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# STABILITY AND BIFURCATION IN FLUID DYNAMICS

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1. **Introduction.** During the last decade some of the instability phenomena in fluid dynamics discovered in the famous early experiments of Bénard [3] and Taylor [59] have found a mathematical interpretation in the proof that the stationary Navier-Stokes boundary-value problem is not in general uniquely solvable. Even for moderate values of the underlying similarity parameter (Reynolds number, Rayleigh number), infinitely many solutions exist which bifurcate from some basic flow at definite values of this parameter. The mathematical interest in this subject was stimulated by the paper of Velte [61], who proved by topological arguments the existence of bifurcating solutions for a certain convection problem. Later, Iudovich obtained the most complete results yet known for the Bénard and the Taylor problem ([21]—[24]). Closely connected to bifurcation phenomena is the question of which, among the many solutions, is the one actually observed. This is generally believed to be a stability problem. The fundamental result relating spectral properties of the Stokes equations and the stability of solutions of the Navier-Stokes equations is due to Prodi [45].

Simultaneous with the above-mentioned research, considerable effort was spent on the development of analytical methods for the study of those problems. Stuart proposed an amplitude expansion by which many of the results known today were obtained before they were rigorously proved [14]. Only recently has his method found mathematical justification to some extent [19].

This survey covers exclusively the mathematical part of bifurcation and stability in fluid dynamics, although other results are mentioned for comparison and stimulation of research. Another branch is excluded as well, time periodic motions of Tollmien-Schlichting type, since no concrete models are known for the theoretical results published recently.

The conception of this survey is to display the common functional analytic structure of the Taylor and Bénard problem. Most of the bifurcation models studied in fluid dynamics have the same structure.

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Proofs are omitted where we believe them to be easily accessible in literature. Some new results on the class of possible solutions for the Bénard problem (Lemma 2.1 and Corollary 2.2), and on stability and instability (Theorem 5.5 and Corollary 5.4) are reported.

In §2 the explicit equations for the Taylor and the Bénard models are given and, in the latter case, all possible solutions with cell structure are classified. In §3 a joint formulation of the two problems as an evolution equation is given (see (3.22)), and the properties of the operators appearing in (3.22) are listed. §4 contains a discussion of the stationary problem (Theorem 4.1), an extension to more general nonlinearities (Theorem 4.2), and special applications to the Taylor and Bénard models. In §5, stability and instability of a stationary solution is connected to the spectrum of the Stokes problem (Theorem 5.5 and Corollary 5.4). Special applications follow. Finally we discuss the selection of certain cell sizes by a stability argument (Theorem 5.8).

2. **Two examples.** In this section two examples are given which exhibit features typifying the bifurcation and stability phenomena encountered in fluid mechanics. These models are the "Taylor" and the "Bénard" model. The latter will be studied within the degree of accuracy ensured by the Boussinesq approximation. (For a mathematical justification of this approximation, cf. Fife [**15**].) However, the Hilbert space formulation of the next section includes other models as well.

*The Taylor model.* Two coaxial circular cylinders of infinite length with radii $r_1'$ and $r_2'$ ($r_1' < r_2'$) rotate with constant angular velocities $\omega_1$ and $\omega_2$. Due to the viscosity, an incompressible fluid rotates in the gap between the cylinders. With the notations $\rho$ and $\nu$ for the mass density and the kinematic viscosity, we introduce reference quantities, namely

$$r_2' - r_1', \quad r_1'\omega_1, \quad (r_2' - r_1')^2/\nu, \quad \rho r_1'^2\omega_1^2$$

for length, velocity, time and pressure. Let

$$(2.1) \qquad \lambda := \frac{r_1'\omega_1(r_2' - r_1')}{\nu}, \qquad r_i := \frac{r_i'}{r_2' - r_1'}, \qquad i = 1, 2,$$

denote the dimensionless parameters; $\lambda$ is the Reynolds number.

The Navier-Stokes equations for incompressible flows (cf. [54]) govern the motion of the fluid. Introducing $r, \varphi$, and $z$, the corresponding velocity components for $v = (v_r, v_\varphi, v_z)$, and the scalar pressure function $p$, one obtains a solution independent of $\lambda$, called the Couette flow:

$$v^0 = (0,\ v_\varphi{}^0,\ 0), \qquad p^0(r) = \int_{r_1}^{r} \frac{v_\varphi{}^0(\rho)^2}{\rho} d\rho + \text{const},$$

$$(2.2) \quad v_\varphi{}^0 = ar + b/r, \qquad a = \frac{1}{r_1\omega_1} \frac{\omega_2 r_2{}^2 - \omega_1 r_1{}^2}{r_2{}^2 - r_1{}^2},$$

$$b = \frac{1}{r_1\omega_1} \frac{(\omega_1 - \omega_2) r_1{}^2 r_2{}^2}{r_2{}^2 - r_1{}^2}.$$

For small positive $\lambda$ this solution is unique and asymptotically stable in the large [53]. As shown by Synge in [58] the Couette flow is locally stable if $a \geqq 0$, $v_\varphi{}^0 \geqq 0$ (cf. also [30]). If $v_\varphi{}^0$ changes sign, the spectrum of the linearized part of the boundary-value problem is as yet unknown. Thus, we restrict attention to the case $a < 0$, $v_\varphi{}^0 \geqq 0$.

As $\lambda$ grows, various types of fluid motions are observed, the simplest of which is periodic in $z$ and independent of $\varphi$ (Taylor vortex flow). Existence and quantitative behaviour of this flow can be derived by a bifurcation argument. We restrict the considerations to $\varphi$ independent, $z$-periodic solutions of the Navier-Stokes equations (for more general flows see [10]), and require that the solution $v$ be invariant under the group of translations $T_1$ generated by $z \to z + 2\pi/\sigma$, $\varphi \to \varphi + 2\pi$, $\sigma > 0$. For later use the initial-value problem is formulated for a general "basic" flow $V = (V_r, V_\varphi, V_z)$, $P$. Setting

$$v = V + u, \qquad p = P + q,$$

$$D_t = \frac{\partial}{\partial t}, \qquad \nabla = \left(\frac{\partial}{\partial r}, 0, \frac{\partial}{\partial z}\right),$$

$$\Delta = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

$$(2.3) \quad \begin{aligned} \tilde{\Delta}_{ik} &= (\Delta - (1 - \delta_{i3})/r^2)\delta_{ik}, \\ L^0_{ik}(V) &= -2V_\varphi \delta_{i1}\delta_{2k}/r + (V_r\delta_{2k} + V_\varphi \delta_{1k})\delta_{i2}/r, \\ L(V)u &= L^0(V)u + (V \cdot \nabla)u + (u \cdot \nabla)V, \\ Q(u)_i &= -\, u_\varphi{}^2\delta_{i1}/r + u_\varphi u_r \delta_{i2}/r, \\ N(u) &= (u \cdot \nabla)u + Q(u), \qquad i, k = 1, 2, 3, \end{aligned}$$

where $i = 1, 2, 3$ corresponds to $r, \varphi, z$, one obtains the following initial-value problem (cf. [54]):

$$\text{(2.4)}\quad \begin{aligned} &\text{(a)}\quad D_t u - \tilde{\Delta} u + \lambda L(V) u + \lambda \nabla q = -\lambda N(u), \\ &\text{(b)}\quad \nabla \cdot u = 0, \qquad u|_{r=r_1,r_2} = 0, \qquad u(Tx, t) = u(x, t), \qquad T \in T_1, \\ &\text{(c)}\quad u|_{t=0} = u^0. \end{aligned}$$

For $V = v^0$ the linear differential operator $L(V)$ is denoted by $L$.

$$\text{(2.5)}\quad \begin{aligned} L &= L(v^0), \\ L_{ik} &= -2v_\varphi{}^0 \delta_{i1}\delta_{2k}/r + 2a\delta_{i2}\delta_{1k}. \end{aligned}$$

Ample experimental evidence shows that for Reynolds numbers $\lambda$ greater than some $\lambda_1$, the Couette flow (2.2) becomes unstable, and a new rotationally symmetric stationary flow is observed, which is periodic in the axial direction with a well-defined period $p_0 = 2\pi/\sigma_0$. The particle trajectories are on tori forming a system of closed vortices (Taylor vortices). These vortices become again unstable for sufficiently large $\lambda$ and a "wavy" vortex pattern appears, the wave number depending on $\lambda$. Further increase of $\lambda$ generates more and more complex flow patterns and eventually turbulence (cf. [**10**], [**8**]).

Mathematically, only the simplest part of this phenomenon is understood, namely the appearance of the vortex flow. The linearized stationary problem (LSP), obtained from (2.4) by setting $D_t = 0$, $V = v^0$, deleting (2.4)(c) and the right side of (2.4)(a), has, for almost all $\sigma > 0$, countably many real values $\lambda_j$, $j = \pm 1, \pm 2, \cdots$, for which 0 is a simple eigenvalue of the LSP; furthermore, $\lambda_{-j} = \lambda_j$, $|\lambda_{j+1}| < |\lambda_j|$. Every $(\lambda_j, 0)$ (0 denotes the zero element of the underlying function space) is a point of bifurcation for the stationary problem ($D_t = 0$ in (2.4)). However, the only bifurcation point of physical significance is $(\lambda_1, 0)$. Two nontrivial solutions of the stationary problem emanate from this point. They are Taylor vortex flows differing only by the orientation of the trajectories along the torus. Stability and instability are determined by the geometric behaviour of the branches. Branching to the right (left) implies stability (instability) within the class of solutions of (2.4) for fixed $\sigma$ (§5).

The selection of a well-defined period $p_0$, as observed in experiments, depends on the shape of the curve $\lambda_1(\sigma)$. If $\sigma$ is not a local minimum of $\lambda_1(\sigma)$, the solution $u$ can be embedded into a function space where it is unstable (§5). Thus, if $\lambda_1(\sigma)$ is strictly convex, an assumption strongly supported by numerical calculations, the local minimum $\sigma_0$ gives the only stable mode of the secondary flow.

*The Bénard model.* The second problem to be studied in detail is quite analogous to the Taylor model. However, the class of possible flow patterns is richer and can be characterized algebraically (Lemma

2.1). Furthermore some more information about the bifurcation behaviour can be obtained since the linearized stationary problem is selfadjoint.

A viscous fluid in the strip between two horizontal planes moves under the influence of the viscosity and the buoyancy force, where the latter is caused by heating the lower plane. Denoting the constant temperatures of the upper, resp. lower, plane by $T_1$, resp. $T_0$, $T_0 > T_1$, the gravity force generates a pressure distribution which, for small values of $T_1 - T_0$, is balanced by the viscous stress, resulting in a linear temperature distribution. If, however, $T_1 - T_0$ is above a certain critical value a convection motion is observed.

The convection takes place in a regular pattern of closed cells having the form of strips (rolls) or hexagons. The actual pattern observed depends strongly on the physical conditions imposed, especially on whether rigid or free boundaries are involved. For a good survey of experimental, resp. analytical, results see [36], resp. [51]. The shape of the cells is determined as the manifold where the normal component of the motion vanishes.

In this survey the Bénard problem is treated by using the so-called Boussinesq approximation whose essential feature is that variations in density are neglected throughout the equations except in the buoyant force term; furthermore all other physical quantities are considered to be independent of temperature. Perturbations of this model for small temperature variations have been given in [4], [37], [15], the lowest order approximation of which always coincides with the Boussinesq approximation. Thus, the points of bifurcation are unchanged; however, the qualitative behaviour of the branches may be rather different (see §5). A strict justification of the Boussinesq approximation for slow convection has been given in [15] under rather weak assumptions.

Mathematically, the class of stationary solutions of the Bénard problem contains all cell patterns which form a complete covering of the plane: rolls, triangles, rectangles, and hexagons. That these are the only doubly periodic solutions of cell structure is proven in Lemma 2.1. The preference for certain cell patterns cannot be explained within the Boussinesq approximation. Far-reaching numerical results obtained by power series expansions for models more physically realistic have to be used (see [4], [37] and §5).

Let $\alpha$, $h$, $g$, $\nu$, $\rho$, $\kappa$ be the coefficient of volume expansion, the thickness of the layer, the gravity, the kinematic viscosity, the density and the coefficient of thermometric conductivity respectively and $\nu/h$, $h$, $\rho\nu^2/h^2$, $h^2/\nu$, $T_0 - T_1$ be the reference quantities for velocity, length, pressure, time and temperature. The space vector $\boldsymbol{x} = (x_1, x_2, x_3)$ and the velocity vector $\boldsymbol{v} = (v_1, v_2, v_3)$ are given in Cartesian coordinates where the $x_3$-axis points opposite to the force of gravity, $\tilde{\boldsymbol{\theta}}$ denotes the temperature and $p$ the pressure. The basic stationary solution $\boldsymbol{v}_0$, $\boldsymbol{\theta}_0$, $p_0$ has the form:

$$(2.6)\qquad \boldsymbol{v}_0 = 0, \qquad p_0(x_3) = -\frac{gh^3}{\nu^2}(x_3 + \alpha(T_0 - T_1)), \qquad \boldsymbol{\theta}_0(x_3) = -x_3.$$

As for the Taylor model the initial-value problem for the convection flow is formulated for an arbitrary reference flow $\boldsymbol{V}$, $T$, $P$. Setting

$$(2.7)\qquad \begin{aligned}
&\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{u}, \qquad p = P + q, \qquad \tilde{\boldsymbol{\theta}} = T + \boldsymbol{\theta},\\
&\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{\theta}),\\
&\lambda = \alpha g(T_0 - T_1)h^3/\nu^2 \quad \text{(Grashoff number)},\\
&Pr = \kappa/\nu \quad \text{(Prandtl number)},\\
&\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3, 0),\\
&\tilde{\Delta}_{ik} = \left(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2}\right)\left(\delta_{ik} + \frac{1}{Pr}\,\delta_{i4}\right),\\
&\overset{0}{L}_{ik} = -\,\delta_{i3}\delta_{k4} - \frac{1}{Pr}\,\delta_{i4}\delta_{k3}, \qquad i, k = 1, \cdots, 4.\\
&L(\boldsymbol{V})\boldsymbol{w} = L^0\boldsymbol{w} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{V},\\
&N(\boldsymbol{w}) = (\boldsymbol{u}\cdot\nabla)\boldsymbol{w},
\end{aligned}$$

one obtains the following initial-value problem (cf. [**54**]):

$$(2.8)\qquad \begin{aligned}
&\text{(a)}\quad D_t\boldsymbol{w} - \tilde{\Delta}\boldsymbol{w} + \lambda L(\boldsymbol{V})\boldsymbol{w} + \nabla q = -N(\boldsymbol{w}),\\
&\text{(b)}\quad \nabla\cdot\boldsymbol{w} = 0, \qquad \boldsymbol{w}|_{x_3=0,1} = 0,\\
&\text{(c)}\quad \boldsymbol{w}|_{t=0} = \boldsymbol{w}^0.
\end{aligned}$$

For $\boldsymbol{V} = \boldsymbol{v}^0$, $P = p_0$, $T = \boldsymbol{\theta}_0$, the linear differential operator $L(\boldsymbol{V})$ is denoted by $L$ and coincides with $L^0$.

$$(2.9)\qquad L = L(\boldsymbol{v}^0) = L^0.$$

Problem (2.8) is not yet well posed; various invariance conditions have to be imposed to yield a definite cell pattern. Firstly, $\boldsymbol{w}$, $q$ has to be doubly periodic, i.e.,

$$(2.8\text{d})\qquad \begin{aligned}
&\boldsymbol{w}(Tx, t) = \boldsymbol{w}(x, t),\\
&q(Tx, t) = q(x, t), \qquad T \in \boldsymbol{T}_1, (x, t) \in \boldsymbol{R}^3 \times \boldsymbol{R}^+,
\end{aligned}$$

where $T_1$ is the group generated by the translations

$$x_1 \to x_1 + 2\pi/\alpha, \qquad x_2 \to x_2 + 2\pi/\beta, \qquad \alpha^2 + \beta^2 \neq 0.$$

Moreover, the cell pattern is determined by

$$\begin{aligned} &u(Tx, t) = Tu(x, t), && q(Tx, t) = q(x, t), \\ &\theta(Tx, t) = \theta(x, t), && T \in T_2, (x, t) \in R^3 \times R^+, \end{aligned} \tag{2.8e}$$

where $T_2$ is the group of rotations generated by

$$T_\omega = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \omega \in (0, 2\pi).$$

In addition to the boundary conditions (2.8b), $w$ is required to satisfy (2.8d) and (2.8e).

