.1 above, we need merely minimize (Lu, u) subject to the constraints

$\mathcal{N}(u) = R$ and $(\mathcal{N}'(u), w) = 0$ for all $w \in \operatorname{Ker} L$. Again the "Euler-Lagrange equation" is $Lu = \beta\mathcal{N}'(u) + \mathcal{N}''(u)w$, where $w \in \operatorname{Ker} L$ and $\mathcal{N}''$ is a selfadjoint nonnegative operator. Consequently, taking the inner product of this equation with $w \in \operatorname{Ker} L$ we find, as before, $(\mathcal{N}''(u)w, w) = 0$. Hence $w = 0$, so that the desired critical points satisfy the equation $Lu = \beta\mathcal{N}'(u)$ and $\beta > 0$.

Nonetheless Theorem 1.1 is weak in the sense that one would like to assert the existence of at least N distinct families of periodic solutions, as in the simple linear case when

$$U(x) = \sum_{i=1}^{N} \beta_i^2 x_i^2, \qquad \beta_i > 0.$$

A first step in this direction is to assume that the function $U(x)$ is an even function of x such that $x \cdot \nabla U(x) > 0$ for $x \neq 0$. In that case one can prove, by the "standard" arguments of nonlinear eigenvalue theory that (1.7) has a countably infinite number of families of periodic solutions. Unfortunately, after reparametrization, these families may coincide. Thus a more detailed investigation is called for.

To this end we consider the special case of problem (B) obtained by studying the periodic solutions of (1.7) near the stationary point $x = 0$ with $\nabla U(x) = Ax + o(|x|)$ where A is a nonsingular selfadjoint $(N \times N)$ matrix with positive eigenvalues $\lambda_1^2 \leqq \lambda_2^2 \leqq \cdots \leqq \lambda_k^2$ $(1 \leqq k \leqq N)$. A classic result [4, p. 217] in this direction is due to Liapunov and in the present case asserts that the system (1.7) possesses k distinct periodic families provided $U(x)$ is real analytic and the eigenvalues $(\lambda_1^2, \cdots, \lambda_k^2)$ satisfy the stringent irrationality conditions: $\lambda_i\lambda_j^{-1} \neq$ integer $(i, j = 1, \cdots, k; i \neq j)$. (They are required because the "majorant method" is used in the proof.) Our results in bifurcation theory and nonlinear eigenvalue problems yield the following extension of Liapunov's result (cf. [3], [4]).

THEOREM 1.2. *If $\nabla U(x)$ is real analytic and $\nabla U(x) = Ax + o(|x|)$, then the system* (1.7) *will possess k one-parameter families of nontrivial periodic solutions $x_i(R)$ $(i = 1, \cdots, k)$ with associated periods tending to $2\pi/\lambda_i$ as $R \to 0$. These families will be distinct if $U(x)$ is even or if the numbers $(\lambda_1, \cdots, \lambda_k)$ satisfy the irrationality conditions, $\lambda_j\lambda_i^{-1} \neq$ positive integer $(j \neq i)$, of Liapunov.*

IDEA OF THE PROOF. We intend to find an analytic meaning for the condition $\lambda_i/\lambda_k \neq$ integer and to determine invariants which (i) describe the appearance of one-parameter families of periodic solutions of (1.7) and (ii) which are valid irrespective of the condition $\lambda_i/\lambda_k =$ integer.

Step 1. Make the reparametrization $t = \lambda s$. Then the equation can be written

$$(1.7') \qquad \ddot{x} + \lambda^2 \nabla U = 0$$

where differentiation is now taken with respect to s. Furthermore we need only consider solutions of (1.7′) defined on $[0, \pi]$ and satisfying $\dot{x}(0) = \dot{x}(\pi) = 0$, as these correspond to $2\pi\lambda$ periodic even solutions of (1.7) in t.

Step 2. Note that the eigenvalues of the linearized system

$$(1.7'') \qquad \ddot{x} + \lambda^2 Ax = 0$$

over the class of even 2π periodic vector functions x are all the numbers N^2/λ_i^2 where N varies over the positive integers $1, 2, \cdots$ and $i = 1, \cdots, k$. Then $\lambda_j/\lambda_i = N$ means that the eigenvalue $1/\lambda_i^2$ is a *degenerate eigenvalue* (of multiplicity > 1) for the system (1.7″). Thus we are concerned with the problem of finding invariants describing the stability of eigenvalues of nonlinear perturbations of degenerate linear eigenvalue problems.

Step 3. Denote by H the direct product of n copies of the Hilbert space of absolutely continuous real-valued functions defined on $[0, \pi]$ and which possess one square integrable derivative $\dot{x}_i(s)$ over $[0, \pi]$;

$$H = \prod_{i=1}^{n} H_{1,2}^{(i)} [0, \pi] .$$

H is a Hilbert space with respect to the inner product

$$(x, y) = \sum_{i=1}^{n} \left(\int_0^\pi \dot{x}_i(s)\dot{y}_i(s)\, ds + \int_0^\pi x_i(s)y_i(s)\, ds \right) .$$

NOTATION. For simplicity, vectors in H will also be denoted x. It will be clear from the context when x denotes a vector in H or a vector in $\boldsymbol{R}^n$.

Let H_0 be the closed subspace of H consisting of n vectors $x(t)$ whose mean value over $[0, \pi]$ is zero. Then we shall show that the periodic solutions of (1.7′) in question can be found as the solutions near $x = 0$ of an operator equation of the form $x = \lambda^2\{Lx + Nx\}$ in H_0. Since L is selfadjoint and N is a gradient operator, the abstract results on bifurcation theory mentioned in the introduction become applicable. The real analyticity of $U(x)$ is used to insure that in a small neighborhood of $x = 0$, the periodic solutions obtained lie on continuous curves.

PROOF (PART I). We first reduce the equation (1.7) to an equation of the form $\ddot{x}_0 + \lambda^2[Ax_0 + \tilde{f}(x_0)] = 0$. Here $x_0(s)$ is a function of mean value zero over $[0, \pi]$, $\tilde{f}$ is a higher order term in x with $\tilde{f}(x_0) = \nabla\tilde{F}(x_0)$ and $\tilde{F}(x_0)$ is a C^1 real-valued function. To this end, we write a tentative solution $x(s)$ of (1.7) as $x(s) = x_0(s) + x_m$ where x_m denotes the mean value of x over $[0, \pi]$. If $H = \{x(s) \mid x(s)$ absolutely continuous on $[0, \pi]$ and $\dot{x}(s) \in L_2[0, \pi]\}$, a Hilbert space structure can be defined on H by setting $(x, y) = (x, y)_{L_2} + (\dot{x}, \dot{y})_{L_2}$. $H = H_0 \oplus R^N$ is an orthogonal decomposition of H, $x_0 \in H_0$, $x_m \in R^N$. Thus (1.7) is equivalent to the system

$$(\alpha) \qquad \ddot{x}_0 + \lambda^2\left[Ax_0 + f(x_0 + x_m) - \frac{1}{\pi}\int_0^\pi f(x_0 + x_m)\right] = 0,$$

$$(\beta) \qquad Ax_m + \frac{1}{\pi}\int_0^\pi f(x_0 + x_m) = 0.$$

Now given x_0 with $\|x_0\|$ sufficiently small, (β) can be solved uniquely for x_m with $\|x_m\|$ sufficiently small, say $x_m = g(x_0)$ where $g(x_0) = o(\|x_0\|)$ is a continuously differentiable function of x_0 (in the Fréchet sense). Consequently (α) and (β) are equivalent to the system $\ddot{x}_0 + \lambda^2[Ax_0 + \tilde{f}(x_0)]$ where $\tilde{f}(x_0)$ is the orthogonal projection of $f(x_0 + g(x_0))$ on H_0. Now we set $\tilde{F}(x_0) = F(x_0 + g(x_0))$ where $f(x) = \nabla F(x)$. Then, for $y_0 \in H_0$, $(\nabla\tilde{F}(x_0), y_0) = (\nabla F(x_0 + g(x_0)), \{I + g'(x_0)\}(y_0)) = (\nabla F(x_0 + g(x_0)), y_0)$ since $g'(x_0)(y_0) = 0$ as $g'(x_0)$: $H \to H_0^\perp$. Hence $\tilde{f}(x_0) = \nabla\tilde{f}(x_0)$.

Now as in [4, p. 519], the solutions of (α), (β) in H near $x \equiv 0$ can be determined by finding the solutions in H_0 of the operator equation $x_0 = \lambda^2(Lx_0 + Nx_0)$ where L is a compact selfadjoint operator of $H_0 \to H_0$ defined by the formula

$$(Lx_0, y_0) = \int_0^\pi Ax_0 \cdot y_0,$$

and N is defined by

$$(Nx_0, y_0) = \int_0^\pi \tilde{f}(x_0) \cdot y_0.$$

Furthermore, N is a compact gradient operator since the Fréchet derivative in H_0 of $\int_0^\pi \tilde{F}(x_0)\, ds$ is Nx_0 itself. We now apply the theorem of Krasnoselskiĭ [(3) of the introduction]. In particular $\beta = 1/\lambda_k^2$ is a characteristic value for the linear equation $x = \lambda Lx$. Thus the operator equation above has a family of solutions (x_R, β_R) with $\|x_R\| = R \to 0$ as $\beta_R \to 1/\lambda_k^2$. This family corresponds to the desired one-parameter family of periodic solutions. The families $x_1(R), \cdots, x_k(R)$ so generated correspond to distinct one-parameter families provided, after reparametrization, $x_k(t/\lambda_k, R) \not\equiv x_j(t/\lambda_j, R)$ $(j \neq k)$. This last condition is certainly met if $\lambda_j\lambda_k^{-1} \neq$ integer $(j \neq k)$, for then the period of $x_j(R) \to 2\pi/\lambda_j$ as $R \to 0$, while the period of $x_k(R) \to 2\pi/\lambda_k \neq$ integral multiple of $2\pi/\lambda_j$.

