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THEORY OF FREDHOLM OPERATORS AND VECTOR BUNDLES RELATIVE TO A VON NEUMANN ALGEBRA[1]

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Introduction. Let H be a complex separable infinite dimensional Hilbert space. A bounded linear operator T of H is Fredholm if its range $\mathcal{R}_T$ is closed and if its null space $\mathcal{N}_T$ and the orthogonal complement $\mathcal{R}_T{}^{\perp}$ of its range are finite dimensional. The index of such an operator T is the integer

(0.1) $$\text{Index } T = \text{Dim } \mathcal{N}_T - \text{Dim } \mathcal{R}_T{}^{\perp},$$

where Dim denotes the complex dimension. The properties of the index map (additivity, homotopy invariance etc.) were investigated by Atkinson [5], Gohberg-Krein [14], Cordes-Labrousse [11] a.o. from 1950 to 1963. Let $\mathfrak{F}(H)$ be the monoid of Fredholm operators of H with the norm topology. Then one of the main results of Cordes-Labrousse [11] is that the index map induces an isomorphism

(0.2) $$\pi_0 \mathfrak{F}(H) \cong Z$$

between the group $\pi_0 \mathfrak{F}(H)$ of connected components of $\mathfrak{F}(H)$ and the additive group Z of integers. In the following various generalizations of (0.2) are discussed.

In 1964 Atiyah [1] and Jänich [16] defined the index of a continuous map T of a compact space X into $\mathfrak{F}(H)$. Having deformed T properly, its index is the difference of the vector bundle $(\mathcal{N}_{T_x})_{x \in X}$ of null spaces and the vector bundle $(\mathcal{R}_{T_x}^{\perp})_{x \in X}$ of orthogonal complements of range spaces, in the sense of K-theory. Atiyah [1] and Jänich [16] prove that the index induces an isomorphism

(0.3) $$[X, \mathfrak{F}(H)] \cong K(X),$$

where $[X, \mathfrak{F}(H)]$ denotes the group of homotopy classes of continuous maps of X into $\mathfrak{F}(H)$ and $K(X)$ is the Grothendieck group of the monoid of finite dimensional complex vector bundles over X. If X has one point only, then (0.3) specializes to (0.2).

In 1968 Breuer ([8], [9]) generalized the concept of a Fredholm operator to wider classes of Hilbert space operators that are Fred-

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holm relative to a given von Neumann algebra of operators of a complex Hilbert space. Let $\mathfrak{M}$ be a von Neumann subalgebra of $\mathcal{L}(H)$. Then $T \in \mathfrak{M}$ is Fredholm relative to $\mathfrak{M}$ if the following hold: (i) there is an $\mathfrak{M}$-finite projection $E \in \mathfrak{M}$ such that range $(1 - E) \subset$ range T, (ii) the orthogonal projection N_T of H on the null space of T is an $\mathfrak{M}$-finite projection. It follows from (i) that the orthogonal projection $R_T{}^{\perp}$ of H on $\mathcal{R}_T{}^{\perp}$ is also $\mathfrak{M}$-finite. Hence

$$\text{Index } T = \text{Dim } N_T - \text{Dim } R_T{}^{\perp} \tag{0.4}$$

is a well-defined element of the index group $I(\mathfrak{M})$ of $\mathfrak{M}$. Equip the monoid $\mathfrak{F}(\mathfrak{M})$ of Fredholm elements of $\mathfrak{M}$ with the norm topology. In Breuer [9] it is shown that the index map (0.4) induces a group isomorphism

$$\pi_0 \mathfrak{F}(\mathfrak{M}) \cong I(\mathfrak{M}) \tag{0.5}$$

if $\mathfrak{M}$ is properly infinite. If $\mathfrak{M} = \mathcal{L}(H)$, then (0.5) also specializes to (0.2).