It is easy to show that all differential operators in (2.8a) preserve invariance under $T_1$ and $T_2$. We indicate the proof for the nonlinearity and for $T \in T_2$, using the summation convention:

$$\begin{aligned} [u(Tx, t) \cdot \nabla u(Tx, t)]_i &= u_k(Tx, t)u_{i,k}(Tx, t) \\ &= T_{ij}u_{j,k}(x, t)T_{mk}T_{mn}u_n(x, t) \\ &= [T(u(x, t) \cdot \nabla u(x, t))]_i, \\ u(Tx, t) \cdot \nabla\, \theta(Tx, t) &= u_k(Tx, t)\,\theta_{,k}(Tx, t) \\ &= \theta_{,k}(x, t)T_{mk}T_{mn}u_n(x, t) = u(x, t) \cdot \nabla\, \theta(x, t). \end{aligned}$$

The cases $\omega = 2\pi/n$, $n = 2, 3, 4, 6$, are of special interest. If $n = 2$ and $\beta = 0$, $u_1$ and $u_2$ vanish for $x_1 = n\pi/\alpha$, $n \in Z$, yielding solutions of roll type; $n = 2$ and $\alpha, \beta \neq 0$ implies that $u_1$ and $u_2$ vanish for $x_1 = n\pi/\alpha$, $x_2 = m\pi/\beta$, $n, m \in Z$, which gives rectangular cells. For $n = 3$, Corollary 2.2 yields that $\beta = \alpha\sqrt{3}$ and an easy geometric consideration gives hexagonal cells; $n = 4$ implies $\alpha = \beta$ and the cell pattern consists of squares; $n = 6$ yields again $\beta = \alpha\sqrt{3}$ and the cells are triangles. It can be shown that all nontrivial stationary solutions of (2.8) with cell structure are either rolls, hexagons, rectangles or triangles.

Lemma 2.1. *Necessary for the existence of nontrivial solutions of* (2.8) *is* $\omega = 2\pi/n$, $n \in E_0 = \{1, 2, 3, 4, 6\}$.

Proof. First it is shown that $\cos\omega$ has to be a rational number. Let

$$w(x) = \sum_{n,m=-\infty}^{\infty} w_{nm}(x_3) e^{i(\alpha n x_1 + \beta m x_2)}$$

be the Fourier expansion of $w$ and

$$\tilde{T}_\omega = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}.$$

If $w_{11} \neq 0$, then according to (2.8e) it follows, for all $n \in Z$, that

$$\text{(2.10)} \qquad \begin{aligned} &\tilde{T}_\omega{}'^n \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \nu_n \alpha \\ \mu_n \beta \end{pmatrix} \quad \text{for some } \nu_n, \mu_n \in Z, \\ &w_{\nu_n \mu_n} = w_{11}, \end{aligned}$$

$\tilde{T}_\omega{}'$ being the transposed matrix of $\tilde{T}_\omega$, which implies $\nu_n \mu_n - (\nu_n + \mu_n)\cos\omega + 1 = 0$, thus $\cos\omega \in Q$. If $w_{nm} \neq 0$, set $\alpha' = n\alpha$, $\beta' = m\beta$, $n^2 + m^2 \neq 0$ and the same result follows. $w_{00}$ cannot be the only nonvanishing term since this would imply $w = 0$.

Next we show that $\omega = 2\pi/n$, $n \in N$. If $\omega = 2\pi r$, $r$ irrational, $\tilde{T}_\omega{}'^n\binom{\alpha}{\beta}$ is dense on the circle with radius $(\alpha^2 + \beta^2)^{1/2}$, contradicting (2.10) which can only be satisfied for finitely many $(\nu_n, \mu_n)$. Let $\omega = 2\pi r$, $r \in Q$, $r = n/m$. If $np = m$ for some $p \in Z$, then $T_{2\pi/p}$ is a generator of $T_2$; otherwise $T_{2\pi/m}$ generates $T_2$. Thus, we can restrict the considerations to $\omega = 2\pi/n$, $n \in Z$.

$\alpha = \cos(2\pi/n) \in Q$ if and only if $n \in E_0$. (We owe this proof to A. Grundmann.) $e^{\pm 2\pi i/n}$ satisfies $z^2 - 2\alpha z + 1 = 0$ which therefore divides $z^n - 1$. The quotient is a polynomial of order $n - 2$ with integer coefficients. Thus $2\alpha \in N_0$, which implies $\cos(2\pi/n) = \pm 1$, or $\pm \frac{1}{2}$, or 0. Q.E.D.

In general, $\alpha$, $\beta$ and $\omega$ cannot be chosen independently. It is easy to derive from (2.7) the relations given in the following corollary. Evidently the role of $\alpha$ and $\beta$ can be interchanged.

COROLLARY 2.2. *Let $\omega = 2\pi/n$, $n = 1, 2, 3, 4, 6$, then the only possible combinations of $n$, $\alpha$, $\beta$ are*

| | | | |
|---|---|---|---|
| $n = 1$, | $\alpha \geqq 0$, | $\beta \geqq 0$, | *no cell structure*; |
| $n = 2$, | $\alpha > 0$, | $\beta = 0$, | *rolls*; |
| | $\alpha > 0$, | $\beta > 0$, | *rectangles*; |
| $n = 3$, | $\alpha = \beta/\sqrt{3}$, | $\beta > 0$, | *hexagons*; |
| $n = 4$, | $\alpha = \beta$, | $\beta > 0$, | *squares*; |
| $n = 6$, | $\alpha = \beta/\sqrt{3}$, | $\beta > 0$, | *triangles*. |

For some $\lambda_1 > 0$, $\lambda \in [0, \lambda_1]$, $w = 0$ is the only solution of the stationary problem (2.8), and is stable. $\lambda_1$ depends on $\sigma = (\alpha^2 + \beta^2)^{1/2}$ but not on $\omega$. Moreover, there exist infinitely many $\lambda_j(\sigma) > 0$, $0 < \lambda_1(\sigma) < \lambda_2(\sigma) < \cdots$, such that 0 is a simple eigenvalue of the LSP (§§4, 5). Every $(\lambda_j, 0)$ is a point of bifurcation; however, only $(\lambda_1, 0)$ is of physical interest. Two nontrivial solutions branch off to the right of this point for every cell pattern possible. They gain stability from the trivial solution within the class of solutions having the same invariance properties. The question of which cell pattern is preferred by nature remains open in this approximation. Far-reaching physical and analytical results are discussed in §§4, 5. As for the Taylor vortex flow, it can be shown that the local minima $\sigma_0$ of $\lambda_1(\sigma)$ determine the only stable cell size (§5).

3. **A functional-analytic approach.** The formal analogy between (2.4) and (2.8) suggests an abstract formulation of the bifurcation and stability problem. Therefore, we derive in this section an evolution equation in a suitable Hilbert space which includes the models under discussion and some other problems as well, studied in fluid dynamics. The properties of the differential operators appearing in this evolution equation are listed in Lemma 3, and are heavily used in §5, which deals with stability and instability questions. The stationary problem discussed in §4 allows some generalizations, especially as regards the assumptions about the nonlinearity. The first part of this section is strongly influenced by the paper of Iooss [**19**].

Let $D \subset R^3$ be open with boundary $\partial D$ which is supposed to be a two-dimensional $C^2$-manifold. $T_1$ denotes a group of translations and $\Omega$ its fundamental region of periodicity, which has to be bounded. We assume

$$D = \bigcup_{T \in T_1} T\Omega.$$

If $T_1 = \{1_{R^3}\}$, $D$ is bounded. For the Taylor problem we have $n = 3$, $D = (r_1, r_2) \times [0, 2\pi) \times R$, $\Omega = (r_1, r_2) \times [0, 2\pi) \times [0, 2\pi/\sigma)$, while for the Bénard problem $n = 4$, and $D = R^2 \times (0, 1)$, and $\Omega = [0, 2\pi/\alpha) \times [0, 2\pi/\beta) \times (0, 1)$. We introduce the following classes of functions for $n \in N$ (here $\mathrm{cl}(D)$ means the closure of $D$):

$$(3.1)\quad \begin{aligned} C^{T,\infty}(\bar{D}) &= \{w \mid w\colon \mathrm{cl}(D) \to R^n, \text{ infinitely often differentiable} \\ &\qquad\qquad \text{in } \mathrm{cl}(D), w(Tx) = w(x), T \in T_1\}, \\ C_0{}^{T,\infty}(D) &= \{w \mid w \in C^{T,\infty}(\bar{D}), \operatorname{supp} w \subset D\}, \\ C_{0,\sigma}^{T,\infty}(D) &= \{w \mid w \in C_0{}^{T,\infty}(D), \nabla \cdot u(x) = 0, w = (u, v), u \in R^3\}. \end{aligned}$$

Defining the scalar product and norm

$$(3.2)\qquad \begin{aligned}(\boldsymbol{v}, \boldsymbol{w})_m &= \sum_{|\gamma| \leqq m} \sum_{i=1}^{n} \int_\Omega D^\gamma v_i(x) D^\gamma w_i(x)\, dx,\\ |\boldsymbol{v}|_m &= \{(\boldsymbol{v}, \boldsymbol{v})_m\}^{1/2},\end{aligned}$$

where $\gamma$ is a multiindex of length 3, one obtains the following Hilbert spaces:

$$(3.3)\qquad \begin{aligned}&L_2{}^T = \mathrm{cl}_{|\ |_0} C_0{}^{T,\infty}(D), \qquad \mathring{J}^T = \mathrm{cl}_{|\ |_0} C_{0,\sigma}^{T,\infty}(D),\\ &\mathring{H}_{1,\sigma}^T = \mathrm{cl}_{|\ |_1} C_{0,\sigma}^{T,\infty}(D), \qquad H_m{}^T = \mathrm{cl}_{|\ |_m} C^{T,\infty}(\bar{D}),\end{aligned}$$

$$\mathfrak{L}(H_m{}^T) = \{L \mid L \text{ is a bounded endomorphism on } H_m{}^T, |\ |_m\}.$$

Occasionally the spaces $L_p(\Omega)$ and the Sobolev spaces $W_p{}^l(\Omega)$ or products of them are used. As for (3.3) we do not distinguish in notation between spaces for different $n$. The norms for $L_p(\Omega)$ and $W_p{}^l(\Omega)$ are written as $|\cdot|_{L_p(\Omega)}$, $|\cdot|_{W_p{}^l(\Omega)}$ resp. $L_2{}^T$ consists of those equivalence classes of functions whose restrictions $\boldsymbol{w}|_\Omega$ are in $L_2(\Omega)$ and satisfy $\boldsymbol{w}(x) = \boldsymbol{w}(Tx)$ a.e. in $D$, for all $T \in \boldsymbol{T}_1$. In view of the smoothness of $\partial D$, Poincaré's inequality is valid [1, p. 73]:

$$(3.4)\qquad |\boldsymbol{w}|_0 \leqq \gamma_1 |\nabla \boldsymbol{w}|_0.$$

LEMMA 3.1 (H. WEYL). *Let* $G_1{}^T := \{\nabla q \mid q : D \to \boldsymbol{R},\ q \in H_{1,\mathrm{loc}}^T\}$, $G_0{}^T := \{\boldsymbol{v} \mid \boldsymbol{v} : D \to \{0\} \subset \boldsymbol{R}^{n-3}\}$, $G^T := G_1{}^T \times G_0{}^T$, *then* $L_2{}^T$ *is the direct sum of the orthogonal subspaces* $\mathring{J}^T$ *and* $G^T$:

$$L_2{}^T = \mathring{J}^T \oplus G^T.$$

PROOF. Set $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{v}) \in L_2{}^T$, $\boldsymbol{u} \in \boldsymbol{R}^3$, then $\boldsymbol{u}|_\Omega = \boldsymbol{u}_1 + \nabla q_1$, where $\boldsymbol{u}_1 \in \mathring{J}(\Omega)$, $q_1 \in H_{1,\mathrm{loc}}(\Omega)$ and $\boldsymbol{u}_1$, $\nabla q_1$ orthogonal in $L_2(\Omega)$ (cf. [39, p. 22]). Define $\boldsymbol{w}_1(Tx) = (\boldsymbol{u}_1(x), \boldsymbol{v}(x))$, $q(Tx) = q_1(x)$, $T \in \boldsymbol{T}_1$, then $\boldsymbol{w}_1 \in \mathring{J}^T$, $q \in H_{1,\mathrm{loc}}^T$, $\boldsymbol{w}_1$ and $\nabla q$ orthogonal in $L_2{}^T$ and $\boldsymbol{w} = \boldsymbol{w}_1 + (\nabla q, 0), 0 \in G_0{}^T$.

The decomposition of $L_2{}^T$ defines an orthogonal projection

$$(3.5)\qquad P : L_2{}^T \to \mathring{J}^T.$$

COROLLARY 3.2. *If* $\partial D$ *is a* $C^{m+2}$*-manifold, then* $P|_{H_m{}^T} \in \mathfrak{L}(H_m{}^T)$, $m \in \boldsymbol{N}$, *holds.*

PROOF. If $\partial D$ is a $C^{m+2}$-manifold the boundary-value problem

$$\begin{aligned}&\Delta p = \nabla \cdot f \quad \text{in } D, \qquad f \in C_m{}^T(\bar{D}),\\ &dp/dn = f_n \quad \text{on } \partial D, \qquad p(Tx) = p(x), \qquad T \in \boldsymbol{T}_1,\end{aligned}$$

where $n$ denotes the outer normal and $f_n$ is the normal component of $f$, possesses a solution $p \in C^T_{m+1}(\bar{D})$. According to Agmon-Douglis-Nirenberg [2, Theorem 14.1, p. 701], the estimate

$$|p|_{m+1} \leqq c_1 \left\{ |\nabla \cdot f|_{m-1} + \inf_{v \in H_m{}^T;\, v|_{\partial D} = f_n} |v|_m \right\} \leqq c_2 |f|_m$$

holds, and $f \in H_m{}^T$ implies $p \in H^T_{m+1}$. Setting $g = f - \nabla p \in H_m{}^T$, then $\nabla \cdot g = 0$ and

$$(g, \nabla q)_0 = \int_\Omega \nabla \cdot (qg)\, dx = \int_{\partial D \cap \bar{\Omega}} q(f_n - dp/dn)\, ds = 0.$$

for all $q \in H_1{}^T$. Therefore $g \in \mathring{J}^T \cap H_m{}^T$ and

$$|Pf|_m = |g|_m \leqq c_3\{|f|_m + |\nabla p|_m\} \leqq c_4 |f|_m$$

holds. Q.E.D.

The operator $-P\tilde{\Delta}$, where $\tilde{\Delta}$ is given by (2.3) or (2.7), is symmetric in $\mathring{J}^T$ and positive definite. Moreover, by standard arguments (cf. [39, p. 31])

$$-P\tilde{\Delta} w = f, \qquad f \in \mathring{J}^T,$$

has a weak solution in $\mathring{H}^T_{1,\sigma}$, for which $|w|_1 \leqq c_0 |f|_0$ holds.

Let $\Omega'$, cl $\Omega' \subset \Omega$; then the local estimates in [39, p. 33] guarantee that

$$\int_{\Omega'} |\tilde{\Delta} w(x)|^2\, dx \leqq c_1 \int_\Omega |f(x)|^2\, dx.$$

Using the estimate [39, p. 14], for $w' = w|_{\Omega'}$, $f' = f|_{\Omega}$, one obtains

$$|w'|_{W_2{}^2(\Omega')} \leqq c_2 |f'|_{L_2(\Omega)}.$$

This estimate is equally valid for every $T\Omega$, thus yielding for $w_1 = w|_{D'}$, cl $D' \subset D$, cl $D'$ compact, $f_1 = f|_{D'}$:

$$|w_1|_{W_2{}^2(D')} \leqq c_3 |f_1|_{L_2(D')}.$$

The above estimate can be guaranteed even for $\Omega' = \Omega$ if the boundary of $\Omega$ is a $C^2$-manifold (see [39, p. 54]). Since only local estimates are involved, the assumed smoothness of $\partial D$ can be used for a direct translation of those arguments to our case. But $\Omega' = \Omega$ implies

$$|w|_2 \leqq \gamma_2 |f|_0. \tag{3.6}$$

Therefore, $-P\tilde{\Delta}$ is symmetric and surjective, hence selfadjoint. In view of (3.6) and Rellich's theorem, $(-P\tilde{\Delta})^{-1}$ is compact.

LEMMA 3.3. *The operator $A$ defined by $A = -P\tilde{\Delta}$ with the domain of definition $D(A) = H_2{}^T \cap \mathring{H}{}^T_{1,\sigma}$ is selfadjoint and positive definite in $\mathring{J}^T$. $A^{-1} \in \mathfrak{L}(\mathring{J}^T)$ is compact.*

COROLLARY 3.4. (i) $D(A^{1/2}) = \mathring{H}{}^T_{1,\sigma}$;
(ii) $|A^{1/2}w|_0$, $|w|_1$, $|\nabla w|_0$ *are equivalent norms in* $\mathring{H}{}^T_{1,\sigma}$.

PROOF. Let $w \in D(A)$, then the form of $\tilde{\Delta}$ in (2.3) and (2.7) implies

$$c_1|\nabla w|_0^2 \leqq (Aw, w)_0 = |A^{1/2}w|_0^2 \leqq c_2|w|_1^2.$$

Using (3.4) and the denseness of $D(A)$ in $\mathring{H}{}^T_{1,\sigma}$ gives (ii) which again proves (i).

We introduce the following notations

$$\begin{aligned} ((v, w)) &= (A^{1/2}v, A^{1/2}w)_0, \\ \|v\| &= |A^{1/2}v|_0, \end{aligned} \tag{3.7}$$

and obtain, from (3.4) and Corollary 3.4,

$$|w|_0 \leqq \gamma_0\|w\|. \tag{3.8}$$

Occasionally we use a stronger regularity result for the Stokes equations due to Cattabriga [**6**], which we state in a form convenient for the intended application.