SKETCH OF PROOF IN CASE $U(x)$ IS EVEN AND $\lambda_1^2 = \lambda_2^2 = \cdots = \lambda_k^2$. We wish to show that there are k distinct one-parameter families in this case. To this end as in Part I we reformulate the desired periodic families as solutions of the operator gradient equation $x = \lambda(Lx + Nx)$ in the space of odd functions in $W_{i,2}(-\pi, \pi)$. Then applying the abstract result (2) of the introduction we find k distinct one-parameter families of periodic solutions $x_i(s)$, tending to $(0, 1/\lambda_k^2)$ as $R = \|x\|^2 \to 0$. The difficulty now reduces to showing that after reparametrization $s \to t$, these families remain distinct. Thus suppose that the ith and jth families are such that $x_i(s) = y(s)$ while $x_j(s) = y(h(s))$ where $h(s)$ is a smooth function $\not\equiv s \pmod{2\pi}$. Then since both x_i and x_j satisfy equation (1.7), we find

$$\ddot{y} + \lambda^2\dot{h}^2(s)\nabla U(y) + \dot{y}(h(s))\ddot{h}(s) = 0.$$

Thus $\ddot{h}(s) \equiv 0$ and $h(s) = as + b$ where a and b are constants. Next we show that $a = 1$. Indeed if $y(s)$ satisfies $\ddot{y}(s) + \lambda^2\nabla U(y) = 0$ and $x_j(s) = y(as + b)$, then $\ddot{x}_j + a^2\lambda^2\nabla U(x_j) = 0$. Since both $a\lambda$ and a come arbitrarily close to $1/\lambda_k$ as $R \to 0$, $a = 1$. Now it is fairly straightforward to show since $y(s)$ is odd in s, and must have a minimal period 2π that $b = 2n\pi$ so that $x_i(s) = x_j(s)$.

When one tries to apply these results to the (regularized) problems of celestial mechanics (C), one arrives at a fundamental difficulty, namely that the problems of celestial mechanics are often nonlinear eigenvalue problems in two respects. For example, equation (1.5) after the "recommended transformation" becomes

$$x_{ss} + \lambda B(x)x_s + \lambda^2\nabla U(x) = 0 \tag{1.8}$$

so that not only are the functions $B(x)$ and $\nabla U(x)$ not linear, but the eigenvalue parameter λ also appears nonlinearly. To illustrate a way to obviate this difficulty, consider the so-called regularized Hill's equations of lunar motion [25, p. 388]

$$\left\{\begin{aligned} &\ddot{x} - 2(x^2 + y^2)\dot{y} - hx = V_x, \\ & \qquad\qquad\qquad\qquad \text{where } V = \tfrac{3}{4}\,[(x^2 + y^2)(x^2 - y^2)^2]. \\ &\ddot{y} + 2(x^2 + y^2)\dot{x} - hy = V_y, \end{aligned}\right. \tag{1.9}$$

Note that if one sets $(x, y) = \beta(w, z)$ in (1.9) and choosing $\lambda\beta^2 = 1$, (1.9) becomes

$$\text{(1.10)} \qquad \begin{cases} \ddot{w} - (w^2 + z^2)\dot{z} - V_w - \lambda^2 hw = 0, \\ \ddot{z} + (w^2 + z^2)\dot{w} - V_z - \lambda^2 hz = 0. \end{cases}$$

Now our methods apply to (1.10) for $h < 0$. Note also that, as in most difficult examples of periodicity phenomena in celestial mechanics, the linearized equation at $(w, z) = (0, 0)$ has an even-dimensional subspace of periodic solutions.

Another important example in celestial mechanics is the problem of the preservation of periodic orbits of the Kepler (two-body) problem under an autonomous perturbation $\nabla V(x)$. The resulting Hamiltonian system becomes

$$\text{(1.11)} \qquad \ddot{u} + u/|u|^3 + \nabla V(u) = 0 \quad \text{where } u = (u_1, \cdots, u_N).$$

This problem was considered by Jürgen Moser in [**20**]; however, his results were all based on stringent nondegeneracy hypotheses. Our results and methods can be used to weaken these nondegeneracy assumptions. A simple result in this direction is

If the equation (†) (*below*) *satisfies the hypotheses of Theorem* 1.1, *then* (1.11) *has a one-parameter family of periodic solutions* $x_i(h)$, *where the parameter* h *varies over* $(0, -\infty)$ *and denotes the total energy of* $x_i(h)$.

Idea of Proof for $N = 2$. Let the system (1.11) have fixed energy h. Then by regularization theory, one finds that periodic solutions of (1.11) with total energy h are in (1-1) correspondence with the periodic solutions of

$$\text{(†)} \qquad 4\ddot{u} = hu + \nabla W(u), \quad \text{where } W(u) = |u|^2 V(u^2),$$

$$\text{(‡‡)} \qquad \text{provided that } 2|u'|^2 - u\bar{u}\{V(u^2) + h\} = 1.$$

Here we have used the obvious notation $u = u_1 + iu_2$, $|u|^2 = u_1{}^2 + u_2{}^2$. Now we apply Theorem (1.1) to (†) to define a one-parameter family of solutions $(u_R(t), \tau_R)$ for (†), and we find that value of R such that (‡‡) holds by taking the mean value of (‡‡) over the period $\tau_i(R)$ and noting that the mean value of $u\bar{u}\{V(u^2) + h\}$ over the period $\tau_i(R)$ is R.

Remark. Clearly result (2) of the introduction can be used to prove much stronger results for (1.11) than that given above. This work will be carried out in a subsequent paper.

It is clear that the problems of celestial mechanics will be a fruitful future source of research problems in nonlinear eigenvalue theory.

Finally, we note the possibility of obtaining the periodic solutions of (1.6) [problem (D)] as limits of (discrete) systems of the form (1.4). Such results are reasonable if one regards a continuous system as a limit of discrete ones. However fundamental difficulties still arise in studying problem (D), since the discrete approximations to the system (1.6) are highly "degenerate" (in the sense of Liapunov's theorem mentioned above).

Let Ω be a bounded open set in R^N. Then there are an infinite number of periodic solutions of the system:

$$u_{tt} - \Delta u = 0, \qquad u|_{\partial\Omega} = 0.$$

These periodic solutions are linear combinations of the "normal modes", which in turn are obtained by considering the eigenfunctions of the Laplacian Δ over Ω subject to null boundary conditions; i.e.

$$\Delta u_i + \lambda_i^2 u_i = 0, \qquad u_i|_{\partial\Omega} = 0.$$

We now consider the periodic solutions of the *nonlinear* system (1.6) where $f(x, u)$ is locally Lipschitz continuous in x and u, with $|f(x,u)| = o(|u|)$. Let us suppose a solution of (1.4), $u(x, t)$, can be written

$$(\,*\,) \qquad u(x, t) = \sum_{i=1}^{\infty} q_i(t)u_i(x)$$

where $u_i(x)$ are the eigenfunctions mentioned above. Then we show that finite-dimensional approximations to the solutions of (1.6) are obtained by studying the solutions of an equation of the type (1.4). Indeed by completeness of the eigenfunctions

$$f(x, u) = \sum_{i=1}^{\infty} a_i u_i(x)$$

where $a_i = \int_\Omega f(x, \sum_{j=1}^{\infty} q_j u_j) u_i$. Thus substituting ($*$) into (1.6) and equating the coefficients of $u_i(x)$ to zero, we obtain the system

$$\ddot{q}_i + \lambda_i^2 q_i + \int_\Omega f\left(x, \sum_{j=1}^{\infty} q_j u_j \right) u_i = 0 \qquad (i = 1, 2, \cdots).$$

Hence finite-dimensional approximations to the solutions of ($*$) are obtained as periodic solutions of the system

$$\text{(1.12)}\quad \ddot{q}_i + \lambda_i^2 q_i + \int_\Omega f\left(x, \sum_{j=1}^n q_j u_j \right) u_i = 0 \qquad (i = 1, \cdots, n).$$

Now this system has the Hamiltonian form of equation (1.4). Indeed, if $F_u(x, u) = f(x, u)$, then $U(q) = \frac{1}{2} \sum_{i=1}^n \lambda_i^2 q_i^2 + \tilde{F}(q)$ so that

$$\tilde{F}_{q_i}(q_1, \cdots, q_n) = \int_\Omega f\left(x, \sum_{j=1}^n q_j u_j \right) u_i(x)$$

where $\tilde{F}(q_1, \cdots, q_n) = \int_\Omega F(x, \sum_{j=1}^n q_j u_j)$. Note however that even for the simplest system $\Omega = (0, 2\pi)$ the irrationality conditions of Liapunov are strongly violated, since $\lambda_N^2 = N^2$ for each integer N. For small initial data and a reasonably large class of functions $f(x, u)$, Sattinger [26] showed that the solutions of (1.12) do approximate the solutions of the Cauchy problem for (1.6) for all time. We conjecture that the periodic solutions of (1.12) near $q = 0$ also approximate the periodic solutions of (1.6) near $u = 0$.

Lecture 2. Periodic water waves. Here we consider the classic problem of proving the existence of steady periodic waves at the free surface $\partial\Gamma$ of an ideal incompressible fluid. Because of their precision and relative simplicity, the results described here represent one of the most successful attempts to apply global analysis to a given difficult nonlinear eigenvalue problem. We suppose the flow is steady, irrotational, and two-dimensional, the fluid occupying a domain Γ in R^2. The points in R^2 are denoted by Cartesian coordinates (x, y). Euler's equation of motion and the equation of continuity for this problem then become

$$\text{(2.1)}\qquad \Delta\zeta = 0 \qquad \text{in } \Gamma,$$

$$\text{(2.2)}\qquad \tfrac{1}{2}|\nabla\zeta|^2 + gy = \text{constant} \qquad \text{on } \partial\Gamma,$$

where ζ denotes the velocity potential for the flow. Hence we are forced to solve a nonlinear free boundary value problem. Following an argument due to Levi-Civita, one introduces the complex variable $z = x + iy$ and two analytic functions of z, $u(z) = \zeta + i\psi$ and $\omega = \log\{\partial\zeta/\partial x - i\partial\zeta/\partial y\} = C(\Phi) + i\Phi$. Here ψ is the stream function for ζ, Φ is the angle formed by the velocity vector V at the point (x, y), and $C(\Phi)$ is the harmonic conjugate of Φ. In order to work in a known domain, one chooses $u = \zeta + i\psi$ as an independent variable and regards ω as a function of u. Assuming, for simplicity, that the fluid is at infinite depth, and after performing the recommended period transformation of Lecture 1, the desired periodic solutions are in (1-1) correspondence with the nontrivial solution of the nonlinear integral equation

$$\Phi(\theta) = \lambda \int_0^\pi K(\theta', \theta) e^{3C(\Phi)} \sin \Phi \, d\theta', \tag{2.3}$$

where $\lambda = (gv)/2\pi c^2$, v is the wave length, and c denotes the constant horizontal velocity of the moving wave. $K(\theta', \theta)$ is the Green's function associated with the Neumann problem for Δ in a circle, and the additive constant in the definition of $C(\Phi)$ is so chosen that $\int_0^{2\pi} C(\Phi(\theta))\, d\theta = 0$. Note that (2.3) is in the form of a nonlinear eigenvalue problem.