To give a common generalization of (0.3) and (0.5) a theory of vector bundles relative to a properly infinite von Neumann algebra $\mathfrak{M}$ is developed in the present paper. The vector bundles in question have transition functions with values in the group of unitary elements of some finite reduced subalgebra of $\mathfrak{M}$. Call such bundles finite $\mathfrak{M}$-vector bundles. There is also a dual characterization of these vector bundles in terms of relatively finite modules over the C^*-algebra of bounded continuous maps of the base space into the commutant $\mathfrak{M}'$ of $\mathfrak{M}$. The equivalence proof of the two definitions would then generalize Swan's theorem [24]. The basic properties of $\mathfrak{M}$-vector bundles are analogous to the ones of vector bundles with finite dimensional complex fibres. To derive these we could either have followed the standard texts of Atiyah [1], Husemoller [15] a.o. or have applied the more recent results of Karoubi on Banach categories (M. Karoubi, R. Gordon, P. Löffler, M. Zisman, *Séminaire Heidelberg-Saarbrücken-Strasbourg sur la K-théorie*, Lecture Notes in Mathematics 136, Springer-Verlag, 1970). In the present paper another approach is given which is based on the generalized Kuiper theorem (Breuer [10]) and some general fibre bundle theory.

The Grassmann spaces of finite projections of $\mathfrak{M}$ (with the norm topology) are shown to be classifying spaces of the finite $\mathfrak{M}$-vector bundles of finite type over a paracompact base space. Subsequently this property of the Grassmannians is used in many proofs, e.g., for the clutching construction. The $\mathfrak{M}$-isomorphism classes of $\mathfrak{M}$-vector bundles over a space X form a commutative monoid under $\oplus$. When X is compact we define $K_{\mathfrak{M}}(X)$ as the universal group of that monoid.

The index of a continuous map of a compact space X into $\mathfrak{F}(\mathfrak{M})$ is defined similarly as in Atiyah [1] and Jänich [15] as the difference of two finite $\mathfrak{M}$-vector bundles in $K_{\mathfrak{M}}(X)$. It is shown that the index induces a homomorphism of the group $[X, \mathfrak{F}(\mathfrak{M})]$ into the group $K_{\mathfrak{M}}(X)$. As in Atiyah [1] the contractibility of the group $\mathfrak{U}\mathfrak{M}$ of unitary elements of $\mathfrak{M}$ in its norm topology is used to prove that this homomorphism is injective. Atiyah [1] and Jänich [16] used elementary operations to show that the index isomorphism is also surjective. In the present paper it is shown that the contractibility of $\mathfrak{U}\mathfrak{M}$ can also be used to prove the surjectivity of this index map. It follows that the index map induces an isomorphism

$$[X, \mathfrak{F}(\mathfrak{M})] \cong K_{\mathfrak{M}}(X) \tag{0.6}$$

for every compact space X. (0.6) is the common generalization of (0.3) and (0.5).

Finally a proof of the periodicity theorem of $K_{\mathfrak{M}}$-theory is given. This theorem is due to Atiyah and Singer. It does not seem to be easy to translate all known proofs of the periodicity theorem of K-theory to $K_{\mathfrak{M}}$-theory, when $\mathfrak{M}$ is of type II. E.g., the proof given by Atiyah and Singer in [4] is not easy to generalize (see in particular the proof of Proposition 3.5 of [4]). As Atiyah and Singer pointed out to me the proof given by Atiyah in [3] lends itself easily to generalization. The proof in [3] is based on (0.3). I have elaborated the von Neumann algebra version of this proof in the present paper by using (0.6) instead of (0.3) and in addition some results on tensor products of C^*-algebras (which are presented in §3 of the first chapter and are all known except, I think, Proposition 5 of that chapter). As in [3] the periodicity theorem is stated and proved in terms of locally compact spaces as follows. For a locally compact space $\dot{Y}$ define $K_{\mathfrak{M}}(Y) = \tilde{K}_{\mathfrak{M}}(\dot{Y})$ where $\tilde{K}_{\mathfrak{M}}$ is the "reduced" $K_{\mathfrak{M}}$-functor and Y the one-point compactification of Y (with the point at infinity as base point). Then one has for each locally compact X a canonical isomorphism

$$K_{\mathfrak{M}}(X) \cong K_{\mathfrak{M}}(R^2 \times X). \tag{0.7}$$

The isomorphisms (0.6) and (0.7) imply that the space $\mathfrak{F}(\mathfrak{M})$ is homotopy periodic of period two. Thus it follows from (0.5) that the even homotopy groups of $\mathfrak{F}(\mathfrak{M})$ are isomorphic to the index group $I(\mathfrak{M})$ and from the simple connectedness of the Grassmann spaces of finite projections of $\mathfrak{M}$ that the odd homotopy groups of $\mathfrak{F}(\mathfrak{M})$ are trivial.