LEMMA 3.5 (CATTABRIGA). *Let $\partial D$ be a two-dimensional $C^s$-manifold, $s = \max(2, m-2)$, $f \in H_m{}^T \cap \mathring{J}^T$, then the solution of $Aw = f$ satisfies*

$$|w|_{m+2} \leqq \gamma_2|f|_m, \qquad m \in N_0.$$

LEMMA 3.6. *Let $L(V)$ be given by* (2.3) *or* (2.7) *and let $V \in H_2{}^T$, then $M(V) := PL(V)$ satisfies*

$$|M(V)w|_0 \leqq \gamma_3|A^{1/2}w|_0, \qquad w \in \mathring{H}{}^T_{1,\sigma}. \tag{3.9}$$

PROOF. If $V \in H_2{}^T$ then $V|_\Omega \in W_2{}^2(\Omega)$ and by Sobolev's embedding theorem $V \in C^0(\bar{\Omega})$. Let $w = (u, v) \in C^{T,\infty}_{0,\sigma}$, $\varphi \in C^{T,\infty}_{0,\sigma}$, then Corollary 3.4 and (2.3) and (2.7) yield

$$\begin{aligned} &|(L^0w + (V \cdot \nabla)w + (u \cdot \nabla)V, \varphi)_0| \\ &\qquad = |(L^0w, \varphi)_0 + ((V \cdot \nabla)w, \varphi)_0 - ((u \cdot \nabla)\varphi, V)_0| \\ &\qquad \leqq c_1 \{(|w|_0 + |A^{1/2}w|_0)|\varphi|_0 + |w|_0|A^{1/2}\varphi|_0\} \\ &\qquad \leqq c_2|A^{1/2}w|_0|A^{1/2}\varphi|_0, \end{aligned}$$

which for $\varphi = Lw$ implies (3.9) for $w \in C^{T,\infty}_{0,\sigma}$. The closure of this class in $D(A^{1/2}) = \mathring{H}^{T}_{1,\sigma}$ satisfies (3.9) as well, which proves the assertion.

COROLLARY 3.7. *Let* $V \in C^{T,\infty}(\bar{\Omega})$ *and* $w \in H^{T}_{m+1}$; *then*

$$|M(V)w|_m \leqq \gamma_4 |w|_{m+1}, \qquad m \in N_0.$$

PROOF.

$$|P\{L^0 w + (V \cdot \nabla)w + (u \cdot \nabla)V\}|_m \leqq c_1(|w|_m + |w|_{m+1}) \leqq \gamma_4 |w|_{m+1}.$$

LEMMA 3.8 (KATO-FUJITA [28, p. 258]). *Let* $N(w)$ *be given by* (2.3) *or* (2.7). *Setting* $R(w) = PN(w)$ *one obtains*

$$\begin{aligned} &|R(w_1) - R(w_2)|_0 \\ &\quad \leqq \gamma_5\{|A^{3/4}w_1|_0 |A^{1/2}(w_1 - w_2)|_0 \\ &\qquad + |A^{3/4}(w_1 - w_2)|_0 |A^{1/2}w_2|_0\}, \quad w_1, w_2 \in D(A^{3/4}). \end{aligned} \tag{3.10}$$

PROOF. The proof uses properties of the trace spaces $T(L_6(\Omega), L_2(\Omega))$ and $T(D(A), D(A^{1/2}))$. The arguments given by Kato-Fujita can be applied literally in our case. In order to obtain (3.10) one takes advantage of the following three inequalities:

$$\begin{aligned} &|w|_{L_6(\Omega)} \leqq c_1 |A^{1/2}w|_0, \qquad |w|_2 \leqq c_2 |Aw|_0, \\ &|R(w)|_0 \leqq c_3 |w|_{L_6(\Omega)} |\nabla w|_{L_3(\Omega)}. \end{aligned} \tag{3.11}$$

The first inequality is a consequence of Sobolev's embedding theorem, (3.4) and Corollary 3.4. The second inequality coincides with (3.6) for $f = Aw$. The third inequality follows for (2.3) by applying Hölder's inequality with exponents 3 and $\frac{3}{2}$ to $(w \cdot \nabla)w$ and $Q(w)$:

$$\begin{aligned} &|(w \cdot \nabla)w|_0^2 \leqq c|w|^2_{L_6(\Omega)} |\nabla w|^2_{L_3(\Omega)}, \\ &|Q(w)|_0^2 \leqq c'|w|^4_{L_4(\Omega)} \leqq c''|w|^2_{L_6} |\nabla w|^2_{L_3}. \end{aligned}$$

The proof of (3.11) for (2.7) is analogous.

The second part of this section contains results for the linear part of the equations (2.4) and (2.8) which will be used in the stability analysis of §5. We introduce a new notation for $A + \lambda M(V)$, where for convenience, the dependence on $V$ is suppressed.

$$\tilde{A}(\lambda, V) = \tilde{A}(\lambda) = A + \lambda M(V), \qquad \lambda \in R. \tag{3.12}$$

The following lemma is essentially due to Prodi [45].

LEMMA 3.9. *Let* $V \in D(A)$.

(i) $\tilde{A}(\lambda)$ *with* $D(\tilde{A}(\lambda)) = D(A)$ *is a closed operator in* $\mathring{J}^T$.

(ii) *For every* $\mu$ *in the spectrum* $\sigma(\tilde{A}(\lambda))$,

$$\operatorname{re}\mu \geqq (\operatorname{im}\mu)^2/4(\lambda\gamma_3)^2 - (\lambda\gamma_3)^2 =: F(\lambda, \mu)$$

*holds.*

(iii) *For* $\mu \in C$ *with* $-\operatorname{re}\mu + F(\lambda, \mu) > 0$, *we have*

$$|(\tilde{A}(\lambda) - \mu 1)^{-1}|_0 \leqq 1/|\lambda|\gamma_3(-\operatorname{re}\mu + F(\lambda, \mu))^{1/2}.$$

(iv) *For* $\mu \in C$ *with* $G(\lambda, \mu) := -\operatorname{re}\mu + \frac{1}{2}(1/\gamma_0^2 - (\lambda\gamma_3)^2) > 0$, $\mu$ *is in the resolvent set of* $\tilde{A}(\lambda)$ *and*

$$|(\tilde{A}(\lambda) - \mu 1)^{-1}|_0 \leqq 1/G(\lambda, \mu)$$

*holds.*

PROOF. Consider the equation

$$Aw + \lambda M(V)w - \mu w = g, \qquad w \in D(A). \tag{3.13}$$

With the aid of (3.7), (3.8) and Lemma 3.6 one obtains

$$\begin{aligned}\left\{ \tfrac{1}{2}\left(\frac{1}{\gamma_0^2} - (\lambda\gamma_3)^2\right) - \operatorname{re}\mu \right\}|w|_0^2 \\ \leqq \tfrac{1}{2}\|w\|^2 - \{\tfrac{1}{2}(\lambda\gamma_3)^2 + \operatorname{re}\mu\}|w|_0^2 \leqq |g|_0|w|_0,\end{aligned} \tag{3.14}$$

which implies (iv). Moreover, Lemma 3.6 yields

$$\operatorname{im}\mu|w|_0 \leqq |g|_0 + |\lambda|\gamma_3\|w\|$$

$$(\operatorname{im}\mu)^2|w|_0^2/4(\lambda\gamma_3)^2 \leqq |g|_0^2/2(\lambda\gamma_3)^2 + \tfrac{1}{2}\|w\|^2.$$

On the other hand it follows from (3.14) that

$$\tfrac{1}{2}\|w\|^2 - \operatorname{re}\mu|w|_0^2 \leqq (\lambda\gamma_3)^2|w|_0^2 + |g|_0^2/2(\lambda\gamma_3)^2.$$

Addition of the last two inequalities gives

$$\left( -\operatorname{re}\mu + \frac{(\operatorname{im}\mu)^2}{4(\lambda\gamma_3)^2} - (\lambda\gamma_3)^2 \right)|w|_0^2 \leqq |g|_0^2/(\lambda\gamma_3)^2$$

which proves (ii) and (iii). To prove (i), consider $w_\nu \in D(A)$ with $w_\nu \to w$, $\tilde{A}(\lambda)w_\nu = v_\nu \to v$. Setting $\mu = 0$ and $g = v_\nu$, (3.14) implies the convergence of $w_\nu$ in $\mathring{H}^T_{1,\sigma}$. Thus, by Lemma 3.6: $\lambda M(V)w_\nu \to u$ and by (3.12) one obtains $Aw_\nu \to v - u$. Since $A$ is closed, $w \in D(A)$ and $\tilde{A}(\lambda)w = v$. Q.E.D.

REMARK. Let $0 < \alpha < 2/\gamma_0^2$. If

$$|\lambda| \leqq (1/\gamma_3)(1/\gamma_0^2 - 2\alpha)^{1/2}$$

we have, for every $\mu \in \sigma(\tilde{A}(\lambda))$, re $\mu \geqq \alpha > 0$.

COROLLARY 3.10. (i) *The spectrum of* $\tilde{A}(\lambda)$ *consists of eigenvalues of finite multiplicities only with a single cluster point at infinity.*

(ii) *The operator* $-\tilde{A}(\lambda)$ *generates a strongly continuous semigroup in* $\mathring{J}^T$.

PROOF. Let $\mu$ be in the resolvent set of $\tilde{A}(\lambda)$. Then, by (3.14) and Corollary 3.4, the resolvent maps bounded sets in $\mathring{J}^T$ into bounded sets in $\mathring{H}^T_{1,\sigma}$. By Rellich's theorem, the compactness follows in $\mathring{J}^T$ which implies (i) (see [27, p. 185]). (ii) follows from Lemma 3.9(iv) and the Hille-Yosida theorem.

LEMMA 3.11. *Let* $V \in D(A)$ *and* re $\mu \geqq \alpha > 0$ *for all* $\mu \in \sigma(\tilde{A}(\lambda))$. *Then* $-\tilde{A}(\lambda)$ *is the generator of a holomorphic semigroup* $\exp(-\tilde{A}(\lambda)t)$ (*see* [27, *p.* 487]), *which satisfies the following estimates*:

(i) $|\exp(-\tilde{A}(\lambda)t)|_0 \leqq \gamma_6(\alpha')e^{-\alpha' t}$, $t \geqq 0$,

(ii) $|\tilde{A}(\lambda)\exp(-\tilde{A}(\lambda)t)|_0 \leqq \gamma_6(\alpha')t^{-1}e^{-\alpha' t}$, $t > 0$, $0 < \alpha' < \alpha$.

PROOF. It suffices to show the inequality

$$(3.15)\qquad |(\tilde{A}(\lambda) - \mu 1)^{-1}|_0 \leqq \frac{c}{|\mu| + 1} \quad \text{for } \frac{\pi}{2} - \epsilon \leqq \arg \mu \leqq \tfrac{3}{2}\pi + \epsilon,$$

for some positive constants $c$ and $\epsilon$ (cf. [57, p. 10]). The proof of (3.15) proceeds in 4 steps.

(1) Consider the set $|\mu| \geqq C_1$, re $\mu \leqq 0$, $|\text{im}\,\mu| \geqq 2|\lambda|\gamma_3|\,\text{re}\,\mu|$. Choose $C_1 = 2(1 + (\lambda\gamma_3)^2)(1 + 4(\lambda\gamma_3)^2)^{1/2}$, then

$$(3.16)\qquad -\text{re}\,\mu + \frac{(\text{im}\,\mu)^2}{4(\lambda\gamma_3)^2} - (\lambda\gamma_3)^2 \geqq \frac{(1 + |\mu|)^2}{4(1 + 4(\lambda\gamma_3)^2)}$$

holds, yielding, by Lemma 3.9(iii),

$$(3.17)\qquad |(\tilde{A}(\lambda) - \mu 1)^{-1}|_0 \leqq \frac{2(1 + 4(\lambda\gamma_3)^2)^{1/2}}{|\lambda|\gamma_3(|\mu| + 1)} = \frac{c_1}{|\mu| + 1}.$$

(2) Consider $|\mu| \geqq C_2$, re $\mu \leqq 0$, $|\text{im}\,\mu| \leqq 2|\lambda|\gamma_3|\text{re}\,\mu|$, with $C_2 = 1 + (\lambda\gamma_3)^2(1 + 4(\lambda\gamma_3)^2)^{1/2}$, then

$$-\text{re}\,\mu - \frac{(\lambda\gamma_3)^2}{2} \geqq \frac{1 + |\mu|}{2(1 + 4(\lambda\gamma_3)^2)^{1/2}}$$

holds, and Lemma 3.9(iv) implies

$$|(\tilde{A}(\lambda) - \mu 1)^{-1}|_0 \leqq \frac{2(1 + 4(\lambda\gamma_3)^2)^{1/2}}{|\mu| + 1} = \frac{c_2}{|\mu| + 1}.$$

(3) Consider $|\mu| \geqq C_3$, re $\mu \geqq 0$, $|\text{im } \mu| \geqq 2|\lambda|\gamma_3$ re $\mu$ and choose $C_3 = 18(\lambda\gamma_3)^2 + 3$, then (3.16) holds, which yields (3.17).

(4) Define $C = \max(C_1, C_2, C_3)$, then the set $S = \{\mu | \mu \in C, |\mu| \leqq C, \text{re } \mu \leqq \alpha/2\}$ is compact in $C$ and belongs to the resolvent set of $\tilde{A}(\lambda)$. Since the resolvent is continuous in $\mu$, there is a constant $c_3$ such that

$$|(\tilde{A}(\lambda) - \mu 1)^{-1}|_0 \leqq c_3/(|\mu| + 1), \qquad \mu \in S.$$

Define $\epsilon$ by $\sin \epsilon = \alpha/2C$, $0 < \epsilon < \pi/2$, $c = \max(c_1, c_2, c_3)$, then (3.15) holds in the sector $\pi/2 - \epsilon \leqq \arg \mu \leqq 3\pi/2 + \epsilon$. Q.E.D.

Under the conditions and the results of the preceding lemma, fractional powers of $\tilde{A}(\lambda)$ can be defined [57].

$$(3.18) \quad \tilde{A}(\lambda)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty \exp(-\tilde{A}(\lambda)t) t^{\beta-1}\, dt, \qquad \beta > 0.$$

$\tilde{A}(\lambda)^{-\beta} \in \mathfrak{L}(\mathring{J}^T)$ is invertible and thus we may set

$$\tilde{A}(\lambda)^{\beta} = (\tilde{A}(\lambda)^{-\beta})^{-1}, \qquad D(\tilde{A}(\lambda)^{\beta}) = R(\tilde{A}(\lambda)^{-\beta}),$$

where $R(\tilde{A}(\lambda)^{-\beta})$ denotes the range of $\tilde{A}(\lambda)^{-\beta}$. According to [57], the fractional powers of $\tilde{A}(\lambda)$ have the following properties:

$$(3.19) \quad \begin{aligned} &\text{(a)} \quad \tilde{A}(\lambda)^{\delta}\tilde{A}(\lambda)^{\beta} w = \tilde{A}(\lambda)^{\delta+\beta} w, \qquad w \in D(\tilde{A}(\lambda)^{\delta+\beta}),\\ &\text{(b)} \quad |\tilde{A}(\lambda)^{\beta} \exp(-\tilde{A}(\lambda)t)|_0 \leqq \gamma_7(\alpha') t^{-\beta} e^{-\alpha' t}, \qquad t > 0,\\ &\text{(c)} \quad \tilde{A}(\lambda)^{\beta} \exp(-\tilde{A}(\lambda)t) w = \exp(-\tilde{A}(\lambda)t)\tilde{A}(\lambda)^{\beta} w,\\ &\qquad\qquad w \in D(\tilde{A}(\lambda)^{\beta}),\ t \geqq 0,\\ &\text{(d)} \quad |\tilde{A}(\lambda)^{\beta} w|_0 \leqq \gamma_8(\alpha) |w|_0^{1-\beta} |\tilde{A}(\lambda) w|_0^{\beta},\\ &\qquad\qquad w \in D(\tilde{A}(\lambda)),\ 0 < \beta \leqq 1. \end{aligned}$$

LEMMA 3.12. *Let* $V \in D(A)$ *and* re $\mu \geqq \alpha > 0$ *for* $\mu \in \sigma(\tilde{A}(\lambda))$. *Then* $A^{\beta}\tilde{A}(\lambda)^{-\delta} \in \mathfrak{L}(\mathring{J}^T)$ *if* $0 \leqq \beta < \delta < 1, \beta = \delta = 1$.

PROOF. (1) $\beta = \delta = 1$. $\tilde{A}(\lambda) = (1 + \lambda M(V)A^{-1})A$. $(-1)$ is in the resolvent set of the compact operator $\lambda M A^{-1}$, since $v + \lambda M(V)A^{-1}v = 0$ implies $\tilde{A}(\lambda)w = 0$ for $w = A^{-1}v$, which, according to the assumption on $\sigma(\tilde{A}(\lambda))$, yields $w = 0$ and thus $v = 0$. Therefore $A\tilde{A}(\lambda)^{-1} = (1 + \lambda M(V)A^{-1})^{-1}$ is a bounded linear operator in $\mathring{J}^T$.