There are basically two types of problems associated with (2.3): (i) a local bifurcation problem for Φ very small, and (ii) a general global problem for $|\Phi|$ unrestricted. The local problem was "solved" in 1925 by Levi-Civita [5], but the global problem (which we discuss here) remained only partially solved until 1961 when the Russian mathematician J. P. Krasovskiĭ proved the following results.

THEOREM 2.1 [6]. *There exist steady periodic waves satisfying* (2.1) *and* (2.2) *for which the maximum angle of inclination of the tangent to the wave profile takes any value in the open interval* $(0, \pi/6)$. *The wave is symmetric relative to a vertical axis passing through the peak of the wave. Furthermore waves of this type with arbitrarily large Froude number* λ *cannot exist.*

Before sketching the proof of this interesting result, we note that the number $\pi/6$ appearing in the theorem is sharp in the sense that (i) Stokes' periodic "limit" waves have $\max|\Phi| = \pi/6$ and possess cusps [7], (ii) the solutions of (2.1), (2.2) show that steady periodic waves with $\max|\Phi| > \pi/6$ do not exist (see Wehausen [27] for further information). Actually, Krasovskiĭ proved a sharp analogue of Theorem 2.1 for waves of finite depth and periodic bottom, by slightly modifying the proof given below.

SKETCH OF THE PROOF OF THEOREM 2.1. The proof breaks down into the usual steps:

1. Representation of equation (2.3) as an operator equation of the form $x = \lambda Ax$ in a suitable Banach space X.
2. Proof of complete continuity of the map A in X.
3. Application of the Leray-Schauder degree to the operator equation.
4. Proof of the estimates necessary to calculate the Leray-Schauder degree.

In order to carry out the steps 1–4, we need to know the following analytic facts concerning the conjugation operator C of a harmonic function and the kernel $K(\theta', \theta)$.

Fact 1 (*M. Riesz's theorem*). For $1 < p < \infty$, $C(\Phi)$ is a bounded mapping of $L_p[0, 2\pi] \to L_p[0, 2\pi]$ and $\|C(\Phi)\|_p \leqq (p/(p-1))\|\Phi\|_p$.

Fact 2 (*Zygmund's theorem*). If $|\Phi| \leqq 1$ and $0 < \lambda < \pi/2$, then $\int_0^{2\pi} \exp(\lambda C(\Phi))\, d\theta \leqq 4\pi/\cos\lambda$.

Fact 3. $\max_\theta \int_0^\pi |K(\theta', \theta)|^p\, d\theta' \leqq C_p$, and for $1 < p < \infty$ and fixed θ, $(\partial K/\partial\theta)$ maps $L_p[0, 2\pi] \to L_p[0, 2\pi]$ boundedly.

(A sketch of the proof of the first two facts will be found in Zygmund — *Trigonometric series* (Cambridge, 1968), pp. 254-257, whereas the third fact is a well-known property of the Green's function for Δ.)

(*Steps* 1 *and* 2). Now let $X = C_0[0, \pi]$, i.e. the continuous functions on $[0, \pi]$ that vanish at 0 and π. Let $\|\Phi\|_X = \sup_{[0,\pi]}|\Phi(\theta)|$ and define the operator

$$A\Phi(\theta) = \int_0^\pi K_1(\theta', \theta) e^{3C(\Phi)} \sin\Phi\, d\theta'. \tag{2.4}$$

One shows that A is a completely continuous map defined on the sphere $S(0, \rho)$ of radius $\rho < \pi/6$ in X. Clearly, by Facts 1 and 3 above, A is a well-defined and continuous map from $S(0, \rho) \to X$ for $\rho < \pi/6$. In fact, under Hölder's inequality, one easily shows that, for Φ_1, $\Phi_2 \in S(0, \rho)$, $\rho = \pi/6 - d\ (d > 0)$,

$$\|A\Phi_1 - A\Phi_2\| \leqq K_d \|\Phi_1 - \Phi_2\|.$$

To verify the compactness of A, we again use Facts 1 and 3 to show that if $\tilde\Phi(\theta) = A\Phi$, then for some $s > 1$, $\|d\tilde\Phi/d\theta\|_{L_s} \leqq K_{d,s}$ for $\Phi \in S(0, \rho)$ with $\rho = \pi/6 - d$ (as above). Consequently, $\|\tilde\Phi\|_{C_{0,\mu}} \leqq M_\rho$ for some $\mu > 0$. The desired compactness of A thus follows. (Here $C_{0,\mu}$ is the Banach space of Hölder continuous functions of exponent μ.)

(*Step* 3). In order to apply the Leray-Schauder degree to prove the existence of a solution of (2.3), we let

$$A_{\epsilon,\lambda}\Phi = \lambda \left[A\Phi + \epsilon \int_0^\pi K(\theta', \theta) \sin\theta'\, d\theta' \right].$$

Note that A_ϵ is compact (and positive). We prove

(†) the Leray-Schauder degree of $I - A_{\epsilon,\lambda}$ on the positive cone $K_\beta = \{\Phi(\theta) | \Phi \in C_0[0, \pi],\ \Phi \geqq 0,\ \|\Phi\|_{C_0} \leqq \beta\}$, $0 < \beta < \pi/6$, is different for large and for small λ.

The last part of Theorem 2.1 and (†) suffices to prove the existence part of Theorem 2.1. To see this, we first note that (†) implies that there are sequences $\{\lambda_n\}$, $\{\epsilon_n\}$, $\{\Phi_n\}$ with $\lambda_n > 0$, $\epsilon_n \to 0$, and $\Phi_n \in C_0[0,\pi]$ such that

$$\Phi_n = \lambda_n A_{\epsilon_n} \Phi_n, \qquad \|\Phi_n\|_C = \beta\,. \tag{2.5}$$

By the compactness of A and the boundedness of $|\lambda_n|$ (due to the nonexistence part of Theorem 2.1) there is a (strongly) convergent subsequence $\{\lambda_{n_j}\}$ and $\{\Phi_{n_j}\}$ with limits $(\lambda_\beta, \Phi_\beta)$ such that

$$\Phi_\beta = \lambda_\beta A \Phi_\beta, \qquad \|\Phi_\beta\|_C = \beta, \qquad \Phi_\beta(\theta) \geqq 0 \quad \text{on } [0,\pi]\,. \tag{2.6}$$

Thus one can extend $\Phi(\theta)$ to an odd 2π-periodic function of θ.

(*Step* 4). First we prove (†). For λ very small, $d(I - A_{\epsilon,\lambda}, 0, K_\beta) = 1$ since for $\lambda = 0$, $A_{\epsilon,\lambda} \equiv 0$. On the other hand, for λ very large, $\Phi(\theta) - A_{\epsilon,\lambda}\Phi(\theta)$ cannot be positive for max $|\Phi(\theta)| \leqq \beta$, so

$$d(I - A_{\epsilon,\lambda}, 0, K_\beta) = 0$$

in that case.

The nonexistence result of Theorem 2.1 is somewhat more difficult. It is based on the following two a priori estimates for solution $\Phi(\theta)$ of (2.3):

(2.7) There are absolute positive constants γ and δ such that $\Phi^\gamma(\theta) \geqq (\lambda/\delta)^\gamma L(\Phi^\gamma)$ where $L\Phi = \int_0^\pi K(\theta', \theta)\Phi(\theta')\,d\theta'$ provided $\|\Phi\|_{C_0} \leqq \pi/2$.

(2.8) There is an absolute constant $\beta > 0$ such that $\Phi^\gamma(\theta) \geqq \beta \sin\theta$.

Assuming (2.7) and (2.8) with β maximal, the proof of nonexistence is as follows, applying the operator L to (2.8) and using (2.7) we have

$$(\delta/\lambda)^\gamma \Phi^\gamma \geqq L\Phi^\gamma(\theta) \geqq \beta L(\sin\theta) = \beta \sin\theta,$$

i.e. $\Phi^\gamma \geqq (\lambda/\delta)^\gamma \beta \sin\theta$, so that $(\lambda/\delta)^\gamma \leqq 1$. Hence for $\lambda > \delta$, (2.3) can have no solution. To end our sketch of the proof of Theorem 2.1, we prove (2.7) and (2.8). To demonstrate (2.7), it suffices to show that, for $\Phi \in K_\beta$,

$$L(e^{3C(\Phi)} \sin\Phi) \geqq (1/\delta) L(\Phi^\gamma)^{1/\gamma}. \tag{2.9}$$

Now (2.9) follows from Hölder's (inverse) inequality, since

$$\begin{aligned} L(e^{3C(\Phi)} \sin\Phi) &\geqq L(e^{3C(\Phi)}\,\Phi) \\ &\geqq \{L(e^{3qC(\Phi)})\}^{1/q}\{L(\Phi^\gamma)\}^{1/\gamma} \end{aligned}$$

with $1/q + 1/\gamma = 1$ $(q < 0)$. Then the basic Facts 2 and 3 imply that for $q = -1/10$ and $|\Phi| \leqq \pi/2$, $|L(e^{3|q|C(\Phi)})| < \delta^{-|q|}$. Finally, we prove (2.8). Applying the inequality (2.7) k times and letting $\Phi^\gamma(\theta) = \sum_{n=1}^{\infty} a_n \sin n\theta$, we find

$$L^k\Phi^\gamma = \sum_{n=1}^{\infty} \frac{a_n}{n^k} \sin n\theta \leqq \left(\frac{\delta}{\lambda}\right)^{\gamma k} \Phi^\gamma(\theta). \tag{2.10}$$

Furthermore

$$\sum (a_n/n^k) \sin n\theta \geqq a_1 \sin\theta - \left| \sum_{n=2}^{\infty} \cdots \right|$$

and $\left|\sum_{n=2}^{\infty}(a_n/n^k) \sin n\theta\right| \leqq \max_n |a_n| \sum_{n=2}^{\infty} |\sin\theta| n^{k-1}$. Thus (2.10) implies that

$$\Phi^\gamma(\theta) \geqq \left(\frac{\lambda}{\delta}\right)^{\gamma k} \left\{a_1 - \max_n |a_n| \sum_{n=2}^{\infty} \frac{1}{n^{k-1}}\right\} \sin\theta.$$

Since $\Phi(\theta) \geqq 0$ on $[0, \pi]$, $a_1 > 0$ and choosing k sufficiently large we can choose $\{a_1 - \max |a_n| \sum_{n=2}^{\infty} (1/n^{k-1})\} \geqq a_1/2$.

Paul Garabedian has observed that the results discussed here can be reformulated in terms of critical point theory. It is an open problem to obtain Theorem 2.1 by this approach. Another open problem of this type is to prove the existence of "solitary" waves as a limit of the periodic waves proved in Theorem 2.1.