(2) $0 \leqq \beta < \delta < 1$. We observe that, in view of Lemma 3.11(ii),

$$
\begin{aligned}
|A \exp(-\tilde{A}(\lambda)t)w|_0 &= |A\tilde{A}(\lambda)^{-1}\tilde{A}(\lambda)\exp(-\tilde{A}(\lambda)t)w|_0 \\
&\leqq c_1 t^{-1} e^{-\alpha' t}|w|_0, \qquad t > 0, 0 < \alpha' < \alpha,
\end{aligned}
$$

holds. The interpolation inequality

$$|A^\beta w|_0 \leqq c_2 |w|_0^{1-\beta}|Aw|_0^\beta, \qquad w \in D(A),$$

yields, together with Lemma 3.11(i),

$$|A^\beta \exp(-\tilde{A}(\lambda)t)|_0 \leqq c_3 e^{-\alpha' t} t^{-\beta}, \qquad t > 0.$$

Since $A^{-1}$ is compact, the spectrum of $A$ consists of eigenvalues of finite multiplicities. The spectral representation has the form $A^\beta w = \sum_{\nu=1}^\infty \mu_\nu{}^\beta (w, \varphi_\nu)_0 \varphi_\nu$ where $\varphi_\nu$ are the normalized eigenelements to the positive eigenvalues $\mu_\nu$ of $A$. Thus, we obtain

$$
\begin{aligned}
|A^\beta \exp(-\tilde{A}(\lambda)t)w|_0^2 &= \sum_{\nu=1}^\infty \mu_\nu{}^{2\beta}(\exp(-\tilde{A}(\lambda)t)w, \varphi_\nu)_0{}^2 \\
&\leqq c_3{}^2 e^{-2\alpha' t} t^{-2\beta}|w|_0^2.
\end{aligned} \tag{3.20}
$$

Further, we need the following estimate:

$$
\begin{aligned}
&\left\{\int_0^\infty \mu_\nu{}^\beta (\exp(-\tilde{A}(\lambda)t)w, \varphi_\nu)_0 t^{\delta-1}\,dt\right\}^2 \\
&\leqq \int_0^\infty \mu_\nu{}^{2\beta}(\exp(-\tilde{A}(\lambda)t), \varphi_\nu)_0{}^2 e^{2bt} t^{2(\delta-c)}\,dt \int_0^\infty e^{-2bt} t^{-2(1-c)}\,dt
\end{aligned} \tag{3.21}
$$

with $0 < b < \alpha'$, $c = \{1 + (\delta - \beta)\}/2$. Setting

$$C = \int_0^\infty e^{-2bt} t^{-2(1-c)}\,dt,$$

one can estimate $A^\beta \tilde{A}(\lambda)^{-\delta}$ by using (3.20) and (3.21) as follows:

$$
\begin{aligned}
|A^\beta \tilde{A}(\lambda)^{-\delta} w|_0^2 &= \sum_{\nu=1}^\infty \mu_\nu{}^{2\beta}(\tilde{A}(\lambda)^{-\delta} w, \varphi_\nu)_0{}^2 \\
&= \frac{1}{\Gamma(\delta)^2} \sum_{\nu=1}^\infty \left\{\int_0^\infty \mu_\nu{}^\beta (\exp(-\tilde{A}(\lambda)t)w, \varphi_\nu)_0 t^{\delta-1}\,dt\right\}^2 \\
&\leqq \frac{C}{\Gamma(\delta)^2} \sum_{\nu=1}^\infty \int_0^\infty \mu_\nu{}^{2\beta}(\exp(-\tilde{A}(\lambda)t)w, \varphi_\nu)_0{}^2 e^{2bt} t^{2(\delta-c)}\,dt \\
&\leqq \frac{Cc_3{}^2}{\Gamma(\delta)^2} \int_0^\infty e^{-2(\alpha'-b)t} t^{(\delta-\beta)-1}\,dt |w|_0^2 \leqq c_4{}^2 |w|_0^2. \quad \text{Q.E.D.}
\end{aligned}
$$

Finally, we are able to formulate (2.4) and (2.8) as an evolution equation in $\mathring{J}^T$:

$$dw/dt + \tilde{A}(\lambda)w + h(\lambda)R(w) = 0, \qquad w|_{t=0} = w^0, \tag{3.22}$$

where $h(\lambda) = \lambda$ for (2.4) and $h(\lambda) = 1$ for (2.8). The operators $\tilde{A}(\lambda)$ and $R$ have the properties listed in Lemma 3.4 to Lemma 3.8. They strongly depend on the special form of the original equations and will be used in §5, where stability and instability results are given. For the stationary problem, discussed in the next section, the assumptions on $R$ can be weakened considerably.

4. **The stationary problem, bifurcation.** In this section the stationary part of the equation (3.22) is discussed. It is easy to show the existence of nontrivial solutions as branches emanating from points of bifurcation (Theorem 4.1). The assumptions on $R$ are weakened thereafter in order to include more general equations which are important in fluid dynamics (Theorem 4.2). Thus, concrete information is obtained from the Taylor and the Bénard model on the bifurcation picture (Theorems 4.3 and 4.6).

In the stationary equation of (3.22),

$$Aw + \lambda M(V)w + h(\lambda)R(w) = 0, \tag{4.1}$$

where $V$ is any known stationary solution with $V \in D(A)$, which may depend on $\lambda$, we make the following substitution: $A^{3/4}w = v$ and obtain:

$$\begin{gathered} v + \lambda K(V) + h(\lambda)T(v) = 0, \qquad v \in \mathring{J}^T, \\ K(V) = A^{-1/4}M(V)A^{-3/4}, \qquad T(v) = A^{-1/4}R(A^{-3/4}v). \end{gathered} \tag{4.2}$$

Since $A^{-1/4} \in \mathfrak{L}(\mathring{J}^T)$, the following estimates can be derived from Lemmas 3.6 and 3.8:

$$|K(V)v|_0 \leqq \gamma_3 |v|_0, \qquad |T(v)|_0 \leqq \gamma_9 |v|_0^2. \tag{4.3}$$

$A^{-1}$ and thus $A^{-1/4}$ are compact, implying that $K$ and $T$ are completely continuous. $1 + \lambda K$ is the $F$-derivative of $1 + \lambda K + h(\lambda)T$ at $v = 0$. Now, well-known theorems on bifurcation can be applied.

THEOREM 4.1. *Let be $\lambda_j \in R$, $\lambda_j \neq 0$ and $(-\lambda_j)^{-1}$ be an eigenvalue of $K$ of odd multiplicity, then*

(i) *in every neighbourhood of $(\lambda_j, 0)$ in $R \times \mathring{J}^T$ there exist $(\lambda, w)$, $0 \neq w \in D(A)$, $w$ solves (4.1);*

(ii) *if* $(-\lambda_j)^{-1}$ *is a simple eigenvalue, then there exists a unique curve* $(\lambda(\alpha), w(\alpha))$, *such that* $w(\alpha) \neq 0$ *for* $\alpha \neq 0$, *which solves* (4.1), *moreover* $(\lambda(0), w(0)) = (\lambda_j, 0)$.

The proof of (i) follows by using a theorem of Krasnoselskiĭ [35, p. 196] for equation (4.2) and $v \in D(A^{1/4})$ which implies $w \in D(A)$. (ii) is a consequence of Remark 2.6 in [9].

In the following part of this section a generalization of Theorem 4.1 is indicated with the purpose of showing that strong regularity for the branching solutions can be obtained and that the nonlinearity can be an arbitrary polynomial operator including differentiation operators up to the order 2. We assume for this part that $V \in C^{T,\infty}(\bar{\Omega})$ and $\partial D$ is a $C^\infty$-manifold which is satisfied for the Taylor and the Bénard problem.

We remark that $w \in H^T_{m+2} \cap \mathring{H}^T_{1,\sigma}$, $m > \frac{3}{2}$, implies by Sobolev's embedding theorem that $w \in C_2{}^T \cap \mathring{H}^T_{1,\sigma}$ and thus $w|_{\partial D} = 0$. Consider $Au = Pf$, $f \in H_m{}^T$, then, by Lemma 3.5, $u \in H^T_{m+2} \cap D(A) = H^T_{m+2} \cap \mathring{H}^T_{1,\sigma}$ and, by Corollary 3.2,

$$|u|_{m+2} \leqq c_1 |Pf|_m \leqq c_2 |f|_m.$$

Setting $\hat{H}_m{}^T = \mathrm{cl}_{|\ |_m} PH_m{}^T$, let $h_\nu \in PH_m{}^T$, $h_\nu \to h$ in $|\ |_m$. Then, $A^{-1}h_\nu = u_\nu \to u \in H^T_{m+2} \cap \mathring{H}^T_{1,\sigma}$, where the convergence is in $|\ |_{m+2}$ and $Au = h$. Since $h \in H_m{}^T \cap \mathring{J}^T$ it follows that $\hat{H}_m{}^T \subset H_m{}^T \cap \mathring{J}^T \subset PH_m{}^T$, so $PH_m{}^T$ is closed.

Let $D(A_m) = H^T_{m+2} \cap \mathring{H}^T_{1,\sigma} \subset PH_m{}^T$, $A_m w = Aw$, $w \in D(A_m)$, then $A_m$ is a surjective, compactly invertible operator in $PH_m{}^T$ and $|A_m{}^{-1}w|_{m+2} \leqq c_3|w|_m$ holds for $w \in PH_m{}^T$. In view of the assumed regularity of $V$, Corollary 3.7 yields for $M = PL(V)$ restricted to $D(A_m)$:

$$\begin{aligned} &|Mw|_{m+1} \leqq c_4 |w|_{m+2}, && w \in D(A_m), \\ &|MA_m{}^{-1}w|_{m+1} \leqq c_4|A_m{}^{-1}w|_{m+2} \leqq c_5|w|_m, && w \in PH_m{}^T. \end{aligned} \tag{4.4}$$

Therefore, $K_m = MA_m{}^{-1}$ is a compact operator in $PH_m{}^T$.

Now, we discuss the feasible structure of the nonlinearity $R = PN$. Observe, that for $m > \frac{3}{2}$, $H_m{}^T$ is a Banach algebra, i.e., $|vw|_m \leqq c_6|v|_m|w|_m$. Let the components of $N(w)$ be of the following polynomial type:

$$\begin{aligned} N(w)(x)_i &= \sum_{j_1, \cdots, j_n = 1}^{n} \sum_{|\gamma_{j_k}| \leq 2} a_{ij_1 \cdots j_n}(x) D_{\gamma_{j_1}} w_{j_1}^{\kappa_{ij_1}} \cdots D_{\gamma_{j_n}} w_{j_n}^{\kappa_{ij_n}}, \\ a_{ij_1 \cdots j_n} &\in C^{T,m}(\bar{D}), \qquad \sum_{l=1}^{n} \kappa_{ij_l} \geqq 2, \qquad i = 1, \cdots, n. \end{aligned} \tag{4.5}$$

Note that (2.3) and (2.7) are of this form. Now, $m > \frac{3}{2}$ yields

$$|N(v) - N(w)|_m = \left\{ \sum_{i=1}^{n} |N(v)_i - N(w)_i|_m^2 \right\}^{1/2}$$

$$\leqq c_7 |v - w|_{m+2} g(|v|_{m+2}, |w|_{m+2}), \quad \text{for } v, w \in H_{m+2}^T,$$

where $g$ is continuous and $g(0, 0) = 0$. Thus, $R(A_m{}^{-1}w) = PN(A_m{}^{-1}w)$ is a locally Lipschitz continuous mapping in $PH_m{}^T$.

Considering

$$(4.6) \qquad A_m w + \lambda M w + h(\lambda) R(w) = 0, \qquad w \in D(A_m), \lambda \in R,$$

this equation is equivalent to

$$v + \lambda K_m v + h(\lambda) R(A_m{}^{-1}v) = 0$$

by $A_n{}^{-1}v = w$. $K_m$ and $R(A_m{}^{-1}v)$ satisfy the assumption of Theorem 2.3 in [**35**, p. 205], and we obtain

THEOREM 4.2. *Let be $V \in C^{T,\infty}(\bar{D})$, $\partial D$ be a $C^\infty$-manifold; let $M$ satisfy* (4.4) *and $N$ be of the form* (4.5). *Then for every eigenvalue $(-\lambda_j)^{-1}$ of $K_m$, $\lambda_j \neq 0$, of odd multiplicity, $(\lambda_j, 0)$ is a bifurcation point of* (4.6). *The solutions $w$ are in $C^{T,\infty}(\bar{D})$ and fulfill the boundary condition $w|_{\partial D} = 0$.*

(4.6) includes equations for the generalized Boussinesq equation, where the physical constants are allowed to depend on the temperature, and also the case of heat sources as discussed in [**15**], [**16**].

Having reduced the existence of nontrivial solutions of (4.1) to the investigation of the spectrum of $K$, we now discuss this operator in detail for the examples given in §2.

*Taylor model.* Consider $K = A^{-1/4}M(v^0)A^{-3/4}$ where $v^0$ is the Couette flow defined in (2.2). Since $\partial D$ is a $C^\infty$-manifold, every eigenfunction $\varphi$ belongs to $C^{T,\infty}(\bar{D})$ (Lemma 3.5) and satisfies (4.1) with $R = 0$ and the boundary conditions $\varphi|_{\partial D} = 0$ pointwise.

$$(4.7) \quad -\tilde{\Delta}\varphi + \lambda L(v^0)\varphi + \lambda \nabla q = 0, \qquad \nabla \cdot \varphi = 0, \qquad \varphi|_{\partial D} = 0,$$

for some $q \in C^{T,\infty}(\bar{D})$. $\varphi$ and $q$ are periodic with period $2\pi/\sigma$, $\sigma > 0$. In order to exclude a two-dimensional eigenspace, we require the solutions of (4.7) to be "even", i.e.,

$$\varphi(r, -z) = (-\varphi_1(r, z), -\varphi_2(r, z), \varphi_3(r, z)),$$

$$q(r, -z) = -q(r, z).$$

It is easily seen that the differential operators in (2.4) preserve this invariance condition. Now, the eigenfunctions have the form

$$\boldsymbol{\varphi}(r, z) = (u(r) \sin k\sigma z, v(r) \sin k\sigma z, w(r) \cos k\sigma z),$$
$$q(r, z) = q(r) \sin k\sigma z, \qquad k \in N.$$

Elimination of $w$ and $q$ in (4.7) yields (cf. [31])

$$(4.8)\qquad \begin{aligned} &\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - \frac{1}{r^2} - k^2\sigma^2\right)^2 u = 2k^2\sigma^2\lambda\frac{v_\varphi^{\;0}}{r}v, \\ &\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - \frac{1}{r^2} - k^2\sigma^2\right)v = 2a\lambda u, \\ &w(r) = (k\sigma)^{-1}\left(\frac{d}{dr} + \frac{1}{r}\right)u(r), \\ &u(r_j) = \frac{d}{dr}u(r_j) = v(r_j) = 0, \qquad j = 1, 2. \end{aligned}$$

If $a \geqq 0$, $v_\varphi^{\;0} \geqq 0$, (4.8) has no real eigenvalues [58]. If $v_\varphi^{\;0}$ changes sign, nothing is known about the spectrum. Therefore, we restrict the following considerations to the case $a < 0$, $v_\varphi^{\;0}(r) \geqq 0$.

With the aid of the Green's functions, $G$ resp. $H$, of the fourth-, resp. second-, order differential operator in (4.8), we transform (4.8) into a system of integral equations

$$(4.9)\qquad \begin{aligned} &\text{(a)}\quad u(r) = -2k^2\sigma^2\lambda \int_{r_1}^{r_2} H(r, \hat{r}; k\sigma)\frac{v_\varphi^{\;0}(\hat{r})}{\hat{r}}v(\hat{r})\, d\hat{r}, \\ &\text{(b)}\quad v(r) = -2a\lambda \int_{r_1}^{r_2} G(r, \hat{r}; k\sigma)u(\hat{r})\, d\hat{r}, \\ &\text{(c)}\quad w(r) = -2k\sigma\lambda \int_{r_1}^{r_2} \hat{H}(r, \hat{r}; k\sigma)\frac{v_\varphi^{\;0}(\hat{r})}{\hat{r}}v(\hat{r})\, d\hat{r}, \end{aligned}$$

with $\hat{H}(r, \hat{r}; k\sigma) = \partial H(r, \hat{r}; k\sigma)/\partial r + H(r, \hat{r}; k\sigma)$.