Lecture 3. Equilibrium states in elasticity. Perhaps the oldest nonlinear eigenvalue problem, dating back to 1744, is the "Elastica" problem of Euler in which one considers the equilibrium states of a thin, flat, narrow, elastic rod compressed uniformly along its length [8]. In this lecture, I hope to show that even simple problems concerning equilibrium states of thin elastic structures lead to a class of extremely interesting unsolved nonlinear eigenvalue problems.

1. **Some terms of elasticity.** A solid body is *elastic* if it changes size and shape when a force (sufficiently restricted in magnitude) acts on it, but returns to its original size and shape when the force is withdrawn. *Elasticity* is the study of deformations of elastic bodies. The mathematical foundations of the subject have been carefully studied since the basic work of Euler, Lagrange and Cauchy; and the associated problems in partial differential equations have been studied by Hadamard, Friedrichs, and John, among many others.

One approach to the study of elasticity proceeds by attempting to obtain global results on deformations from local information. Two basic tensor quantities enter at this point: *strain* a measure of the change of an element of length when a body is deformed (a purely geometric quantity), and *stress* a measure of internal forces acting in a body.

A *stress-strain law* is a functional relationship between these two tensors, and characterizes the elastic properties of a given body. The simplest possible relation is the linear one (Hooke's law).

We shall focus attention on equilibrium states of elastic bodies acted on by forces. The partial differential equations describing such states are obtained as Euler-Lagrange equations of the potential energy of the given elastic system. Problems arise in one, two or three dimensions depending on the relative dimensions of the elastic body.

The solutions of the Euler-Lagrange equations, subject to the appropriate boundary conditions, determine "global" equilibrium states. The joint assumptions of small deformations and Hooke's law lead to linear equations and boundary conditions. However, if either of these assumptions is given up, nonlinear equations arise and the usual methods for studying the resulting problems become inapplicable. It is at this stage that the qualitative methods of global analysis come to the fore.

2. **A 1-dimensional example.** Consider a thin, flat, narrow elastic rod of unit length compressed uniformly along its ends which are constrained to lie on a fixed line. In 1744 Euler showed that the equilibrium states of this rod correspond to the solutions of the nonlinear equation

$$d^2y/dx^2 + \lambda y(1 + \dot{y}^2)^{3/2} = 0$$

subject to the boundary conditions $y(0) = y(1) = 0$. (Euler obtained this equation by minimizing the integral of the square of the curvature over the deformed rod.) Here λ is a measure of the compressive force acting on the rod $\{(x, 0) \mid 0 \leqq x \leqq 1\}$ and $y(x)$ measures the vertical displacement of the point $(x, 0)$ from its original flat state. This equation was solved by Euler by explicit integration using elliptic functions. He found that, for $\lambda \leqq \lambda_1$ (the smallest eigenvalue of the linearized problem), $y(x) \equiv 0$ is the only possible solution, but, for $\lambda > \lambda_1$, the rod deforms out of its flat state. Furthermore as $\lambda \to \infty$ a countably infinite number of distinct equilibrium states are possible for the rod. Which of these states, existing for a given λ, is preferred by nature? Following Friedrichs, we add

Hypothesis E. Nature prefers the state with the least potential energy.

3. **The two-dimensional problems of von Karman.** The equations defining the two-dimensional analogue of Euler's example were formulated over 50 years ago by T. von Karman. We consider a thin elastic body B which is flat in its undeformed state subjected to a compressive force (of magnitude λ) acting on the boundary of B. Then the stresses produced in B, as measured by the Airy stress function, $f(x, y) + \lambda F_0(x, y)$, and the displacement of B from its flat state $u(x, y)$ are defined by the following quasilinear elliptic system:

$$\begin{aligned} \Delta^2 f &= -\tfrac{1}{2}[u, u], \\ \Delta^2 u &= \lambda[F_0, u] + [f, u], \end{aligned} \tag{3.1}$$

where Δ^2 denotes the biharmonic operator and $[f, g] = f_{xx}g_{yy} + f_{yy}g_{xx} - 2f_{xy}g_{xy}$. If we represent B as a bounded domain G in R^2 and the boundary of B as ∂G, we may consider the following boundary conditions associated with (3.1):

$$\left.\begin{aligned} u = u_x = u_y = 0 \\ f = f_x = f_y = 0 \end{aligned}\right. \quad \text{on } \partial G. \tag{3.2}$$

Here $F_0(x, y)$ is the function obtained by solving an associated inhomogeneous linear problem, and is a measure of the stress produced in the undeflected plate, if it were prevented from deflecting.

The resulting equilibrium states are called "buckled" states, and the problem is referred to as "elastic buckling".

Problem. Determine the totality of solutions of (3.1) and (3.2) as a function of λ and decide which among them are physically relevant.

In order to answer this question we note first abstract reformulations of (3.1) and (3.2) (see [9]).

(a) The solutions of (3.1) and (3.2) are in (1-1) correspondence with the solutions of the following operator equation in the Sobolev space $\dot{W}_{2,2}(G)$, henceforth denoted by H:

$$u + Cu = \lambda Lu. \tag{3.3}$$

Here L and C denote certain completely continuous mappings of $\dot{W}_{2,2}(G)$ into itself, with L linear and selfadjoint while C is homogeneous of degree 3 and a gradient map.

(b) The solutions of (3.3) in turn are in (1-1) correspondence with the critical points of the functional (Lu, u) subject to the constraint $\|u\|^2 + \frac{1}{2}(Cu, u) = R$ $(0 \leqq R < \infty)$, that is, critical points of the functional (Lu, u) on the one-parameter family of Hilbert manifolds

$$\partial A_R = \{u \mid \|u\|^2 + \tfrac{1}{2}(Cu, u) = R\}.$$

PROOF OF (a). By virtue of the facts that H is a Hilbert space with respect to the inner product

$$(u, v)_{2,2} = \int_G \{u_{xx}v_{xx} + 2u_{xy}v_{xy} + u_{yy}v_{yy}\},$$

and that

$$[f, g] = (f_{yy}g_x - f_{xy}g_y)_x + (f_{xx}g_y - f_{xy}g_x)_y,$$

weak solutions of (3.1) and (3.2) can be defined as pairs of functions u, f, each an element of H, which satisfy the following integral identities, for all $\phi, \eta \in H$:

$$(*) \qquad (u, \eta)_{2,2} = \int_G [(\bar{f}_{xy}u_y - \bar{f}_{yy}u_x)\eta_x + (\bar{f}_{xy}u_x - \bar{f}_{xx}u_y)\eta_y],$$

$$(**) \qquad (f, \phi)_{2,2} = 2\int_G (u_x u_{yy}\phi_x - u_x u_{xy}\phi_y),$$

where $\bar{f} = \lambda F_0 + f$. Now it is a standard result of the regularity theory of elliptic partial differential equations that the solutions of (3.1) and (3.2) are in (1-1) correspondence with those of (*) and (**) (see [**9**, Theorem 2.1]). We now show that the solutions of (*) and (**) are in (1-1) correspondence with the solutions of (3.3). To this end, we employ Sobolev's imbedding theorem and Riesz's representation theorem for linear functionals in the Hilbert space H. Define the operator C for $g, w, \phi \in H$ by

$$(C(w, g), \phi)_{2,2} = \int_G (g_{xy}w_y - g_{yy}w_x)\phi_x + (g_{xy}w_x - g_{xx}w_y)\phi_y$$

then, (i) $(C(w, g), \phi)_{2,2} \leqq K\|g\|_{2,2}\|w\|_{1,4}\|\phi\|_{1,4}$ where K is a constant independent of w, g, ϕ; and (ii) $C(w, g)$ is a bounded bilinear mapping of $H \times H$ into itself. Furthermore if $w_n \to w$ weakly in H, $C(w_n, w_n) \to C(w, w)$ strongly in H and

$$\|C(w, w) - C(u, u)\| \leqq K\{\|u\| + \|w\|\}\|u - w\|.$$

Hence equations (*) and (**) can be rewritten

$$(u, \eta)_{2,2} = (C(u, \bar{f}), \eta)_{2,2},$$

$$(f, \phi)_{2,2} = (C(u, u), \phi)_{2,2}.$$

Since these equations hold for all $\eta, \phi \in H$, we have that $f = -C(u, u)$ and $w = C(u, \lambda F_0 + f)$. Define $C(u) = C(u, C(u, u))$ and $L(u) = C(u, F_0)$; then the above equations can be written $f = -C(u, u)$ and $u + Cu = \lambda Lu$. Since u uniquely determines f, the solutions of equation (3.3) are in (1-1) correspondence with the solutions of (*)

and (**), as required. Note the stated properties of C and L follow immediately from the definition; the fact that C is a gradient map follows because the form $(C(u, g), \phi)$ is symmetric in u, g and ϕ so that if we set $I(u) = \frac{1}{4}(C(u), u)$, a short computation shows $\lim_{\epsilon \to 0} (I(u + \epsilon\phi) - I(u))/\epsilon = (C(u), \phi)$ for all $\phi \in H$.

PROOF OF (b). First we note that for fixed positive R the set $\partial A_R = \{u \mid \|u\|^2 + \frac{1}{2}(Cu, u) = R\}$ is a Hilbert manifold in H. Indeed setting $F(u) = \|u\|^2 + \frac{1}{2}(Cu, u)$, so that grad $F(u) = \frac{1}{2}(u + Cu) = 0$ implies $u = 0$ because $(Cu, u) = \|C(u, u)\|^2 \geqq 0$. Thus the critical points of the functional (Lu, u) on ∂A_R for some R coincide with the solutions of $Lu = k\{u + Cu\}$. Now $k \neq 0$ as $Lu = 0$ implies $u = 0$. So setting $k = \lambda^{-1}$, the critical points coincide with the solutions of the equation (3.3), $u + Cu = \lambda Lu$, on ∂A_R for some R.

In studying this problem, the following result is of some interest. We assume for simplicity $(Lu, u) > 0$ for $u \neq 0$, so that the eigenvalues of $u = \lambda Lu$ can be written $0 < \lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots$.

THEOREM 3.1. *The solutions of the system* (3.1), (3.2) *have the following properties*:

(1) *For* $\lambda \leqq \lambda_1$, *the only solution is* $u \equiv f \equiv 0$ (*the trivial solution*).