The adjoint problem has the following form:

$$(4.9)'\qquad \begin{aligned} &\text{(a)}\quad \tilde{u}(r) = -2a\lambda \int_{r_1}^{r_2} G'(r, \hat{r}; k\sigma)\tilde{v}(\hat{r})\, d\hat{r}, \\ &\text{(b)}\quad \tilde{v}(r) = -2k\sigma\lambda\frac{v_\varphi^{\;0}(r)}{r}\left\{k\sigma\int_{r_1}^{r_2} H'(r, \hat{r}; k\sigma)\tilde{u}(\hat{r})\, d\hat{r} + \int_{r_1}^{r_2} \hat{H}(r, \hat{r}; k\sigma)\tilde{w}(\hat{r})\, d\hat{r}\right\}, \\ &\text{(c)}\quad \tilde{w}(r) = 0, \end{aligned}$$

where $G'$, $H'$, $\hat{H}'$ denote the transposed kernels. Obviously, $\mu = \lambda^2$ is the characteristic parameter. $\mu$ is simple if and only if

$$(\varphi, \psi)_0 = \frac{\pi}{k\sigma} \int_{r_1}^{r_2} (u(r)\tilde{u}(r) + v(r)\tilde{v}(r) + w(r)\tilde{w}(r))\, dr \neq 0$$

holds. Thus, by $\tilde{w}(r) = 0$, the simplicity of a characteristic value is determined by the equations (4.9a) and (4.9b) only. Before the number of characteristic values is counted, we mention that (4.9) gives an explicit representation of the equations

$$\varphi + \lambda A^{-1}M(v^0)\varphi = 0, \quad \psi + \lambda M'(v^0)A^{-1}\psi = 0,$$

$$\varphi(r, z) = (u(r) \sin k\sigma z, v(r) \sin k\sigma z, w(r) \cos k\sigma z),$$

$$\psi(r, z) = (\tilde{u}(r) \sin k\sigma z, \tilde{v}(r) \sin k\sigma z, 0).$$

The solutions of $\tilde{\varphi} + \lambda K(v^0)\tilde{\varphi} = 0$ and $\tilde{\psi} + \lambda K'(v^0)\tilde{\psi} = 0$ are connected to $\varphi$ and $\psi$ by $\varphi = A^{-3/4}\tilde{\varphi}$, $\psi = A^{3/4}\tilde{\psi}$. Hence $(\varphi, \psi)_0 = (\varphi, \tilde{\psi})_0$ and the simplicity of a characteristic value of $K$ can be determined by considering (4.9a,b) only.

Iudovich [21] has shown—using the theory of oscillation kernels in the sense of Krein [29]—that the spectrum of (4.9a,b) consists of a sequence of simple positive characteristic eigenvalues $\mu_i(k\sigma) = \lambda_i^2(k\sigma)$, $0 < \mu_1(k\sigma) < \mu_2(k\sigma) < \cdots$. The corresponding functions $u$, $v$ have exactly $i - 1$ zeros in $(r_1, r_2)$ which are simple. This property is used for the case $i = 1$ in the next section. Hence, $K$ has an infinite sequence of simple characteristic values $\lambda_i(k\sigma) = \pm(\mu_i(k\sigma))^{1/2}$. Regarding the analyticity of $\lambda_i$ with respect to $\sigma > 0$ Iudovich [21] was able to prove that $\lambda_i(k\sigma) \neq \lambda_r(s\sigma)$ for $i, k, r, s \in N$, not all equal, and for all $\sigma > 0$, except at most countably many. This property implies the simplicity of the characteristic value $\lambda_i$ of $K(v^0)$ for "almost all" $\sigma$.

THEOREM 4.3. *Let be $a < 0$, $v_\varphi^0(r) > 0$ for $r \in (r, r_2)-a$ and $v_\varphi^0$ defined in* (2.2). *Then, for all $\sigma > 0$—except at most a countable set of positive numbers—there exist countably many real simple eigenvalues $(-\lambda_i)^{-1}$ of $K$. Every point $(\lambda_i, 0) \in R \times D(A)$ is a bifurcation point of* (4.1) *where exactly one nontrivial solution branch $(\lambda(\alpha), w(\alpha))$ emanates. These solutions are Taylor vortices.*

Recent results of Rabinowitz [47] show that every nontrivial solution branch is either connected to infinity or to another bifurcation point. For $\lambda_1$, the smallest characteristic value, Theorem 4.3 was first proven by Velte [62]; its general form is due to Iudovich [21].

In view of the definition of $\lambda$ in (2.1), the negative characteristic values $\lambda_i$ are of no physical significance. Moreover, strong experimental evidence suggests that all solutions branching off $(\lambda_i, 0)$, where $\lambda_i \neq \lambda_1$, are unstable, however no proof is known.

The general form of the eigenfunction $\varphi$ contains the parameter $k\sigma$. However, the solution branch corresponding to this eigenfunction is periodic with period $2\pi/k\sigma$. Thus, no generality is lost if $k$ is assumed to be equal to 1.

COROLLARY 4.4. *Let the assumptions of Theorem* 4.3 *be satisfied and let* $\sigma > 0$ *be such that* $\lambda_k(\sigma) > 0$ *is a simple characteristic value of* (4.8), *resp.* (4.7). *Then the nontrivial solution branch with period* $2\pi/\sigma$

(i) *either exists to the left, resp. right, of* $\lambda_k$ *and permits the parametrization*

$$\lambda(\alpha) = \lambda, \qquad w(\lambda) = \pm |\lambda_k - \lambda|^{1/r} F(\lambda)$$

*where* $r \in N$ *is even and* $F : R \to D(A)$ *is analytic in* $(\lambda_k - \lambda)^{1/r}$, *resp.* $(\lambda - \lambda_k)^{1/r}$;

(ii) *or exists for all* $\lambda$ *in a full neighbourhood of* $\lambda_k$ *and permits the parametrization*

$$\lambda(\alpha) = \lambda, \qquad w(\lambda) = (\lambda - \lambda_k)^{1/r} F(\lambda),$$

*where* $r \in N$ *is odd,* $F$ *as above.*

PROOF. Let $Q$ denote the eigenprojection to $\lambda_k$. Setting $Qv = z\varphi$, $z \in R$, $v_1 = v - z\varphi$, $\varphi$ denotes the normalized (in $L_2{}^T$) eigenfunction of $K$ in (4.2), then, for sufficiently small $|\lambda - \lambda_k|$, $|z|$ (cf. [33]),

$$v_1 = \sum_{\mu=2;\,\nu=0}^{\infty} v_{\mu\nu} z^\mu \tau^\nu, \qquad \tau = \lambda - \lambda_k,$$

where the series converges in $\mathring{J}^T$. The "bifurcation equation" for the determination of $z$ is obtained by applying $Q$ to (4.2), yielding

$$\begin{aligned} & -\frac{\lambda - \lambda_k}{\lambda_k} z\varphi + h(\lambda) T(z\varphi + v_1) \\ (4.10) \qquad & = \left\{ -\frac{\lambda - \lambda_k}{\lambda_k} z + \sum_{\mu=2}^{\infty} a_{\mu 0} z^\mu + \sum_{\mu=2;\ \nu=1}^{\infty} a_{\mu\nu} z^\mu (\lambda - \lambda_k)^\nu \right\} \varphi = 0. \end{aligned}$$

$z = 0$ corresponds to the trivial solution $v = 0$. It is well known that

$a_{20} = 0$ [33]. If all $a_{\mu 0} = 0$ then the nontrivial solutions of (4.10) satisfy $z(0) \neq 0$. Since the corresponding $v$ does not bifurcate from the trivial solution, $a_{\mu 0} = 0$ for all $\mu$, yielding a contradiction to Theorems 4.1 or 4.3. Thus $a_{r+1,0} \neq 0$, $a_{20} = \cdots = a_{r0} = 0$, which by Newton's diagram (cf. [11]) implies that the real nontrivial solutions of (4.10) are power series in $(\lambda - \lambda_k)^{1/r}$, resp. $(\lambda_k - \lambda)^{1/r}$, depending on the sign of $a_{r+1,0}$. By (4.2) and (4.3), the power series for $v$ converges in $D(A^{1/4})$ and therefore, $w = A^{-3/4}v$ possesses the asserted representation.

Of special physical interest is the smallest positive characteristic value $\lambda_1(\sigma)$ which determines the critical Reynolds number where Taylor vortices are observed. Numerical calculations show that $a_{30} < 0$ for $\lambda = \lambda_1(\sigma)$ (cf. [33]). Thus, bifurcation is expected to the right, and the branching solution behaves like $\pm(\lambda - \lambda_1)^{1/2}$ near $\lambda_1$; however no proof is known. The "upper" and "lower" part of the branch, corresponding to the plus, resp. minus sign are Taylor vortices differing only in the orientation of particle paths.

Secondary bifurcation of Taylor vortices in form of "wavy" vortices [10], [38] has to be expected on experimental evidence [8], [13], [56] and analytical calculations. However, due to the complexity of the linear eigenvalue problems, no proofs are known.

*Bénard model.* The analysis of this case proceeds analogously to that of the Taylor model. However, since two free parameters, $\alpha$ and $\beta$, are involved, the set of solutions is richer. Moreover, quantitative information about the bifurcation behaviour can be obtained.

Consider (3.22) with $\tilde{A}(\lambda) = A + \lambda M(v^0) = A + \lambda PL(v^0)$, where $v^0$ and $L$ are given by (2.6) and (2.7), $h(\lambda) = 1$, $w = (u, \theta)$. Suppose $\lambda > 0$ and replace $u$ by $u\sqrt{\lambda}$, then, by (2.7), one obtains the explicit form of (3.22):

$$
\begin{aligned}
&D_t u - \Delta u - \sqrt{\lambda}\,\theta e + \nabla q = -\sqrt{\lambda}(u \cdot \nabla)u,\\
(4.11)\qquad &D_t \theta - Pr^{-1}\Delta\theta - \sqrt{\lambda}u_3 = -\sqrt{\lambda}(u \cdot \nabla)\theta,\\
&\nabla \cdot u = 0, \qquad w|_{\partial D} = 0, \qquad w|_{t=0} = w^0,
\end{aligned}
$$

with $e = (0, 0, 1)$. If

$$w \in L_{2,\mathrm{loc}}([0, \infty); D(A)), \qquad dw/dt \in L_{2,\mathrm{loc}}([0, \infty); \mathring{J}^T),$$

$u$ solves (4.11), scalar multiplication by $(u, \theta)$ yields (cf. (3.8))

$$(4.12)\qquad \frac{1}{2}\frac{d}{dt}|w|_0^2 + \|w\|^2 - 2\sqrt{\lambda}(u_3, \theta)_0 = 0.$$

The functional in the variational problem

$$\sup_{w \in \mathring{H}^T_{1,\sigma}} 2(u_3, \theta)_0/\|w\|^2 = 1/\mu_1 > 0$$

is weakly upper semicontinuous. By the same arguments as used in Lemma 3.3, its solution belongs to $D(A)$. It satisfies the Euler-Lagrange equations:

$$(4.13) \qquad \begin{aligned} -\Delta u - \mu\theta e + \nabla q = 0, \qquad -Pr^{-1}\Delta\theta - \mu u_3 = 0,\\ u|_{\partial D} = \theta|_{\partial D} = 0 \end{aligned}$$

for $\mu = \mu_1$ which, for $\sqrt{\lambda} = \mu_1$, coincides with the linearized stationary problem corresponding to (4.11). Thus, $\mu_1$ is the smallest positive characteristic value of (4.13). Moreover, for $0 \leqq \lambda < \lambda_1 = \mu_1{}^2$ one obtains, using (3.4), (4.12) and Corollary 3.4,

$$\frac{1}{2}\frac{d}{dt}|w|_0^2 + c_1|w|_0^2\left(1 - \left(\frac{\lambda}{\lambda_1}\right)^{1/2}\right)$$
$$\leqq \frac{1}{2}\frac{d}{dt}|w|_0^2 + \|w\|^2\left(1 - \left(\frac{\lambda}{\lambda_1}\right)^{1/2}\right) \leqq 0$$

which implies $|w(t)| \to 0$ as $t \to \infty$. This is a well-known result by Joseph [25]. We gave the proof, since it can be applied to the study of the equation $dw/dt + \tilde{A}(\lambda)w = 0$ as well. From the Hille-Yosida theorem it follows that the spectrum of $\tilde{A}(\lambda)$, consisting of eigenvalues only, lies in the positive complex half plane.

LEMMA 4.5. *Let be $\tilde{A}(\lambda) = A + \lambda M(v^0)$, $M = PL$, where $v^0$ and $L$ are given by (2.6), resp. (2.7), and let $\lambda_1 = \mu_1{}^2$, $\mu_1$ the smallest positive characteristic value of (4.13). Then, for $0 \leqq \lambda < \lambda_1$,*

(i) *$w = 0$ is the unique stationary solution of (3.22),*

(ii) *$w = 0$ is asymptotically stable for $w^0 \in \mathring{J}^T$, arbitrary, $w \in L_{2,\mathrm{loc}}([0, \infty); D(A))$, $dw/dt \in L_2([0, \infty), \mathring{J}^T)$,*

(iii) *there exists a positive constant $\delta$ such that* re $\mu \geqq \delta$ *for every $\mu \in \sigma(\tilde{A}(\lambda))$.*

Through Theorem 4.1 the bifurcation picture is determined by the spectrum of $K = A^{-1/4}M(v^0)A^{-3/4}$, which consists of eigenvalues only. Since (4.2) is equivalent to (4.1), the eigenvalue problem can be derived by using the linear part of (4.1) which, after the substitution $u \to u\sqrt{\lambda}$, for $\lambda > 0$, $w = (u, \theta)$, $\mu = \sqrt{\lambda}$, coincides with equation (4.13).

In order to obtain simple eigenvalues we again have to introduce "even" solutions [23], [61] by requiring

$$u(-x) = (-u_1(x), -u_2(x), u_3(x)),$$
$$\theta(-x) = \theta(x), q(-x) = -q(x).$$

It is easily verified that the differential operators in (2.8) preserve these invariance conditions.

The function $w \in C^{T,\infty}(\bar{D})$ (Lemma 3.5) satisfies the boundary condition pointwise. Elimination of $q$ transforms (4.13) into (cf. [7])

$$\begin{aligned} \Delta^2 u_3 &= \sqrt{\lambda}(\partial^2 \theta/\partial x_1{}^2 + \partial^2 \theta/\partial x_2{}^2), \\ \Delta\, \theta &= Pr\sqrt{\lambda} u_3, \\ u_3|_{\partial D} &= \theta|_{\partial D} = 0. \end{aligned} \tag{4.14}$$

Solutions of (4.14) are sought having the invariance properties (2.8d) and (2.8e). By Lemma 2.1 and Corollary 2.2, only some exceptional combinations of $\omega = 2\pi/n$, $\alpha$ and $\beta$ are possible. We give the general form of the eigenfunction $u_3$ for the case of even solutions; $\theta$ has an analogous representation whereas the forms of $u_1$ and $u_2$ are obtained by replacing sin- by cos-terms.

$n = 1$ *or* 2: no cell structure, rolls ($\beta = 0$), rectangles

$$u_3(x) = u_{3\mu}(x_3) \cos(\nu\alpha x_1 + \mu\beta x_2).$$

$n = 3$ *or* 6, $\alpha = \beta\sqrt{3}$: hexagons or triangles

$$\begin{aligned} u_3(x) = u_{\nu\mu}(x_3)\Big[&\cos\{\beta(\nu\sqrt{3}x_1 + \mu x_2)\} \\ &+ \cos\left\{\beta\left(-\frac{\nu-\mu}{2}\sqrt{3}x_1 - \frac{3\nu+\mu}{2}x_2\right)\right\} \\ &+ \cos\left\{\beta\left(-\frac{\nu+\mu}{2}\sqrt{3}x_1 + \frac{3\nu-\mu}{2}x_2\right)\right\}\Big]. \end{aligned}$$

$n = 4$, $\alpha = \beta$: squares

$$u_3(x) = u_{\nu\mu}(x_3)[\cos\{\beta(\nu x_1 + \mu x_2)\} + \cos\{\beta(\mu x_1 - \nu x_2)\}].$$

In either case (4.14) leads to

$$\begin{aligned} (d^2/dx_3{}^2 - \sigma^2)^2 u_{\nu\mu} &= \sigma^2\sqrt{\lambda}\,\theta_{\nu\mu}, \\ (d^2/dx_3{}^2 - \sigma^2)\,\theta_{\nu\mu} &= Pr\sqrt{\lambda} u_{\nu\mu}, \end{aligned} \tag{4.15}$$

$$u_{\nu\mu}(k) = \frac{du_{\nu\mu}}{dx_3}(k) = \theta_{\nu\mu}(k) = 0, \qquad k = 0, 1, \sigma^2 = \nu^2\alpha^2 + \mu^2\beta^2.$$

The analogy to (4.8) is obvious now. Iudovich [22] has shown that

(4.15) is equivalent to an integral equation of the form (4.9) ($\lambda^2 \to \lambda$, $r_k \to k-1$, $k = 0, 1$) with an oscillation kernel $C$. Thus (4.15) has a countable sequence of positive simple characteristic values $0 < \lambda_1(\sigma) < \lambda_2(\sigma) < \cdots$. To ensure that $\lambda_k(\sigma)$ is a simple characteristic value of (4.14) one has to verify $\lambda_k(\sqrt{\nu^2\alpha^2 + \mu^2\beta^2}) \neq \lambda_i(\sqrt{n^2\alpha^2 + m^2\beta^2})$ for all $i, k, \nu, \mu, n, m \in N$, except for $i = k$, $\nu = n$, $\mu = m$. Using the analyticity of $\lambda_i(\sigma)$, Iudovich [22] proved this to be true for all $\alpha, \beta$ except a set S whose intersection with every analytic curve in the $(\alpha, \beta)$-plane consists of at most countably many points (nomenclature: approximately everywhere).

The investigation of the quantitative behaviour in the neighbourhood of a bifurcation point proceeds as in the proof of Corollary 4.4. In the Bénard case however, one can show that for $k = 1$, $a_{20} < 0$ in (4.10) [23].