(2) *If* $\lambda > \lambda_1$, *nontrivial solutions exist.*

(3) *The potential energy of any nontrivial solution is strictly negative, so that if* $\lambda > \lambda_1$, *a nontrivial solution is preferred by nature.*

(4) *There are at least a countably infinite number of distinct one-parameter families of solutions* $(u_n(R), f_n(R), \lambda_n(R))$ *existing for all* $R \geqq 0$ *which tend as* $R \to 0$ *to the vector* $(0, 0, \lambda_n)$.

PROOF OF (1). If $(\bar{u}, \bar{f})$ satisfies (3.1) and (3.2) for $\lambda < \lambda_1$, then $\bar{f} = -C(\bar{u}, \bar{u})$ and $\bar{u} + C\bar{u} = \lambda L\bar{u}$ with $\lambda < \lambda_1$. Hence $(\bar{u}, \bar{u}) + (C\bar{u}, \bar{u}) = \lambda(L\bar{u}, \bar{u})$. By the variational characterization of λ_1, for $\lambda \leqq \lambda_1$, $(\bar{u}, \bar{u}) - \lambda(L\bar{u}, \bar{u}) \geqq 0$. Combining these equations $(C\bar{u}, \bar{u}) = \|C(\bar{u}, \bar{u})\|^2$ and hence $C(\bar{u}, \bar{u}) = 0$. Since $\bar{u}$ is smooth, this implies $\int_G [\bar{u}, \bar{u}]\phi = 0$ for all $\phi \in H$. Hence

$$\bar{u}_{xx}\bar{u}_{yy} - \bar{u}_{xy}^2 = 0,$$

$$\bar{u}|_{\partial G} = 0.$$

Thus the surface $\bar{u} = \bar{u}(x, y)$ has zero Gaussian curvature, and is generated by straight lines. Since $\bar{u} = 0$ on ∂G, we obtain $\bar{u} \equiv 0$ so that $\bar{f} = -C(\bar{u}, \bar{u}) \equiv 0$ also.

PROOF OF (2). The numbers $c_1(R) = \sup_{\partial A_R}(Lu, u)$ are a one-parameter family of critical values of the variational problem (b). These critical values determine solutions $(u_1(R), \lambda_1(R))$ of (3.3) and consequently solutions $(u_1(R), f_1(R), \lambda_1(R))$ of (3.1) and (3.2). Now as $R \to 0$, $(u_1(R), \lambda_1(R)) \to (0, \lambda_1)$. Indeed as $R \to 0$ $\|u_1(R)\| \to 0$ so $\sup_G |u_1(R)| \to 0$ and $\lambda_1(R) \to \lambda_1$. [See Appendix at the end of the lecture.]

PROOF OF (3). The potential energy of a nontrivial solution (u, f, λ) is proportional to $V(u) = \|u\|^2 + \frac{1}{2}(Cu, u) - \lambda(Lu, u)$. For a solution $\|u\|^2 + (Cu, u) = \lambda(Lu, u)$. Thus $V(u) = -\frac{1}{2}(Cu, u) < 0$. Hence the result follows by Hypothesis E.

PROOF OF (4). To prove the existence of a countable number of distinct solutions $u_n(R)$ of (3.3) on A_R we use the results discussed in [17]. Since both u and $-u \in A_R$, A_R/Z_2 is homeomorphic to infinite-dimensional real projective space over H, $P(H)$, and $\operatorname{cat} P(H) = \infty$. Furthermore, it is immediate from the results of [3] that the variational problem (b) satisfies the Palais-Smale condition. Indeed it is sufficient to show that $\operatorname{grad} A(u) = u + Cu$ satisfies condition S, i.e., if $u_n \to u$ weakly and $(\operatorname{grad} Au_n - \operatorname{grad} Au, u_n - u) \to 0$, then $u_n \to u$ strongly. But this is clear as C is a completely continuous mapping. More precisely, $u_n(R)$ can be characterized by the minimax principle of [2], setting $B(u) = (Lu, u)$, $B(u_n(R)) = \sup_{[V]_n} \inf_V B(u)$ where $[V]_n = \{V \mid V \in A_R/Z_2, \operatorname{cat}(V, A_R/Z_2) \geqq n\}$. As this result is true for each $R > 0$, we obtain a countably infinite number of distinct one-parameter families of solutions $(u_n(R), \lambda_n(R))$. Furthermore, we note that, by [9], as $R \to 0$, $(u_n(R), \lambda_n(R)) \to (0, \lambda_n)$.

The following graph of the norm of solutions u versus λ summarizes the contents of the above results.

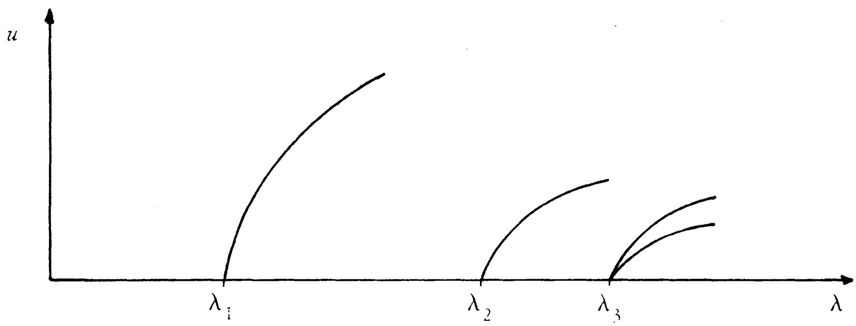

FIGURE 1. Buckling of a thin elastic plate.

The above diagram suggests the following:

THEOREM 3.2. *For* $\lambda \in (\lambda_N, \lambda_{N+1}]$, *the system* (3.1), (3.2) *possesses* N *distinct solution pairs* $(\pm u_i(\lambda), f(\lambda))$, $i = 1, 2, \cdots, N$.

Sketch of Proof. Arguing as in Hempel [21], we consider the critical points of the functional (Cu, u) subject to the constraint of lying in $S = \{u \mid (u, u) + (Cu, u) = \lambda(Lu, u), u \neq 0\}$. Since S does not contain 0, the critical points of this isoperimetric variational problem satisfy $Cu + g(u + 2Cu - \lambda Lu) = 0$ where g is a constant $\neq 0$. We can show $g = -1$ by taking the inner product of this last equation with u. Thus the main difficulty is demonstrating the existence of nonzero critical points. To accomplish this, one uses the characterization of the eigenvalues of a compact selfadjoint operator defined on a Hilbert space to show that for $\lambda \in (\lambda_N, \lambda_{N+1})$, S contains sets of category $\leqq N$ but not sets of category $> N$. One now shows that the Ljusternik-Schnirelmann principle applies to this isoperimetric problem. Of course the main difficulty in this step is the singular point $u = 0$ for S. Full details will appear in a future publication.

Another problem of importance is "combined buckling-bending" where in addition to the compressive force acting on ∂B there is a force f on the plate normal to B. Again the von Karman equations defining equilibrium states can be written as operator equations in the Hilbert space H. Indeed (using the same notations as above), the equation is

$$u + Cu = \lambda Lu + f. \tag{3.4}$$

Clearly the solutions of this equation in H coincide with the critical points of the potential energy functional

$$V(u) = \|u\|^2 + \tfrac{1}{2}(Cu, u) - 2(f, u) - \lambda(Lu, u) \tag{3.5}$$

and, by Hypothesis E, the physically relevant solution u can be characterized by

$$V(u) = \min_H V(u). \tag{3.6}$$

In this connection the following theorem is of interest.

Theorem 3.3. *The problem of elastic bending, defined by equation* (3.4), *always has a solution u, characterized by* (3.6). *For $\lambda \in [0, \lambda_1)$ and sufficiently small f, the solutions of* (3.4) *are unique, but (in general) not otherwise.*

Proof. The functional $V(u)$ is (i) lower semicontinuous with respect to weak convergence in H, since $C(u)$ is completely continuous in H; and (ii) coercive in the sense that $V(u) \to \infty$ as $\|u\| \to \infty$. Indeed since $Lu = C(u, f)$,

$$\begin{aligned} V(u) &= \|u\|^2 + \tfrac{1}{2}(Cu, u) - 2(f, u) - \lambda(Lu, u) \\ &= \|u\|^2 + \tfrac{1}{2}\|C(u, u)\|^2 - 2(f, u) - \lambda(f, C(u, u)) \\ &\geqq \|u\|^2 - 2\|f\|\,\|u\| - \tfrac{1}{2}|\lambda|\,\|f\|^2. \end{aligned}$$

To prove uniqueness for small f, the following inequality is useful:

$$\|Cu - Cv\| \leqq k\{\|u\|^2 + \|v\|^2\}\|u - v\|$$

where k is a constant independent of $u, v \in H$. First, for $\lambda = 0$, if $u - Cu = f$, $(u, u) + (Cu, u) = (f, u)$ so that as $(Cu, u) \geqq 0$, $\|u\| \leqq \|f\|$. Hence if u, v are solutions of (3.4) for $\lambda = 0$, then $(u - v) - (Cu - Cv) = 0$. Hence $\|u - v\| = \|Cu - Cv\| \leqq k\{\|u\|^2 + \|v\|^2\}\|u - v\|$ so that if $u \neq v$, then $1 \leqq k(2\|f\|^2)$, i.e. $\|f\|^2 \geqq 1/2k$. Hence if $\|f\|^2 < 1/2k$, the solutions of (3.4) are unique.

To illustrate nonuniqueness for large f, we consider the example of a thin circular plate B deformed under a large force normal to the plane of B. Yanowitch [22] has shown that (3.4) has a radially symmetric solution $u = u(r)$, but that the critical point of (3.6) is *not* radially symmetric.

4. **The effect of curvature.** A thin elastic body which is curved in its undeformed state is called a shell. The buckling problem for a thin, shallow shell B, subjected to a force (sufficiently restricted) of magnitude λ acting on the clamped boundary of B, can be formulated as the following system of nonlinear elliptic partial differential equations and boundary conditions (the so-called von Karman equations) (see [32]):

$$(3.7) \qquad \Delta^2 f/E = -\tfrac{1}{2}[u, u] - (k_1 w_x)_x - (k_2 w_y)_y,$$

$$(3.8) \qquad D\Delta^2 u/h = [f, u] + (k_1 f_x)_x + (k_2 f_y)_y + Z$$

$$\text{where } [f, g] = f_{xx}g_{yy} + f_{yy}g_{xx} - 2f_{xy}g_{xy},$$

$$(3.9) \qquad D^\alpha u|_{\partial\Omega} = 0, \qquad |\alpha| \leqq 1,$$

$$(3.10) \qquad D^\alpha f|_{\partial\Omega} = \lambda\Psi_\alpha, \qquad |\alpha| \leqq 1.$$

Here k_i are the principal curvatures. Again we can translate the problem into a variational problem on an infinite-dimensional manifold ∂A_R in the Sobolev space $\dot{W}_{2,2}(G)$. However in this case the structure of the critical points for the problem is quite different if $k_1, k_2 \neq 0$. This can be seen most easily by the following graph showing the norm of solutions u versus λ.