The general form of the eigenfunctions contains cosine-terms with the argument $\nu\alpha x_1 + \mu\beta x_2$. However, the Fourier expansion of the corresponding branching solution consists of terms with argument $n\nu\alpha x_1 + m\mu\beta x_2$, $n, m \in Z$, only. Thus, without loss of generality, we may assume $\nu = \mu = 1$ and $\sigma^2 = \alpha^2 + \beta^2$.

THEOREM 4.6 [23]. (i) *The Bénard problem* (2.8) *possesses for approximately all* $\alpha$ *and* $\beta$ *countably many simple positive characteristic values* $\lambda_i$. *Furthermore* $(\lambda_i, 0) \in \boldsymbol{R} \times D(A)$ *is a bifurcation point.*

(ii) *If* $n$, $\alpha$, $\beta$ *are chosen according to Corollary* 2.2, *the branches emanating from* $(\lambda_i, 0)$ *are doubly periodic, rolls, hexagons, rectangles or triangles.*

(iii) *If* $\lambda_1$ *denotes the smallest characteristic value, then the nontrivial solution branches to the right of* $\lambda_1$ *and permits the parametrization*

$$w(\lambda) = \pm(\lambda - \lambda_1)^{1/2}F(\lambda)$$

*where* $F : \boldsymbol{R} \to D(A)$ *is holomorphic in* $(\lambda - \lambda_1)^{1/2}$.

The positive and negative signs in the above representation give solutions differing in the flow direction at $x_1 = x_2 = 0$ (center of the cell). Since the characteristic values $\lambda_k$ are determined by $\sigma = (\nu^2\alpha^2 + \mu^2\beta^2)^{1/2}$ only, solutions of every possible cell structure emanate from each bifurcation point $(\lambda_k, 0)$, the structure depending on the choice of $\alpha$ and $\beta$. Thus, the solution set of the stationary problem (4.1) is so rich that stability properties must explain the selection of distinct cell patterns by nature.

Certain differences in the qualitative behaviour of the bifurcation solutions considered in Theorem 4.6 have been found in generalized Bénard models for the convection flow in a heated fluid layer. We conclude this section with a short survey of those results. Generalized problems may be formed by allowing the material parameters, such as viscosity, to depend on the temperature — the rate of variation is measured by some small positive parameter $\gamma$ — ([4], [41], [52], [60]), or by introducing a steady change of the mean temperature at the rate $\gamma$ caused by a steadily increasing heat supply (free or poorly conducting surfaces) ([36], [37], [16]). In all these cases solutions have been constructed in the form of double power series in $\epsilon$ and $\gamma$, where $\epsilon$ measures the amplitude, in a neighbourhood of the eigenvalue $\lambda_1$. The method has been justified in [46], [16] by a convergence proof.

The results for $\gamma = 0$ are in agreement with the assertions of Theorem 4.6 [50]. For $\gamma \neq 0$ however, there exist "subcritical" solutions, i.e. solutions for $\lambda < \lambda_1$ with hexagonal structure, the direction of the flow in the cell-center being determined by the sign of $\gamma$; all other solutions bifurcate to the right of $\lambda_1$. The selection of certain cell patterns is explained by considering the behaviour of the spectrum of $\tilde{A}(\lambda_1; V(\lambda))$ near $\lambda_1$ and by drawing conclusions about the stability and instability. A critical survey of these arguments is given at the end of §5.

5. **Stability and bifurcation.** In this section the relation between bifurcation, stability and instability is studied. It is well known that the basic solution loses stability for some $\lambda_c \in (0, \lambda_1]$, where $\lambda_1$ was defined in the preceding section as the smallest parameter with $0 \in \sigma(\tilde{A}(\lambda))$. If $\lambda_c \in (0, \lambda_1)$ then two conjugate complex eigenvalues of $\tilde{A}(\lambda_c)$ lie on the imaginary axis. In this case time periodic solutions of (3.22) may exist, as was proven by Joseph and Sattinger recently [26]. The condition $\lambda_c = \lambda_1$ is known as the "principle of exchange of stabilities" (PES). The PES holds for the Bénard problem, yet its validity for the Taylor model is an open problem. Under the assumption that $\lambda_c = \lambda_1$, and $\lambda_1$ simple, the nontrivial solution branch emanating from $(\lambda_1, 0)$ gains stability for $\lambda > \lambda_1$ and is unstable for $\lambda < \lambda_1$ (Lemma 5.6). This result can be derived using Leray-Schauder degree ([24], [49]) or by analytic perturbation methods [32]. The applications to the Bénard and Taylor problem are formulated in Theorem 5.7. At the end of this section we give a proof that every branch, corresponding to a value of $\sigma$ which is not a locus of a local minimum of $\lambda_1(\sigma)$, is unstable in a suitable function space [32]. This argument explains the selection of a preferred cell size in the Bénard case (Theorem 5.8).

For the intended applications we need a general result relating properties of $\sigma(\tilde{A}(\lambda))$ to the stability or instability of the trivial solution (corresponding to the stationary solution $V$). (Stability is understood in the sense of Lyapunov.) The first result of this kind was given by Prodi [45] who proved asymptotic stability in $\mathring{H}^T_{1,\sigma}$ if $\sigma(\tilde{A}(\lambda))$ lies in a half plane $\operatorname{re}\mu \geqq \alpha > 0$ ($\operatorname{re}\sigma(\tilde{A}(\lambda)) \geqq \alpha > 0$). Later, Iudovich announced theorems on stability and instability in [20]. Sattinger [48] has shown that $\operatorname{re}\sigma(\tilde{A}(\lambda)) \geqq 0$ is necessary and $\operatorname{re}\sigma(\tilde{A}(\lambda)) \geqq \alpha > 0$ is sufficient for stability in $\mathring{J}^T$, if the class of solutions considered consists of weak solutions of Hopf type. (For necessity one needs a slightly stronger condition on $\sigma(\tilde{A}(\lambda))$ (see Theorem 5.5).) Asymptotic stability in $D(A)$ — including pointwise stability — was proven by Iooss in [19]. Here we show the necessity and sufficiency of the spectral stability conditions for strict solutions in the sense of Kato-Fujita [28] in $\mathring{J}^T$ or $D(A^\beta)$, $\frac{3}{4} \leqq \beta < 1$.

Following Kato-Fujita [28] we call $w : [0,\tau] \to \mathring{J}^T$ a strict solution of (3.22) if

$$
\begin{aligned}
&\text{(i) } w(0) = w_0,\\
&\text{(ii) } w \in C([0,\tau], \mathring{J}^T),\\
&\text{(iii) } w \in C^1((0,\tau], \mathring{J}^T),\\
&\text{(iv) } w(t) \in D(A) \quad \text{for all } t \in (0,\tau] \text{ and } Aw \in C((0,\tau], \mathring{J}^T),\\
&\text{(v) } w \text{ solves (3.22) in } (0,\tau].
\end{aligned}
\tag{5.1}
$$

LEMMA 5.1. *Let be $\beta \in [\frac{3}{4}\, 1)$, $w_0 \in D(A^\beta)$. Then there exists a $\tau_\beta \in (0, \infty]$ such that* (3.22) *possesses a unique strict solution $w$ in every interval $[0,\tau]$, $\tau < \tau_\beta$, with the additional properties*
(i) *$w(t) \in D(A^\beta)$ for all $t \in [0,\tau]$, and $A^\beta w \in C([0,\tau], \mathring{J}^T)$,*
(ii) *$\lim_{\tau \to \tau_\beta} |A^\beta w(\tau)|_0 = \infty$, if $\tau_\beta < \infty$.*

For the proof we refer to the appendix. The case $\lambda = 0$ was considered in [28] under the weaker assumption $w_0 \in D(A^{1/4})$. Our proof guarantees the validity of the conditions (5.1) (iii) to (v) even in the closed interval $[0,\tau]$ under stronger assumptions on $w_0$ (e.g., $A^\beta w_0 \in D(A)$, $Mw_0 \in D(A^\beta)$, $R(w_0) \in D(A^\beta)$), Lemma 5.1 implies moreover that an a priori bound for $|A^\beta w|_0$ yields the global existence of a strict solution.

LEMMA 5.2. *Let* $\operatorname{re}\sigma(\tilde{A}(\lambda)) \geqq \alpha > 0$, $\beta \in (\frac{3}{4}, 1)$, $w_0 \in D(\tilde{A}(\lambda)^\beta)$. *Then there exists a $\tilde{\tau}_\beta \in (0, \infty]$ such that* (3.22) *possesses a unique strict solution $w$ in every interval $[0,\tau]$, $\tau < \tilde{\tau}_\beta$ with the additional properties*

(i) $w(t) \in D(\tilde{A}(\lambda)^\beta)$ *for all* $t \in [0, \tau]$ *and* $\tilde{A}(\lambda)^\beta w \in C([0, \tau], \mathring{J}^T)$,
(ii) $\lim_{\tau \to \tau_\beta} |\tilde{A}(\lambda)^\beta w(\tau)|_0 = \infty$, *if* $\tilde{\tau}_\beta < \infty$.
*If* $|\tilde{A}(\lambda)^\beta w_0|_0$ *is sufficiently small, then* $\tilde{\tau}_\beta = \infty$.

The proof can be found in the appendix. The condition for the spectrum is imposed to ensure the existence of fractional powers of $\tilde{A}(\lambda)$.

THEOREM 5.3. *Let* $\operatorname{re}\sigma(\tilde{A}(\lambda)) \geqq \alpha > 0$, $0 < b < \alpha$, $\beta \in (\frac{3}{4}, 1)$, $w_0 \in D(\tilde{A}(\lambda)^\beta)$. *To every positive* $\epsilon$ *there exists a* $\delta > 0$, $\delta$ *depending on* $\lambda$ *and* $\beta$, *such that* $|\tilde{A}(\lambda)^\beta w_0|_0 \leqq \delta$ *implies*
(i) *the existence of a strict solution* $w$ *in* $[0, \infty)$,
(ii) $|\tilde{A}(\lambda)^\beta w(t)|_0 \leqq \epsilon e^{-bt}$ *for* $t \in [0, \infty)$.

PROOF. In view of property (i) in Lemma 5.2 we may set $v(t) = e^{bt}\tilde{A}(\lambda)^\beta w(t)$, $t \in [0, \tilde{\tau}_\beta)$, $0 < b < \alpha' < \alpha$, and we obtain from (3.19) and the integral representation of (3.22)

$$v(t) = e^{bt} \exp(-\tilde{A}(\lambda)t)\tilde{A}(\lambda)^\beta w_0$$
$$- h(\lambda) \int_0^t e^{bt}\tilde{A}(\lambda)^\beta \exp(-\tilde{A}(\lambda)(t-s))R(e^{-bs}\tilde{A}(\lambda)^{-\beta}v(s))\, ds.$$

Lemmas 3.12 and 3.8 yield the boundedness of the nonlinearity; together with (3.19b) and Lemma 3.11 this implies the estimate

$$|v(t)|_0 \leqq \gamma_6 e^{-(\alpha'-b)t}|\tilde{A}(\lambda)^\beta w_0|_0$$
$$+ \gamma_7 c_1 \int_0^t e^{-(\alpha'-b)(t-s)}(t-s)^{-\beta}|v(s)|_0^2\, ds.$$

Set $x_0 = |\tilde{A}(\lambda)^\beta w_0|_0$. There exists an $\epsilon_0 > 0$ such that, for all $0 < \epsilon \leqq \epsilon_0$ and $x_0 \leqq \delta(\epsilon)$,

$$\gamma_6 x_0 + \epsilon^2 \gamma_7 c_1 \int_0^\infty e^{-(\alpha'-b)s} s^{-\beta}\, ds < \epsilon$$

holds. The set $J = \{t \mid t \in [0, \tilde{\tau}_\beta),\ |v(s)|_0 \leqq \epsilon,\ s \in [0, t]\}$ is closed and open in $[0, \tilde{\tau}_\beta)$. Thus, by Lemma 5.2(ii), $\tilde{\tau}_\beta = \infty$, which yields the assertion.

The trivial solution $w = 0$ of (4.1), and therefore (see (3.12)) the stationary solution $V$, is asymptotically stable in $D(\tilde{A}(\lambda)^\beta)$ if $\operatorname{re}\sigma(\tilde{A}(\lambda)) \geqq \alpha > 0$. An analogous result is valid in $D(A^\beta)$. We state explicitly the notion of stability used in this context: $w = 0$ *is called stable in a Banach space* $(X, \|\ \|)$ if, for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $\|w(t)\| \leqq \epsilon$ holds in $[0, \tau)$ for all strict solutions $w$ of (3.22) with $\|w_0\| \leqq \delta(\epsilon)$, $[0, \tau)$ denoting the maximal interval of existence. $w = 0$ *is called asymptotically stable in* $X$ if it is stable and $\lim_{\tau\to\infty}\|w(t)\| = 0$.

COROLLARY 5.4. *Let be* $\operatorname{re}\sigma(\tilde{A}(\lambda)) \geqq \alpha > 0$. *Then* $w = 0$ *is asymptotically stable in* $D(A^\beta), \beta \in [\frac{3}{4}, 1)$.

PROOF. The solution $w$ solves the integral equations (see (3.12)):

$$(5.2)\quad \begin{aligned} &\text{(a)}\quad w(t) = \exp(-\tilde{A}(\lambda)t)w_0 \\ &\qquad\qquad + \int_0^t \exp(-\tilde{A}(\lambda)(t-s))(-h(\lambda)R(w(s)))\,ds, \\ &\text{(b)}\quad w(t) = \exp(-At)w_0 \\ &\qquad\qquad + \int_0^t \exp(-A(t-s))(-\lambda Mw(s) - h(\lambda)R(w(s)))\,ds. \end{aligned}$$

Setting $v(t) = e^{bt}A^\beta w(t)$, $0 < b < \alpha' < \alpha$, one obtains, from (3.20), Lemmas 3.6 and 3.8 as in the proof of Theorem 5.3,

$$(5.3)\quad \begin{aligned} &\text{(a)}\quad |v(t)|_0 \leqq c_2 e^{-(\alpha'-b)t}t^{-\beta}|w_0|_0 \\ &\qquad\qquad + c_4\int_0^t e^{-(\alpha'-b)(t-s)}(t-s)^{-\beta}|v(s)|_0^2\,ds, \\ &\text{(b)}\quad |v(t)|_0 \leqq e^{-(\alpha'-b)t}|A^\beta w_0|_0 \\ &\qquad\qquad + c_5\int_0^t e^{-(\alpha'-b)(t-s)-\beta}(|\lambda|\,|v(s)|_0 + |h(\lambda)|\,|v(s)|_0^2)\,ds, \end{aligned}$$

By (5.3b) there exist constants $\tau_0 \in (0, \tau_\beta)$, $\epsilon_0 > 0$, such that, to every $\epsilon$, $0 < \epsilon \leqq \epsilon_0$, there is a $\delta = \delta(\epsilon)$ with $|v(t)|_0 \leqq \epsilon$, $t \in [0, \tau_0]$, if only $|A^\beta w_0|_0 \leqq \delta$. For $t \geqq \tau_0$ we get, from (5.3a),

$$|v(t)|_0 \leqq c_6|A^\beta w_0|_0 + c_4\int_0^t e^{-(\alpha'-b)(t-s)}(t-s)^{-\beta}|v(s)|_0^2\,ds.$$

Choose $\epsilon_1 > 0$, $\delta' = \delta(\epsilon')$ so that, if $|A^\beta w_0|_0 \leqq \delta'$,

$$c_6|A^\beta w_0|_0 + c_4\epsilon'^2\int_0^\infty e^{-(\alpha'-b)s}s^{-\beta}\,ds < \epsilon', \qquad 0 < \epsilon' \leqq \epsilon_1,$$

holds. Let be $\epsilon = \min(\epsilon_0, \delta', \epsilon')$ then, $|v(t)|_0 \leqq \epsilon'$ for all $t \in [0, \tau_\beta)$ and for all $w_0$ with $|A^\beta w_0|_0 \leqq \min(\delta, \delta')$. Q.E.D. Moreover, Lemma 5.1(ii) implies $\tau_\beta = \infty$.

Except for a slightly different notion of strict solution, Prodi's result [45] is the case $\beta = \frac{1}{2}$. Sattinger [48] proved asymptotic stability in $\mathring{J}^T$ for weak solutions in the sense of E. Hopf. Hence, the stability estimate of Corollary 5.4 is stronger than those mentioned above; however, whether Sattinger's class of solutions is larger than the one considered here is still an open problem. Iooss' Corollary 1.1 in [19] is the limit case $\beta = 1$. A somewhat complementary version of Corollary 5.4 is stated in the following theorem.

THEOREM 5.5. *If $w = 0$ is stable in $\mathring{J}^T$ and if $\tilde{A}(\lambda)$ has no eigenvalues with vanishing real part, then* $\operatorname{re}\sigma(\tilde{A}(\lambda)) > 0$.

*Note.* The condition that $\tilde{A}(\lambda)$ has no eigenvalue with vanishing real part is obsolete, as was shown by the second author recently.

PROOF. Assume that $\sigma(\tilde{A}(\lambda))$ has spectral points with negative real part. In view of Lemma 3.9 and Corollary 3.10, there exist only finitely many of those spectral points and they are eigenvalues with finite multiplicities. Let $P$, resp. $Q = 1 - P$, denote the eigenprojections corresponding to the negative, resp. positive, part of the spectrum. Then, by Lemma 3.9, the restriction $\tilde{A}(\lambda)_2 = \tilde{A}(\lambda)|_{Q\mathring{J}^T}$ satisfies $\operatorname{re}\sigma(\tilde{A}(\lambda)_2) \geqq \alpha > 0$ for some $\alpha$. $\tilde{A}(\lambda)_2$ is densely defined in $Q\mathring{J}^T$, and generates a holomorphic semigroup with the properties stated in Lemma 3.11. Thus, $\tilde{A}(\lambda)_2{}^\beta$ can be defined as in (3.18) and the estimates (3.19) and Lemma 3.12 hold.

Let $w$ be a strict solution of (3.22) in $[0, \tau)$, where $[0, \tau)$ is the maximal interval of existence, with the initial condition $w_0$, and $Qw_0 = 0$. Since $P$ projects $\mathring{J}^T$ into a subspace spanned by finitely many generalized eigenfunctions, $w_0$ is arbitrarily smooth and, by Lemma 5.1, $w(t) \in D(A^{3/4})$ for all $t \in [0, \tau)$ and $A^{3/4}w \in C([0, \tau), \mathring{J}^T)$. Therefore, $Qw$ solves the integral equation

$$Qw(t) = -h(\lambda) \int_0^t \exp(-\tilde{A}(\lambda)_2(t-s))QR(Pw(s) + Qw(s))\, ds.$$

Set $v = \tilde{A}(\lambda)_2{}^\beta Qw$ for some $\beta \in (\frac{3}{4}, 1)$. Then $v$ satisfies

$$v(t) = -h(\lambda) \int_0^t \tilde{A}(\lambda)_2{}^\beta \exp(-\tilde{A}(\lambda)_2(t-s))QR(Pw(s) + \tilde{A}(\lambda)_2{}^{-\beta}v(s))\, ds$$

since $\tilde{A}(\lambda)_2{}^\beta$ is closed and the following inequality is valid. Observe that in the finite-dimensional space $P\mathring{J}^T$, $|A^{3/4}w|_0 \leqq c_1|w|_0$ holds. In view of the Lemmas 3.8 and 3.12 and by (3.19b) we obtain

$$\text{(a)}\quad |v(t)|_0 \leqq f(t) + c_2 \int_0^t e^{-\alpha'(t-s)}(t-s)^{-\beta}|v(s)|_0^2\, ds,$$

$$(5.4) \qquad\qquad 0 < \alpha' < \alpha,$$

$$\text{(b)}\quad f(t) = c_1{}' \int_0^t e^{-\alpha'(t-s)}(t-s)^{-\beta}|Pw(s)|_0^2\, ds.$$

The integral in (5.4a) exists since $v = \tilde{A}(\lambda)_2{}^{\beta}Qw$ is bounded by $\tilde{A}(\lambda)_2 Qw$ in $[0,\tau)$, and this is bounded by $AQw$. The boundedness of $Aw$ follows from Lemma 6.1 in the appendix for smooth $w_0$.

The assumed stability in $\mathring{J}^T$ guarantees that $|Pw(t)| \leqq \epsilon$, $t \in [0,\tau)$, if only $|w_0|_0 \leqq \delta(\epsilon)$. In the course of the proof we shall dispose of $\epsilon$. Now we assert

$$(5.5)\qquad \begin{aligned} &|v(t)|_0 \leqq 2f(t), \qquad t \in [0,\tau),\\ &\epsilon^2 < (4c_1{}'c_2c_3{}^2)^{-1}, \quad \text{where } c_3 = \int_0^\infty e^{-\alpha' s}s^{-\beta}\,ds. \end{aligned}$$

Define $I = \{t \mid t \in [0,\tau),\ |v(s)|_0 \leqq 2f(t) \text{ for all } s \in [0,t]\}$. Since $|v(t)|_0$ is continuous, $I$ is closed in $[0,\tau)$. The openness of $I$ follows from (5.4a) by the inequality

$$|v(t)|_0 \leqq f(t) + 4c_2c_3f(t)^2 < 2f(t), \qquad t \in I,$$

by the choice of $\epsilon$, therefore $I = [0,\tau)$. Hence, $\tilde{A}^{\beta}Qw = v$ is bounded in $[0,\tau)$. By Lemma 3.12, $A^{3/4}Qw$ and thus $A^{3/4}w$ is bounded in $[0,\tau)$ as well. Lemma 5.1(ii) implies $\tau = \infty$.

We set $u = Pw$ and

$$\tilde{A}(\lambda)_1 = \tilde{A}(\lambda)|_{P\mathring{J}^T}, \qquad \mathrm{re}\,\sigma(\tilde{A}(\lambda)_1) \leqq -q < 0.$$

Choose a basis $\varphi_1, \cdots, \varphi_m$ in $P\mathring{J}^T$ such that

$$(\tilde{A}(\lambda)_1 u, u) \leqq -\tfrac{3}{4}q|u|^2$$

where the scalar product is defined by

$$(u^1, u^2) = \sum_{i=1}^m x_i{}^1 x_i{}^2, \qquad u^j = \sum_{i=1}^m x_i{}^j \varphi_i, \qquad j = 1, 2.$$

Applying $P$ to (3.22) yields

$$(5.6)\qquad du/dt + \tilde{A}(\lambda)_1 u + h(\lambda)PR(u + \tilde{A}(\lambda)_2{}^{-\beta}v) = 0.$$

By the Lemmas 3.8 and 3.12 and the equivalence of the norms $|\cdot|$, $|\cdot|_0$ in $P\mathring{J}^T$ we obtain

$$|h(\lambda)PR(u + \tilde{A}(\lambda)_2{}^{-\beta}v)| \leqq c_4(|u|^2 + |v|_0^2).$$

Scalar multiplication of (5.6) by $u$ implies (in view of (5.5)), for $\epsilon \leqq (4c_4)^{-1}q$,

$$\frac{d}{dt}|u(t)|^2 \geqq \frac{q}{2}|u(t)|^2 - 2c_4f(t)^2|u(t)|.$$

For some $\eta > 0$, $|u(t)|$ strictly increases for $t \in [0, \eta]$, since $f(0) = 0$. As long as $|u(t)|$ is increasing, $f(t) \leqq c_1{}' c_3 |u(t)|^2$ holds by (5.4b). Choose $\epsilon < (4c_1{}'^2 c_3^2 c_4)^{-1} q$, then $\eta = \infty$ and $|u(t)|$ grows exponentially. This contradicts the assumed stability.

Assume that $\sigma(\tilde{A}(\lambda))$ has an eigenvalue with negative real part and no purely imaginary eigenvalue. Then, Theorem 5.5 states that in every $J^T$-neighbourhood of $w = 0$ there exists a $w_0$ and a strict solution $w(t, w_0)$ of (3.22), with the maximal interval of existence $[0, \tau)$, such that $|w(t, w_0)|_0 > \epsilon_0$ for some $t \in [0, \tau)$ and some $\epsilon_0 > 0$. *In this sense* $w = 0$ *is called unstable.* Particularly, Theorem 5.5 implies the result of Sattinger [48] who proved instability in $\mathring{J}^T$ for weak solutions of Hopf-type.

$V = v_0 + w^+$, where $w^+$ is a stationary solution of (4.1), is called stable if, for $\tilde{A}(\lambda) = A + \lambda M(V)$, $w = 0$ is stable with respect to strict solutions in $D(A^\beta)$. $V$ is called unstable, if it is not stable in $\mathring{J}^T$. Since we have to investigate stability and instability for different $V$, we indicate the dependence of $\tilde{A}(\lambda)$ and $K$ on $V$ by the notation

$$(5.7) \qquad \tilde{A}(\lambda; V) = A + \lambda M(V), \qquad K(V) = A^{-1/4} M(V) A^{-3/4}.$$

By Corollary 5.4 and Theorem 5.5, stability and instability of $V$ are essentially determined by $\sigma(\tilde{A}(\lambda; V))$. For small positive $\lambda$ there exists a unique stationary solution $v_0$ which, by Lemma 3.6, Corollary 3.4 and (3.4), is stable since

$$(Aw, w)_0 + \lambda(M(v_0)w, w)_0 \geqq |A^{1/2} w|_0^2 (1 - \lambda \gamma_3 \gamma_1{}')$$

holds. Asymptotic stability can also be proved (see [53]).

Let us denote

$$(5.8) \qquad \lambda_c = \sup \{\lambda \mid \lambda > 0, \operatorname{re} \sigma(\tilde{A}(\rho; v_0)) > 0 \text{ for } \rho \in [0, \lambda)\}.$$

Since $\sigma(\tilde{A}(\lambda; v_0))$ is closed, Lemma 3.9(ii) and Corollary 5.4 imply that $v_0$ is stable for $\lambda < \lambda_c$. If $\tilde{A}(\lambda; v_0)$ has spectral points with negative real parts for $\lambda > \lambda_c$ and no eigenvalues with $\operatorname{re} \lambda = 0$, then $v_0$ becomes unstable. We know that $\lambda_c \leqq \lambda_1$, where $\lambda_1$ is the smallest characteristic value of $K(v_0)$, since $0 \in \sigma(1 + \lambda_1 K(v_0))$ if and only if $0 \in \sigma(\tilde{A}(\lambda_1; v_0))$.

For the case that $\lambda_c$ equals $\lambda_1$ and under some additional assumptions on the nature of the eigenspace, the stability behaviour of $v_0$ and of the bifurcating solution $V$ can be studied if it is possible to parametrize $V$ "piecewisely" in $\lambda$, i.e. if the nontrivial solution branch $(\lambda(\alpha), V(\alpha))$ in $R \times \mathring{J}^T$, $|\alpha| \leqq 1$, $(\lambda(0), V(0)) = (\lambda_1, v_0)$, can be written

$$\begin{aligned}(5.9)\qquad &\alpha(\lambda) = \pm\, c_1|\lambda - \lambda_1|^{1/r}, \\ &V(\lambda) = v_0 + |\lambda - \lambda_1|^{1/r}F(\lambda), \qquad r \in N,\end{aligned}$$

where $F : R \to D(A)$ is analytic in $(\lambda - \lambda_1)^{1/r}$ and $F(\lambda_1) \neq 0$ (see Corollary 4.4 and Theorem 4.6). The following lemma explains the stability behaviour for this case. $R$ in (4.1) is a quadratic operator and maps analytic functions $w : C \to D(A)$ into functions $v : C \to \mathring{J}^T$, analytic near 0 as well. Lemma 3.8 implies that (5.9) and this analyticity property hold for the two models under consideration.

LEMMA 5.6. *Assume* $\lambda_c = \lambda_1$ *and* $\operatorname{re}\mu \geqq \alpha > 0$ *for all nonvanishing* $\mu$ *in* $\sigma(\tilde{A}(\lambda_1; v_0))$. *Let* $\lambda_1$ *be a simple characteristic value of* $K(v_0)$, *let* 0 *be a simple eigenvalue of* $\tilde{A}(\lambda_1; v_0)$, *and let* (5.9) *be satisfied. Then, if* $\lambda$ *is restricted to a suitable neighbourhood of* $\lambda_1$,

(i) $v_0$ *is stable for* $\lambda < \lambda_1$ *and unstable for* $\lambda > \lambda_1$,

(ii) $V(\lambda)$ *is stable for* $\lambda > \lambda_1$ *and unstable for* $\lambda < \lambda_1$.

PROOF. Part (3) of the proof follows an argument given in [32].

(1) Let $\tau_0 > 0$ be sufficiently small and choose

$$K := \{z \mid z \in C, 0 \leqq \operatorname{re} z \leqq \alpha/2, |\operatorname{im} z| \leqq \beta, |z| \geqq \alpha/4\},$$

$$\beta = 2|\lambda_1 + \tau|\gamma_3\{\alpha/2 + (\lambda_1 + \tau_0)^2\gamma_3\}^{1/2},$$

then $K$ is a compact subset of the resolvent set of $\tilde{A}(\lambda_1; v_0)$. By Lemmas 3.6, 3.9, the family of operators $(\tilde{A}(\lambda_1; v_0) - \mu 1)^{-1}M(v_0) : \mathring{H}^T_{1,\sigma} \to \mathring{H}^T_{1,\sigma}$ is uniformly bounded for $|\lambda - \lambda_1| \leqq \tau_0$, $\mu \in K$. Denote the bound by $c_1$. Since $V(\lambda_1) = v_0$ and $V(\lambda) \in D(A)$ is continuous in $\lambda$, we can find a $\tau_1 > 0$, $\tau_1 \leqq \tau_0$ such that

$$\lambda_1\|(\tilde{A}(\lambda_1; v_0) - \mu 1)^{-1}(M(V(\lambda)) - M(v_0))\| + |\lambda - \lambda_1|c_1 < 1$$

for $|\lambda - \lambda_1| \leqq \tau_1$. By the identity

$$\begin{aligned}\{\tilde{A}(\lambda_1; V(\lambda)) - \mu 1\}^{-1} &= \{1 + ((\tilde{A}(\lambda_1; v_0) - \mu 1)^{-1}(\lambda_1 M(V(\lambda)) - \lambda_1 M(v_0)) \\ &\qquad + (\lambda - \lambda_1)M(v_0))\}^{-1} \\ &\quad \cdot (\tilde{A}(\lambda_1; v_0) - \mu 1)^{-1},\end{aligned}$$

it follows that $K$ belongs to the resolvent set of $\tilde{A}(\lambda; V(\lambda))$ for $|\lambda - \lambda_1| \leqq \tau_1$. In view of Lemma 3.9(ii), all eigenvalues of $\tilde{A}(\lambda; V(\lambda))$ belong either to the set $|z| \leqq \alpha/4$ or to the set $\operatorname{re} z \geqq \alpha/2$.

(2) For $V(\lambda) = v_0$, the above consideration yields that $\tilde{A}(\lambda; v_0)$ has a simple eigenvalue $\mu_1(\lambda)$ near 0, and that all other eigenvalues

are to the right of re $\mu = \alpha/2$. Observe, that $\tilde{A}(\lambda; v_0) = A + \lambda M(v_0)$ : $D(A) \to \mathring{J}^T$ depends analytically on $\lambda$ with $D(A)$, independent of $\lambda$, as domain of definition. Thus, $\tilde{A}(\lambda; v_0)$ is of type $A$ (cf. [27, p. 375]) and $\mu_1$ as well as the eigenprojection $P(\lambda)$ are analytic in $\lambda$, $\mu_1(\lambda_1) = 0$, $\mu_1(\lambda) = \mu_{11}(\lambda - \lambda_1) + O((\lambda - \lambda_1)^2)$. One obtains

$$\mu_{11}(\varphi_0, \psi_0) = -(A^{1/2}\varphi_0, A^{1/2}\psi_0)_0/\lambda_1$$

where $\varphi_0$, resp. $\psi_0$, are the eigenvector of $A + \lambda_1 M(v_0)$, resp. the adjoint eigenvector, solving

$$A\psi_0 + \lambda_1 M'(v_0)\psi_0 = 0.$$

The simplicity of $\mu_1(\lambda)$ yields $(\varphi_0, \psi_0) \neq 0$. Setting $f_0 = A^{3/4}\varphi_0$ and $g_0 = A^{1/4}\psi_0$, we get $(1 + \lambda_1 K(v_0))f_0 = 0$, $(1 + \lambda_1 K'(v_0))g_0 = 0$. Since $\lambda_1$ is a simple characteristic value of $K(v_0) : (f_0, g_0) = (A^{1/2}\varphi_0, A^{1/2}\psi_0) \neq 0$. $\mu_1(\lambda)$ is positive for $\lambda < \lambda_1$ according to $\lambda_1 = \lambda_c$. Hence, $\mu_{11} < 0$ and (i) is proved.