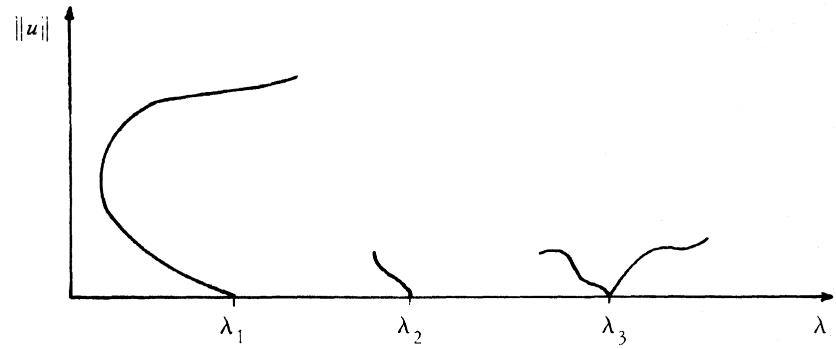

FIGURE 2

More explicitly,

(a) the problem (3.7)–(3.10) is equivalent to an operator equation (3.3′) analogous to (3.3). In fact, as in §3, we find

$$f = -\tfrac{1}{2}C(u, u) - kLu, \tag{3.3a′}$$

$$u = \lambda L_1 u + C(u, f) + kLf, \tag{3.3b′}$$

where we have set $k = (k_1{}^2 + k_2{}^2)^{1/2}$. Substituting (3.3a′) into (3.3b′) and simplifying, we obtain

$$u + \tfrac{1}{2}Cu + kC(u, Lu) + \tfrac{1}{2}kLC(u, u) + k^2L^2u = \lambda L_1 u. \tag{3.3′}$$

(Note that if $k = 0$, then (3.3′) reduces to a variant of (3.3).)

(b) The solutions of (3.3′) are the critical points of (Lu, u) subject to the constraint $u \in \partial A_R$, where

$$\partial A_R = \{u \mid A(u) \equiv \|u\|^2 + \tfrac{1}{4}(Cu, u) + k(C(u, u), Lu) + k^2\|Lu\|^2 = R\}$$

(for small R) is a Hilbert manifold.

PROOF OF (a). Repeat proof of (a) of §3.

PROOF OF (b). Apart from a constant factor, grad Au coincides with the left side of (2.3). Furthermore, the critical points of the functional $B(u) = (L_1u, u)$ coincide with the solutions on ∂A_R of (3.3′). We shall show that for sufficiently small R (say $R \leqq R_0$), ∂A_R is a Hilbert manifold in H. To this end we recall that $I(u, v, w) = (C(u, v), w)$ is a symmetric function of u, v, w. Thus grad Au coincides with the right side of (3.3′). If

$$A(u) = \|u\|^2 + \tfrac{1}{4}C(u, u)^2 + k(C(u, u), Lu) + k^2\|Lu\|^2 = R,$$

then $\|u\|^2 + \|\tfrac{1}{2}C(u, u) + kLu\|^2 = R$. So $\|u\|^2 \leqq R$ for $u \in \partial A_R$.

Also if grad $Au = 0$,

$$(\text{grad } Au, u = \|u\|^2 + \tfrac{1}{2}\|C(u,u)\|^2 + \tfrac{3}{2}k(C(u,u), Lu) + k^2\|Lu\|^2 = 0,$$

from which $\|u\|^2 \leqq \frac{3}{2}k\|Lu\| \; \|C(u,u)\|$. Thus $\|u\|^2 \leqq cc_1\|u\|^3$ where $c = \|L\|$ and $\|C(u,u)\| \leqq c_1\|u\|^2$, so that $1 \leqq cc_1\|u\|$. If $u \in \partial A_R$ and grad $Au = 0$, we have $1 < cc_1\sqrt{R}$. Therefore if $R < (1/cc_1)^2$, then grad $A(u) \neq 0$ on ∂A_R and ∂A_R is a Hilbert manifold.

Furthermore, the functional $B(u)$ defined on A_R (for $R \leqq R_0$) satisfies the Palais-Smale condition. To check this, as in the proof of Theorem 3.1 we need only note that whenever $u_n \to u$ weakly in H, $u_n \in \partial A_R$, and grad $Au_n \to$ grad Au strongly in H, then $u_n \to u$ strongly in H, so that $u \in \partial A_R$. Now the critical points of $B(u)$ on A_R are the solutions of grad $Au = \lambda L_1 u$ for some λ with $\|u\|^2 \leqq R$, and so the stated result follows.

In order to explain Figure 2 the following result, analogous to Theorem 3.1, is of interest. (We assume for simplicity that $L_1 u = 0$ implies $u = 0$.)

THEOREM 3.4. *The solutions of the system* (3.3′) *have the following properties*:

(i) *Let* λ_1^* *be the smallest positive eigenvalue of the problem* $u = \lambda L_1 u$. *Then the potential energy of a nontrivial solution of* (3.3) *is strictly positive for* $\lambda < \lambda_1^*$. *Hence the shell will not buckle for* $\lambda < \lambda_1^*$.

(ii) *Let* λ_n *denote any eigenvalue of* $u + k^2L^2u = \lambda L_1 u$. *Then for* R *sufficiently small, the system* (3.3) *has a one-parameter family of solutions* $(u_n(R), \lambda_n(R)) \to (0, \lambda_n)$ *as* $R \to 0$ *where* $u_n(R) \in \partial A_R$, *for each* R.

PROOF. (i) The potential energy of a solution of (3.3) is proportional to

$$V(u) = \|u\|^2 + \tfrac{1}{4}C(u,u)^2 + k(C(u,u), Lu) + k^2\|Lu\|^2 - \lambda(L_1u, u)$$
$$= \|u\|^2 + \|\tfrac{1}{2}C(u,u) + kLu\|^2 - \lambda(L_1u, u).$$

Now if $\lambda \leqq \lambda_1^*$ then $\|u\|^2 - \lambda(L_1u, u) \geqq 0$, so $V(u) \geqq 0$.

(ii) This result follows from the bifurcation theorem of Appendix A. The solutions are obtained by comparing the critical values of $B(u)$ on ∂A_R ($R \leqq R_0$) with the critical values of $B(u)$ on the set $\partial S_R = \{u \mid \|u\|^2 + k^2\|Lu\|^2 = R\}$. The latter critical values are precisely the numbers $R\lambda_n^{-1}$.

REMARK. A global one-parameter family of solutions for (3.3′) can be found as the critical points of the (conjugate) variational problem:

Minimize $A(u)$ subject to the constraint $B(u) = R$. However the existence of global families of solutions as in Theorem 3.1 is as yet unknown for $k \neq 0$.

Appendix — Proof of the fact that $\lambda_1(R) \to \lambda_1$ in Theorem 3.1(2). Let $c_1(R) = \sup (Lu, u)$ over ∂A_R and $\tilde{c}_1(R) = \sup (Lu, u)$ over $\partial \Sigma_R$, where $\partial \Sigma_R = \{u \mid \|u\|_H^2 = R\}$. Now the sets $\partial \Sigma_R$ and ∂A_R are homeomorphic by means of the natural mapping defined by rays through the origin. Thus if $u \in \partial \Sigma_R$, then there is a unique positive real number t such that $tu \in \partial A_R$ and $t = 1 + o(R)$. Consequently,

$$(\dagger) \qquad c_1(R) = \sup_{\partial \Sigma_R} t^2(Lu, u) \quad \text{and} \quad |c_1(R) - \tilde{c}_1(R)| = o(R).$$

Next we note that if $\sup (Lu, u)$ over ∂A_R is attained at $\bar{u}$, then $\bar{u}$ satisfies $u + Cu = \lambda_1(R)Lu$. Thus

$$\begin{aligned} c_1(R) &= \lambda_1{}^{-1}(R)\{(\bar{u}, \bar{u}) + (C\bar{u}, \bar{u})\} \\ &= \lambda_1{}^{-1}(R)\{R + \tfrac{1}{2}(Cu, u)\} = \lambda_1{}^{-1}(R)\{R + O(R)\}. \end{aligned}$$

Since $\tilde{c}_1(R) = \sup_{\partial \Sigma_R} (Lu, u) = R\lambda_1{}^{-1}$, $(\dagger)$ implies that

$$|\lambda_1{}^{-1}(R) - \lambda_1{}^{-1}| = o(1) \quad \text{as } R \to 0.$$

Consequently $\lambda_1(R) \to \lambda_1$ as $R \to 0$.

Lecture 4. Compactness lost and compactness regained. In our final lecture, we consider two special nonlinear eigenvalue problems in which certain compactness properties are absent. In each case, under certain special circumstances, we shall show that these compactness properties may be regained. This leads one to suspect that many of the abstract results, mentioned in the introduction, can be extended to a class more general than those of the form $Ax = \lambda Bx$ with B compact. (Such results are well known for bifurcation theory, but not for more global problems.)

We shall however restrict attention to problems involving gradient mappings. This limitation enables us to readily discern the essential difficulties in each example and in addition to distinguish two types of loss of compactness: strong compactness (illustrated by the first problem) and weak compactness (illustrated by the second). Of course there is a large body of work attempting to extend the Leray-Schauder degree theory to noncompact perturbations of the identity. We shall not mention applications of this work here, since it will be discussed by other speakers.

The first problem discussed points out the difficulties of studying nonlinear problems on unbounded domains when nonuniqueness is the main object of study. The second problem has a long and interesting history dating back to studies of Klein and Poincaré on the uniformization theorem for Riemann surfaces. In each problem studied, necessary and sufficient conditions for solvability will be obtained. This fact points up the possibility of a precise abstract theory for this class of problems.

Case I. Stationary states for nonlinear wave equations. We seek complex-valued solutions $u(x, t)$ of the following nonlinear wave equation defined on $R^N \times R^1$ $(N > 2)$

$$u_{tt} = \Delta u - f(x, |u|^2)u \tag{4.1}$$

of the form $u(x, t) = e^{i\lambda t}v(x)$. Here $x \in R^N$, λ is a real number, and $v(x)$ is a real-valued exponentially decaying function of x. Thus we seek solutions $(v(x), \lambda)$ of

$$\Delta v + \lambda^2 v - f(x, |v|^2)v = 0 \quad \text{on } R^N \tag{4.2}$$

such that $|v| \to 0$ exponentially as $|x| \to \infty$. Suppose for simplicity that $f(x, |v|^2) = m^2 - |v|^\sigma$, then we prove the following [12]:

THEOREM 4.1. *If* $f(x, |v|^2) = m^2 - |v|^\sigma$, *then if* $\beta = m^2 - \lambda^2$,

(i) *for* $0 < \sigma < 4/(N-2)$ *and each* $\beta > 0$, (4.1) *has a countably infinite number of distinct stationary states,*

(ii) *for* $\sigma \geqq 4/(N-2)$, (4.1) *has no nontrivial stationary state for any* λ.

To prove (i) we restrict attention to radially symmetric states of the form $u(x, t) = e^{i\lambda t}v(|x|)$, $|x| = r$ and $w(r) = r^{(N-1)/2}u(|x|)$. After a simple computation one finds that the stationary states of (4.1) are in (1-1) correspondence with the nontrivial solutions of the following equation on $[0, \infty)$:

$$w_{rr} - \left(\beta + \frac{(N-3)(N-1)}{4r^2} \right) w + \frac{|w|^\sigma}{r^\sigma} w = 0, \tag{4.3}$$

$$w(0) = w(\infty) = 0. \tag{4.4}$$

One then considers the Hilbert space $\dot{W}_{1,2}[0, \infty)$ with inner product $(w, \tilde{w}) = \int_0^\infty (w_r \cdot \tilde{w}_r + w\tilde{w})\, dr$. Now we note that the map $\mathcal{N}$: $\dot{W}_{1,2}(0, \infty) \to \dot{W}_{1,2}(0, \infty)$ defined implicitly by the formula

$$(\mathcal{N} w, \phi) = \int_0^\infty \frac{|w|^\sigma}{|r|^\sigma} w\phi\, dr \qquad (0 < \sigma < 4/(N-2))$$

is compact; whereas the map $\mathcal{P}$: $\dot{W}_{1,2}(R^N) \to \dot{W}_{1,2}(R^N)$ defined by

$(\mathcal{P}u, v) = \int_{R^N} |u|^\sigma uv$ is not compact. Thus we have the first instance of compactness lost and regained.

Suppose, for simplicity, that $N = 3$. Now after scaling by a constant factor we find that the solutions of (4.3) are in (1-1) correspondence with the critical points of the isoperimetric variational problem (π): Find the critical points of the functional $J(w) = \int gr^{-\sigma}|w|^{\sigma+2}\, dr$ subject to the constraint $\Sigma_R \equiv \int_0^\infty (w_r^2 + \beta w^2)\, dr =$ constant (R say). Thus $J(w)$ is weakly continuous when restricted to the sphere Σ_R in the $\mathring{W}_{1,2}(0, \infty)$ topology, so that (π) satisfies the "compactness" hypotheses necessary for the application of the general critical point theories of Ljusternik-Schnirelmann. Consequently making use of the antipodal symmetry of (π) and the results of the introduction, we obtain the desired fact (i).

To prove (ii) we need the following:

LEMMA 4.2. *Any solution $v(x)$ (vanishing exponentially at ∞) of $\Delta v + f(v) = 0$ in R^N satisfies the identity*

$$\left(\frac{2N}{N-2}\right)\int_{R^N} F(v) = \int_{R^N} vf(v)$$

where $F'(s) = f(s)$.

PROOF. The function $v(x)$ is a critical point of $\mathcal{I}(u(x)) = \int_{R^N} [\frac{1}{2}|\nabla u|^2 - F(u)]\, dx$. Thus as a function of c, $(d/dc)\, \mathcal{I}(v(cx))|_{c=1} = 0$. After a simple calculation and a change of variables, we find

$$\mathcal{I}(v(cx)) = \tfrac{1}{2}c^{2-N}\int_{R^N} |\nabla v|^2 - c^{-N}\int_{R^N} F(v),$$

so

$$(N-2)\int_{R^N} |\nabla v|^2 = 2N\int_{R^N} F(v).$$

Since also $\int_{R^N} |\nabla v|^2 = \int_{R^N} f(v)v$, the result follows.

Now (ii) follows immediately from Lemma 4.2. Indeed in the present case $f(v) = -\beta v + |v|^\sigma v$ so that $F(v) = -\frac{1}{2}\beta v^2 + (1/(\sigma+2))|v|^\sigma v^2$. Hence, by the lemma,

$$\beta\left(\frac{2N}{N-2} - 1\right)\int_{R^N} v^2 = \left[\left(\frac{2N}{N-2}\right)\left(\frac{1}{\sigma+2}\right) - 1\right]\int_{R^N} |v|^\sigma v^2. \tag{4.5}$$

Now (4.5) implies that for a nontrivial stationary state, with $\beta > 0$,

$$1/(\sigma + 2) > (N - 2)/2N \quad \text{or} \quad \sigma < 4/(N - 2).$$

Furthermore if $\beta = 0$, the lemma implies $\sigma = 4/(N - 2)$. An application of the Kelvin transformation to transform the point at infinity to the origin and the unique continuation theorem shows that the resulting equation (4.2) has no exponentially decaying solutions. Thus there remains the possibility that $\beta < 0$. However the possibility of a nontrivial solution in this case can be ruled out by applying the results of Kato [28] on the reduced wave equation to (4.2).

Case II. A nonlinear eigenvalue problem in global differential geometry. We consider the following problem:

(π) Let $(\mathcal{M}, g)$ denote a smooth compact two-dimensional manifold equipped with some Riemannian metric g. We seek sufficient conditions for $\mathcal{M}$ to admit a Riemannian metric $\bar{g}$ conformally equivalent to g with arbitrarily prescribed Gauss curvature $K(x)$, assumed Hölder continuous on $(\mathcal{M}, g)$.

Clearly, in this form (π) does not seem to be a nonlinear eigenvalue problem. However we can formulate a semilinear elliptic partial differential equation for $\sigma \in C^2(\mathcal{M}, g)$ by setting $\bar{g} = e^{2\sigma}g$. Indeed, in terms of isothermal parameters (u, v) on $\mathcal{M}$ with $ds^2 = \gamma'\{du^2 + dv^2\}$ the Gaussian curvature can be written

$$K = -\tfrac{1}{2}\gamma'^{-1}\{(\log\gamma')_{uu} + (\log\gamma')_{vv}\} \tag{4.6}$$

so that setting $\gamma' = \gamma \exp 2\sigma$ in (4.6), we find the following equation for the unknown function σ:

$$\Delta\sigma - k(x) + K(x)e^{2\sigma} = 0, \tag{4.7}$$

where Δ is the Laplace-Beltrami operator relative to g on $\mathcal{M}$, and $k(x)$ is the associated Gaussian curvature of $(\mathcal{M}, g)$. Now (4.7) is a nonlinear eigenvalue problem in the following sense:

Lemma 4.3. *If the Euler characteristic of $\mathcal{M}$, $\chi(\mathcal{M}) \neq 0$, then the solutions of* (4.7) *are in* (1-1) *correspondence with the critical points of*

$$\mathcal{G}(\sigma) = \int_{\mathcal{M}} \left(\tfrac{1}{2}|\nabla\sigma|^2 + k(x)\sigma\right) dV$$

subject to the constraint $F(u) \equiv \int K(x)e^{2\sigma} = 2\pi\chi(\mathcal{M})$.

Proof. A smooth critical point u of the isoperimetric problem satisfies the Euler equation

$$(*) \qquad \Delta u - k(x) + \beta K(x)e^{2u} = 0$$

where β is some constant. To determine β, we integrate $(*)$ over $\mathcal{M}$ to find $\int_{\mathcal{M}} k(x)\, dV = \beta \int_{\mathcal{M}} K(x)e^{2u}\, dV$. Thus, since $\chi(\mathcal{M}) \neq 0$, $\beta = 1$, so that any solution of $(*)$ satisfies (4.7).

In order to demonstrate the existence of critical points for the isoperimetric variational problem described above, it is convenient to restrict the admissible class C to an appropriate Hilbert space. To this end, we denote by $W_{1,2}(\mathcal{M}, g)$ the set of functions $u(x)$ defined on $\mathcal{M}$ such that (relative to the Riemannian structure g) u and $\nabla u = \operatorname{grad} u$ are square integrable over $\mathcal{M}$. $W_{1,2}(\mathcal{M}, g)$ is a Hilbert space relative to the inner product

$$(u, v)_{1,2} = \int_{\mathcal{M}} uv\, dV + \int_{\mathcal{M}} \nabla u \cdot \nabla v\, dV.$$

Now one can prove the following result for manifolds $(\mathcal{M}, g)$ with $\chi(\mathcal{M}) < 0$, using the fact [13] that the functional $F(u)$ is weakly continuous in the appropriate Hilbert space $W_{1,2}(\mathcal{M}, g)$.

THEOREM 4.4. *If $\chi(\mathcal{M}) < 0$, then the problem (π) is solvable for any function $K(x) < 0$ on $\mathcal{M}$.*

The reader is referred to the proof in [14].

It is rather surprising that the analogue of Theorem 4.4 is unproven for simply connected $\mathcal{M}$ (i.e. $\chi(\mathcal{M}) > 0$) due to lack of compactness. To see this we argue as follows for $(\mathcal{M}, g) = (S^2, g_1)$, the sphere with metric of constant curvature 1: In accord with Lemma 4.3 we consider minimizing $F(u) = \int_{S^2} (\frac{1}{2}|\nabla\sigma|^2 + k(x)\sigma)\, dV$ over the class $W_{1,2}(S^2, g_1)$ subject to the constraint:

$$(*) \qquad \int_{S^2} K(x)e^{2\sigma} = 4\pi.$$

Set $\sigma = \sigma_0 + \sigma_m$ where $\mu(\mathcal{M})\sigma_m = \int_{\mathcal{M}} \sigma$ and $\int_{\mathcal{M}} \sigma_0 = 0$. Then $(*)$ implies $\log \int K(x)e^{2\sigma_0} = -2\sigma_m$. Thus

$$\inf_{S} F(u) \geqq \inf_{W_{1,2}(S^2, g_1)} \left\{ \tfrac{1}{2} \int |\nabla u_0|^2 - 2\pi \log \int_{S^2} K(x)e^{2\sigma_0} - c \right\}.$$

Since $K(x)$ is bounded, it suffices to bound $\log \int e^{2u_0}$ in terms of $\int |\nabla u_0|^2 = \|u_0\|^2$. To this end, set $2u_0 = 2\|u_0\|v \leqq 4\pi v^2 + \|u_0\|^2/4\pi$ and use the fact proved by Moser [15] that $\sup_{\|v\|=1} \int \exp(4\pi v^2)$

$\leqq c_1$ (where the constant 4π is sharp). Thus

$$\inf_S F(u) \geqq \inf_{\bar{W}_{1,2}(S^2, g_1)} \left\{ \bar{c} + \tfrac{1}{2}\|u_0\|^2 - 2\pi \left[\log \int \exp\left(4\pi v^2 + \frac{1}{4\pi}\|u_0\|^2\right)\right]\right\}$$

$$\geqq \text{const} > -\infty .$$

However, *due to a lack of weak compactness,* to date no one has been able to show precisely when $\inf_S F(u)$ is attained, even though it is bounded from below. (Although recently, J. Moser [30] showed that $\inf_S F(u)$ is attained if $K(-x) = K(x)$.) Hence we arrive at the general question of why, in general, nonlinear eigenvalue problems in global differential geometry are more easily resolved for complicated nonsimply connected manifolds $\mathscr{M}$ than for relatively simple simply connected ones.