(3) By $R(v, w)$ we denote a bilinear operator (not necessarily symmetric) such that $R(v, v) = R(v)$. Since $V(\lambda) = v_0 + w^+(\lambda)$ we obtain

$$M(V(\lambda)) = M(v_0) + h(\lambda)\{R(w^+(\lambda), \cdot\,) + R(\,\cdot\,, w^+(\lambda))\}. \tag{5.10}$$

Consider the case $\lambda > \lambda_1$ and choose $\tau = (\lambda - \lambda_1)^{1/r}$, $w^+(\lambda) = \sum_{\nu=1}^{\infty} \tau^\nu w_\nu$, where the series converges for $|\tau| \leqq \tau_0$, $\tau_0$ sufficiently small, in $D(A)$. $w^+$ is a solution of (4.1). Comparison of powers of $\tau, \lambda = \lambda_1 + \tau^r$ yields

$$\tilde{A}(\lambda_1; v_0)w_1 = 0,$$

$$\begin{aligned} &\tilde{A}(\lambda_1; v_0)w_n + h(\lambda_1)\sum_{\nu=1}^{n-1} R(w_\nu, w_{n-\nu}) = 0, \qquad 2 \leqq n \leqq r, \\ &\tilde{A}(\lambda_1; v_0)w_{r+1} + M(v_0)w_1 + h(\lambda_1)G_r = 0, \end{aligned} \tag{5.11}$$

$$G_r = \sum_{\nu=1}^{r} R(w_\nu, w_{r+1-\nu}).$$

An argument analogous to the one used in part (2) of this proof shows that the eigenvalue $\mu_1(\tau)$, $\mu_1(0) = 0$, of $\tilde{A}(\lambda; V(\lambda))$ and the corresponding eigenprojection $P(\tau)$ are analytic in $\tau$. Choose $\varphi_0 = w_1$; then by an elementary calculation, one obtains, for $P(\tau)\varphi_0 = \sum_{\nu=0}^{\infty} \tau^\nu \varphi_\nu$, $\mu_1(\tau) = \sum_{\nu=1}^{\infty} \mu_\nu \tau^\nu$ from (5.10) and (5.11),

$$\begin{aligned} &\varphi_\nu = (\nu+1)w_{\nu+1}, && 0 \leqq \nu \leqq r, \\ (5.12) \qquad &\mu_\nu = 0, && 1 \leqq \nu \leqq r-1, \\ &\mu_r\varphi_{r+1} = \tilde{A}(\lambda_1; v_0)\varphi_{r+1} + M(v_0)\varphi_0 + (r+1)h(\lambda_1)G_r. \end{aligned}$$

Define $\psi_0$ as in (2). Scalar multiplication of the third equation of (5.11) and (5.12) by $\psi_0$ yields

$$\mu_r(\varphi_0, \psi_0)_0 = \lambda_1(A^{1/2}\varphi_0, A^{1/2}\psi_0)_0.$$

According to (2), $(\varphi_0, \psi_0)_0$ and $(A^{1/2}\varphi_0, A^{1/2}\psi_0)_0$ have the same sign. Thus $\mu_r > 0$ and $\mu(\tau) > 0$ if $\tau > 0$.

The proof for $\lambda < \lambda_1$ coincides with the above proof by setting $\tau = \lambda_1 - \lambda$. Hence, we obtain $\mu_1(\tau) = \mu_r(\lambda - \lambda_1) + o(|\lambda - \lambda_1|)$ which proves the lemma.

In [49] Sattinger has given a proof of Lemma 5.6 for general analytic nonlinearities $R$ using the Leray-Schauder degree. He needs the simplicity of the eigenvalue $\mu = 0$ of $\tilde{A}(\lambda_1; v_0)$, although this is not mentioned explicitly. The above proof could be extended to this general case as well by tedious calculations.

The assumptions of Lemma 5.6 are satisfied for fixed $n, \alpha, \beta$ (see Corollary 2.2) in the Bénard case. The fact that $\lambda_c = \lambda_1$ follows from Lemma 4.5(iii). $\lambda_1$ is a simple characteristic value of $K(v_0)$ by Theorem 4.6. System (4.13) shows that $\tilde{A}(\lambda_1; v_0)$ is selfadjoint. Thus, in part (2) of the above proof we have $\psi_0 = \varphi_0$ which implies $(\varphi_0, \psi_0) \neq 0$ and hence the simplicity of $\mu = 0$ in $\sigma(\tilde{A}(\lambda_1; v_0))$. For the Taylor problem only the simplicity of $\lambda_1$ as a characteristic value of $K(v_0)$ is known. Even if $\lambda_c = \lambda_1$ is assumed, it has not yet been determined generally whether $V(\lambda)$ exists to the right or to the left of $\lambda_1$. The simplicity of $\mu = 0$ in $\sigma(\tilde{A}(\lambda_1; v_0))$ is an open problem as well.

THEOREM 5.7. (1) *For the Bénard problem, every solution with a given cell pattern (fixed $n, \alpha, \beta$) exists in some right neighbourhood of $\lambda_1$ and is asymptotically stable in $D(A^\beta)$, $\beta \in (\frac{3}{4}, 1)$. The basic solution $v_0$ is asymptotically stable for $\lambda < \lambda_1$ and unstable for $\lambda > \lambda_1$.*

(2) *For the Taylor problem, let the assumptions of Lemma 5.6 on the spectrum of $\tilde{A}(\lambda; v_0)$ be valid. Then, for every period ($\sigma$ fixed), $V(\lambda)$ is asymptotically stable if it exists for $\lambda > \lambda_1$ and is unstable if it exists for $\lambda < \lambda_1$.*

Some of the authors mentioned at the end of §4 have considered the "stability" and "instability" of cellular solutions for generalized

Bénard problems among the class of general doubly periodic solutions of (3.22) ([4], [5], [36], [50]). For $\gamma = 0$ (notations from the end of §4), rolls are the only stable cellular solutions. For $\gamma \neq 0$ the solution with hexagonal structure bifurcates "subcritically". The subcritical branch increases in amplitude with decreasing $\lambda$ up to a certain $\lambda_{cr} < \lambda_1$, this part being unstable. At $\lambda_{cr}$ the solution turns into a stable branch which exists for $\lambda > \lambda_{cr}$. For some $\lambda_a > \lambda_1$ the hexagonal solution loses stability to the roll solution. Thus, at $\lambda_{cr}$ the basic solution is predicted to snap into a finite amplitude solution of hexagonal structure.

In this context stability and instability were understood as properties of the spectrum of $\tilde{A}(\lambda; V(\lambda))$. If all eigenvalues have positive real parts $V(\lambda)$ is called stable, whereas if an eigenvalue with negative real part exists, $V(\lambda)$ is called unstable. The sign of the real part is determined by exploiting properties of a double power series in $\epsilon$ and $\gamma$ for the smallest eigenvalue $\mu_1(\epsilon, \gamma)$, $\mu_1(0,0) = 0$. But $\mu_1$ as a function of $\epsilon$ and $\gamma$ might have rather complicated singularities at $\epsilon = \gamma = 0$ even for a linear problem [27, p. 117]. Since superpositions of doubly periodic solutions of (3.22) are considered, $\mu_1$ has multiplicity greater than one and the above remark applies as well. When instability is shown, usually some eigenvalues with vanishing real parts are present. No rigorous instability result is known for this case.

It can be proved, however, that all cellular solutions are unstable for almost all values of $\sigma$ in a suitably chosen function space. The distinguished values are the local minima of $\lambda_1(\sigma)$ and the exceptional set mentioned in the proof of Theorem 4.6. Geometrically the following theorem excludes almost all ratios of wave numbers $\alpha$ and $\beta$.

THEOREM 5.8 [32]. *Let $(\alpha_0, \beta_0)$ be not in the exceptional set mentioned in Theorem 4.6. Moreover, assume that $\lambda_1(\sigma)$ does not have a local minimum at $\sigma_0 = (\alpha_0{}^2 + \beta_0{}^2)^{1/2}$ and let $V(\lambda; \sigma_0)$ be a bifurcation solution of the Bénard problem in $\lambda_1$. Then there exists a space $\mathring{J}^T$ such that $V(\lambda; \sigma_0) \in \mathring{J}^T$ and $V(\lambda; \sigma_0)$ is unstable for all $\lambda$ near $\lambda_1(\sigma_0)$, $\lambda \neq \lambda_1(\sigma_0)$.*

PROOF. Note that $\lambda_1(\sigma) \geqq c_1\sigma^2 + c_2\sigma^{-1}$ [31] and that $\lambda_1(\sigma)$ is a real analytic function for $\sigma > 0$ [22]. Thus, $\lambda_1(\sigma)$ has finitely many local minima. Assume $\lambda_1(\sigma)$ to be strictly increasing at $\sigma_0$. Denote by $\mu_k(\lambda; \sigma)$ the eigenvalues of $\tilde{A}(\lambda; v_0)$ for given $\sigma$—they are real—and let be $\mu_1(\lambda; \sigma) < \mu_k(\lambda; \sigma)$, $k \geqq 2$. Then $\mu_1(\lambda_1(\sigma), \sigma) = 0$ holds. By Lemma 5.6 we have $\mu_1(\lambda_1(\sigma_0), \sigma) < 0$ for $\sigma \in S - (\sigma_0 - \delta, \sigma_0)$, and suitable $\delta > 0$. If $\mu_k(\lambda_1(\sigma_0); \sigma) = 0$, $\mu_k$ is simple (Theorem 4.6(i)) and $\mu_k(\lambda; \sigma) \neq 0$ in some neighborhood of $\lambda_1(\sigma_0)$, $\lambda \neq \lambda_1(\sigma_0)$. (Proof (2) of Lemma 5.6.)

Choose $\rho > 0$ such that $(n-1)\rho \in S$, $n\rho = \sigma_0$, $n \in N$. For at most finitely many $m, k \in N$, $1 \leqq m \leqq n$, $\mu_k(\lambda_1(\sigma_0); m\rho/n) = 0$ holds. An arbitrary small variation of $\lambda$ yields $\mu_k(\lambda_1(\sigma_0); m\rho/n) \neq 0$. Set $\tilde{\lambda} \neq \lambda_1(\sigma_0)$, then $V(\tilde{\lambda}; \sigma_0) \in \mathring{J}^T$, where $\mathring{J}^T$ consists of functions periodic with period $2\pi n/\alpha_0$, resp. $2\pi n/\beta_0$, in the $x_1$-, resp. $x_2$-,direction, and there exists a negative eigenvalue $\mu_1(\tilde{\lambda}; \sigma_0/(n-1))$ of $\tilde{A}(\lambda; v_0)$. Hence, $V(\tilde{\lambda}; \sigma_0)$ is unstable for every $\tilde{\lambda}$ near $\lambda_1(\sigma_0)$, $\tilde{\lambda} \neq \lambda_1(\sigma_0)$. If $\lambda_1(\sigma)$ is strictly increasing in $\sigma_0$, the proof proceeds analogously (replace $n-1$ by $n+1$). Q.E.D.

For the Taylor case it is not known whether all $\mu_k$ are real. Thus, the change of sign as $\lambda$ crosses $\lambda_1(\sigma_0)$ cannot be guaranteed. However, for values of $\sigma$ close to a local minimum the same argument could be applied. For analytical, resp. experimental, results see [12], [34], resp. [55].

Ample numerical evidence suggests that $\lambda_1(\sigma)$ is a convex curve with a unique minimum. The locus of this minimum would determine the only possibly stable cell size of the convection flow.

## 6. Appendix.

(1) PROOF OF LEMMA 5.1. Consider the integral equation

$$v(t) = \exp(-At)v_0 + \int_0^t A^\beta \exp(-A(t-s))F(v(s))\,ds \tag{6.1}$$

with

$$v_0 = A^\beta w_0 \in \mathring{J}^T,$$

$$F(v) = -\lambda M A^{-\beta} v - h(\lambda) R(A^{-\beta} v).$$

(6.1) is solved locally by some $v \in C([0,\tau], \mathring{J}^T)$. The operator $F$ can be written as follows:

$$F(v) = Bv + G(v, v)$$

where, by Lemma 3.6 and Lemma 3.8, $B \in \mathfrak{L}(\mathring{J}^T)$ and $G$ is a bounded bilinear form on $\mathring{J}^T$. Then, for fixed $u \in \mathring{J}^T$,

$$F(u, v) = Bv + G(u, v)$$

is bounded and linear in $v$. The estimate

$$|F(u, v)|_0 \leqq c_1(1 + |A^{1/2-\beta}u|_0)|v|_0 \tag{6.2}$$

yields that the mapping $F(\cdot, v)$ is completely continuous in $\mathring{J}^T$ for fixed $v \in \mathring{J}^T$.

The local solution of (6.1) can be constructed iteratively. Let $X_\tau = C([0,\tau], \mathring{J}^T)$ and $u \in X_\tau$; define

$$v^0(t) = \exp(-At)v_0,$$

$$v^{n+1}(t) = \int_0^t A^\beta \exp(-A(t-s))F(u(s), v^n(s))\,ds, \qquad t \in [0,\tau].$$

Then, for all $n \in N_0$, $v^n \in X_\tau$. For sufficiently small $\tau$ the series $\sum_{n=0}^\infty v^n$ converges normally in $X_\tau$, i.e.,

$$\sum_{n=0}^\infty |v^n|_{X_\tau} < \infty, \qquad |v^n|_{X_\tau} = \sup_{[0,\tau]} |v^n(t)|_0.$$

Setting $v = \sum_{n=0}^\infty v^n \in X_\tau$, $v$ solves

$$v(t) = \exp(-At)v_0 + \int_0^t A^\beta \exp(-A(t-s))F(u(s), v(s))\,ds. \tag{6.3}$$

Setting $H(u) = v$ where $v$ solves (6.3), $H$ maps a closed ball with center 0 in $X_\tau$ completely continuous into itself. Thus, $H$ has a fixed point which solves (6.1) on $[0,\tau]$.

By applying the above process to an interval $[\tau, \tau_1]$ with $v_0 = v(\tau)$, $v$ can be continued to a continuous solution of (6.1) in a maximal interval of existence $[0, \tau_\beta)$. Furthermore

$$\lim_{\tau \to \tau_\beta} |v(t)|_0 = \infty$$

holds if $\tau_\beta < \infty$. Hence $w = A^{-\beta}v$ solves the integral equation

$$\begin{aligned} w(t) = {} & \exp(-At)w_0 \\ & + \int_0^t \exp(-A(t-s))(-\lambda Mw(s) - h(\lambda)R(w(s)))\,ds \end{aligned} \tag{6.4}$$

in $[0, \tau_\beta)$ with the properties (i) and (ii) of Lemma 5.1. As was shown in [17], $w$ is the unique strict solution of (3.22) in every interval $[0,\tau] \subset [0, \tau_\beta)$.

(2) Proof of Lemma 5.2. Since $\tilde{A}(\lambda)$ is the generator of a holomorphic semigroup (Lemma 3.11), fractional powers $\tilde{A}(\lambda)^\beta$ with the properties (3.19) can be defined. Setting $\tilde{A}(\lambda)^\beta w = v$, one obtains the estimate by Lemma 3.8 and Lemma 3.12:

$$|\tilde{F}(u, v)|_0 \leqq c_2(1 + |A^{-1/4}(A^{3/4}A(\lambda)^{-\beta})u|_0)|v|_0 \tag{6.5}$$

with $\tilde{F}(v,v) = \tilde{F}(v) = -h(\lambda)R(\tilde{A}(\lambda)^{-\beta}v)$. The proof proceeds as for Lemma 5.1, since, by (6.5), $\tilde{F}$ has the same properties as $F$.

LEMMA 6.1. *Let $w_0 \in D(A^\beta)$ be such that $A^\beta w_0 \in D(A)$, $Mw_0 \in D(A^\beta)$, $R(w_0) \in D(A^\beta)$, $\beta \in [\frac{3}{4}, 1)$. Then the unique strict solution of* (3.22) *constructed in Lemma* 5.1 *satisfies* (5.1)(iii)—(v) *in the closed interval* $[0,\tau]$, $\tau < \tau_\beta$.

PROOF. We show (5.1)(iii)—(v) for $t = 0$. With the same notations as in the proof of Lemma 5.1, equation (6.1) is solved in $X_\tau{}^1 = C^1([0,\tau], \mathring{J}^T)$. Setting $u(0) = v_0$, $u \in X_\tau{}^1$, the functions $v^n$ belong to $X_\tau{}^1$. This is clear for

$$v^0(t) = \exp(-At)v_0.$$

For general $n$ observe the identity obtained by twice using the substitution $s \to t - s$:

$$\begin{aligned}\frac{d}{dt} v^n(t) = \int_0^t & A^\beta \exp(-A(t-s)) \\ & \cdot \left\{ G\left(\frac{d}{ds} u(s), v^{n-1}(s)\right) + F\left(u(s), \frac{d}{ds} v^{n-1}(s)\right)\right\} ds \\ & + \delta_{n1} \exp(-At) A^\beta F(v_0)\end{aligned}$$

where, for $n = 1$, $F(v_0) \in D(A^\beta)$ is used. $v^n \in X_\tau{}^1$ follows inductively. For sufficiently small $\tau$ the series $v = \sum_{n=0}^\infty v^n$ converges normally in $X_\tau{}^1$, and $v \in X_\tau{}^1$ solves (6.3). The mapping $H_1 : X_\tau{}^1 \to X_\tau{}^1$ maps $u \in X_\tau{}^1$, $u(0) = v_0$, into the constructed solution $v$ of (6.3). For the closed, bounded, convex set

$$K = \{u \in X_\tau{}^1 \mid u(0) = v_0, |u|_{X_\tau} \leqq d, |du/dt|_{X_\tau} \leqq d'\}$$

and appropriate positive numbers $\tau$, $d$, $d'$, we have $H_1(K) \subset K$ and $H_1$ is completely continuous. The existing fixed point $v$ of $H_1$ in $K$ solves (6.1) in $[0,\tau]$ and $w = A^{-\beta}v$ is a solution of (6.4). According to [17], $w$ is a strict solution of (3.22) in $[0,\tau]$, i.e.,

$$dw/dt + Aw + \lambda Mw + h(\lambda)R(w) = 0 \quad \text{on } (0,\tau].$$

Since $w \in C([0,\tau], D(A^\beta))$, $dw/dt \in C([0,\tau], \mathring{J}^T)$ it follows that $w \in D(A)$ for $t \in [0,\tau]$ and $Aw \in C([0,\tau], \mathring{J}^T)$. Thus, (3.22) is satisfied on $[0,\tau]$. Q.E.D.

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