Now in the intermediate case when $\chi(\mathscr{M}) = 0$, we shall show that we can regain weak compactness by the addition of a simple explicit constraint in the isoperimetric problem of Lemma 4.3. In fact, we shall prove the following sharp result.

THEOREM 4.5 ([14], [23]). Suppose $\chi(\mathscr{M}) = 0$. *Then the problem* (π) *is solvable if and only if either* $K(x) \equiv 0$ *or* $K(x)$ *changes sign on* $\mathscr{M}$ *and* $\int_{\mathscr{M}} K(x) \exp(2u_0)\, dV < 0$, *where* u_0 *is any solution of* $\Delta u = k(x)$ *on* $\mathscr{M}$.

PROOF OF NECESSITY. If u satisfies (4.7) and $\chi(\mathscr{M}) = 0$, then $\int_{\mathscr{M}} K(x) \exp(2u)\, dV = 0$. Thus if $K(x)$ is not identically zero, $K(x)$ must change sign on $\mathscr{M}$. On the other hand, if we set $u = u_0 + w$, the function w satisfies the equation

$$\Delta w + K(x) \exp(2u_0 + 2w) = 0.$$

Multiplying this equation by $\exp(-2w)$, integrating over $\mathscr{M}$ and integrating by parts, we find

$$2 \int_{\mathscr{M}} \exp(-2w) |\nabla w|^2\, dV = -\int_{\mathscr{M}} K(x) \exp(2u_0)\, dV > 0.$$

PROOF OF SUFFICIENCY. First we prove an analogue of Lemma 4.3 for the case $\chi(\mathscr{M}) = 0$.

LEMMA 4.6. *Suppose* $\chi(\mathscr{M}) = 0$ *and* $K(x)$ *is a given function defined on* $\mathscr{M}$ *such that relative to some Riemannian metric* g *defined on* $\mathscr{M}$, $\int_{\mathscr{M}} K(x) \exp(2u_0)\, dV < 0$. *Then the (smooth) critical points of the functional* $\mathscr{G}(u)$ *subject to the constraint*

$$S' = \left\{ u \mid u \in W_{1,2}(\mathcal{M}, g), \int_{\mathcal{M}} u \, dV = 0, \int_{\mathcal{M}} K(x)e^{2u} \, dV = 0 \right\}$$

are (apart from a constant) solutions of the equation

$$\Delta u - k(x) + K(x)e^{2u} = 0$$

where $k(x)$ is the Gauss curvature of $(\mathcal{M}, g)$.

PROOF. A smooth critical point u of the isoperimetric variational problem satisfies the Euler equation

$$(\dagger) \qquad \Delta u - k(x) + \beta_1 K(x)e^{2u} = \beta_2$$

where β_1 and β_2 are constants. Since $\int_{\mathcal{M}} K(x) \exp(2u_0) \, dV \neq 0$, both β_1 and β_2 cannot be zero. To show that $\beta_2 = 0$, we integrate $(\dagger)$ over $\mathcal{M}$ and find

$$\int_{\mathcal{M}} k(x) \, dV + \beta_1 \int_{\mathcal{M}} K(x)e^{2u} \, dV = \beta_2 \mu(\mathcal{M}).$$

Since $\int_{\mathcal{M}} k(x) \, dV = 0$, and $u \in S'$, $\beta_2 = 0$. Since $\beta_1 \neq 0$, there is a constant c such that $\pm \exp(2c) = \beta_1$. Hence $\bar{u} = u + c$ satisfies

$$\Delta \bar{u} - k(x) \pm K(x)e^{2\bar{u}} = 0.$$

Now we show that $\beta_1 > 0$ so that $\beta_1 = \exp(2c)$, and consequently $\bar{u} = u + c$ satisfies equation (4.7). Set $u = u_0 + w$ in $(\dagger)$. Then by hypothesis, since $\beta_2 = 0$,

$$\Delta w + \beta_1 K(x) \exp(2u_0) \exp(2w) = 0.$$

Again multiplying by $\exp(-2w)$, integrating over $\mathcal{M}$, and integrating by parts, we find

$$\int_{\mathcal{M}} \exp(-2w) |\nabla w|^2 \, dV = -\beta_1 \int_{\mathcal{M}} K(x) \exp(2u_0) \, dV.$$

Thus $\beta_1 > 0$ since $w \neq 0$.

To prove the existence of a critical point for this variational problem, we set $\sigma = \sigma_0 + \sigma_m$ so that

$$\mathcal{G}(\sigma) = \tfrac{1}{2} \int_{\mathcal{M}} (|\nabla \sigma_0|^2 + k(x)\sigma_0) \, dV \quad \left(\text{since } \int_{\mathcal{M}} k(x) \, dV = 0\right)$$
$$\geqq \tfrac{1}{2} \|\sigma_0\|^2 - c\|k(x)\| \, \|\sigma_0\|.$$

Consequently, $\mathcal{G}(\sigma_0) \to \infty$ for $\sigma \in S'$ as $\|\sigma_0\| \to \infty$, and $\mathcal{G}(\sigma)$ is weakly lower semicontinuous with respect to weak convergence in $W_{1,2}(\mathcal{M}, g)$. Furthermore S' is weakly closed. Thus $\inf \mathcal{G}(\sigma)$ over

S' is attained by an element $u \in S'$, and u is a weak solution of the equation (4.7) in the space $W_{1,2}(\mathcal{M}, g)$. Therefore u is a solution of a linear equation of the form $\Delta u = f$ with $f \in L_p$ for all finite $p > 1$. It follows that u is smooth enough to satisfy equation (4.7) in the classical sense, and the theorem is thereby proved.

Appendix A. A bifurcation theorem for gradient operators. We consider the operator equation

$$f(x, u) \equiv (I - \lambda L)x + T(x, \lambda) = 0 \tag{A.1}$$

defined on a real Hilbert space H. Suppose that L is a selfadjoint bounded operator mapping H into itself, and $T(x, \lambda)$ is a real C^2 higher order gradient mapping, so that $T(0, \lambda) = T_x(0, \lambda) \equiv 0$ and $T(x, \lambda)$ is the Fréchet derivative (w.r.t. x) of the real-valued functional $\mathcal{T}(x, \lambda)$. We now state and refer to the author's recent paper in Proc. Nat. Acad. Sci. U.S.A. **69** (1972), 1737–1738.

THEOREM. *Suppose that the linear operator $f_x(0, \lambda_0) = I - \lambda_0 L$ is a linear Fredholm operator and is not invertible in H. Then* $(0, \lambda_0)$ *is a point of bifurcation relative to the equation* (A.1).

SKETCH OF THE PROOF. There are two main ideas that are essential. First, the well-known observation that the solutions of (A.1) near $(0, \lambda_0)$ are in (1-1) correspondence with the solutions of the "bifurcation equations" for the problem. Relative to (A.1), these equations can be written in the form

$$h(u, \lambda) \equiv (I - \lambda L)u + PT(u + g(u, \lambda), \lambda) = 0, \tag{A.2}$$

where $u \in \operatorname{Ker}(I - \lambda_0 L)$, P is the standard projection of $H \to \operatorname{Ker}(I - \lambda_0 L)$, and $g(u, \lambda)$ is a real C^2 higher order mapping of $H \to [\operatorname{Ker}(I - \lambda_0 L)]^{\perp}$. A further observation is that the operator $h(u, \lambda)$ is also a gradient operator mapping $\operatorname{Ker}(I - \lambda_0 L)$ into itself. Indeed one easily shows that $\mathcal{T}_u(u + g(u, \lambda), \lambda) = PT(u + g(u, \lambda), \lambda)$.

The second idea in the proof is the use of the type numbers of M. Morse [29, p. 149] to describe the isolated critical point of a real-valued function $H(u, \lambda)$ of a finite number N of real variables. If $(0, \lambda)$ is an isolated nondegenerate critical point of $H(u, \lambda)$, its type number M_λ is its Morse index, the number of negative eigenvalues (counted with multiplicity) of the Hessian matrix $H_{uu}(0, \lambda)$. If $(0, \lambda_0)$ is degenerate, its type number $M_{\lambda_0} = (m_0(\lambda_0), \cdots, m_N(\lambda_0))$ is an $(N + 1)$-vector of integers (and in fact are the Betti numbers of certain local homology groups). These numbers are so assigned that if $H(u, \lambda)$ is C^3 and all the critical points of $H(u, \lambda)$ are nondegenerate, for fixed λ in a small deleted neighborhood V of λ_0, then there are at least $m_k(\lambda_0)$ nondegenerate critical points of index k of $H(u, \lambda)$ near $u = 0$ for each λ in V.

With these preliminaries the proof is easily carried out. Suppose $(0, \lambda_0)$ is not a point of bifurcation of (A.1). Then $(0, \lambda_0)$ is not a point of bifurcation of (A.2). Thus there is a small spherical neighborhood of $(0, \lambda_0)$ such that for fixed λ, $(0, \lambda)$ is an isolated critical point of $H(u, \lambda) = (u + g(u, \lambda), \lambda)$. Since $\det |h_u(0, \lambda)| \neq 0$ for λ in a small deleted neighborhood V_1 of λ_0, $(0, \lambda)$ is nondegenerate for $\lambda \in V_1$. Furthermore, a simple computation shows that the type number $m_N(\lambda)$ of $(0, \lambda)$ relative to $H(u, \lambda)$ is 0 for $\lambda < \lambda_0$ and unity for $\lambda > \lambda_0$. On the other hand, $(0, \lambda_0)$ is an isolated degenerate critical point of $H(u, \lambda_0)$ and its type number is inconsistent with these facts since type numbers are homotopy invariant. This is the desired contradiction and the result is thus proved.

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