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Chapter I. Preliminaries

1. **Auxiliary lemmas of functional analysis.** In the following H, K denote complex Hilbert spaces. $\mathcal{L}(H, K)$ is the Banach space of all bounded linear maps of H into K with the usual operator norm

$$\|T\| = \sup\{\|Tv\| \mid v \in H \text{ and } \|v\| \leqq 1\} \tag{1.1}$$

for all $T \in \mathcal{L}(H, K)$. Let

$$\mathcal{G}\mathcal{L}(H, K) = \{T \in \mathcal{L}(H, K) \mid T \text{ bijective}\} \tag{1.2}$$

and

$$\mathcal{J}\mathcal{L}(H, K) = \{T \in \mathcal{L}(H, K) \mid T \text{ injective with closed range}\}. \tag{1.3}$$

$\mathcal{G}\mathcal{L}(H, K)$ is known to be open in $\mathcal{L}(H, K)$. One also has

Proposition 1. *$\mathcal{J}\mathcal{L}(H, K)$ is open in $\mathcal{L}(H, K)$.*

Proof. Let $T \in \mathcal{J}\mathcal{L}(H, K)$ and $L = K \ominus T(H)$ be the orthogonal complement of the range of T in K. Then the map

$$T' : H \oplus L \to K \tag{1.4}$$

defined by

$$T'(u \oplus v) = Tu + v \tag{1.5}$$

is in $\mathcal{G}\mathcal{L}(H \oplus L, K)$. Let $\iota_H : H \to H \oplus L$ be the canonical injection. Then the linear map

$$\pi : \mathcal{L}(H \oplus L, K) \to \mathcal{L}(H, K) \tag{1.6}$$

defined by

$$\pi(S) = S \circ \iota_H \tag{1.7}$$

is continuous and surjective. By the open mapping theorem (Bourbaki [7, Chapter I, §3, Theorem 1]) $\pi\mathcal{G}\mathcal{L}(H \oplus L, K)$ is open in $\mathcal{L}(H, K)$. The relations

$$T \in \pi\mathcal{G}\mathcal{L}(H \oplus L, K) \subseteq \mathcal{J}\mathcal{L}(H, K) \tag{1.8}$$

are obvious.

If $H = K$ we write $\mathcal{L}(H)$ instead of $\mathcal{L}(H, H)$. A Hermitian idempotent of the involutive algebra $\mathcal{L}(H)$ is called a projection of H or of $\mathcal{L}(H)$. Let $T \in \mathcal{L}(H, K)$. The projection of H onto the null space of T is called the null projection of T and denoted by N_T. The projection of K onto the closure of the range of T is called the range projection of T and denoted by R_T.

LEMMA 1. *Let $T \in \mathcal{L}(H, K)$ and let E be a projection of H. If*

$$\text{range } E \subseteq \text{range } T^*, \tag{1.9}$$

then there is a neighborhood $\mathcal{N}$ of T in $\mathcal{L}(H, K)$ such that for all $S \in \mathcal{N}$ one has

$$\inf(E, N_S) = 0 \tag{1.10}$$

and

$$\text{range } (SE) \textit{ is closed in } K. \tag{1.11}$$

PROOF. This follows from Proposition 1 and the classical "alternatives"

$$N_T = 1 - R_{T^*} \tag{1.12}$$

and

$$\text{range } ET^* \text{ closed } \Rightarrow \text{range } TE \text{ closed} \tag{1.13}$$

(see Yosida [27, p. 205]) (1.9), (1.12) and (1.13) imply that TE can be considered as an element of $\mathcal{FL}(E(H), K)$. By Proposition 1 there is an open neighborhood $\mathcal{N}$ of T in $\mathcal{L}(H, K)$ such that SE can also be considered as an element of $\mathcal{FL}(E(H), K)$ for all $S \in \mathcal{N}$. This implies (1.10) and (1.11).

PROPOSITION 2. *The map $S \to R_S$ of $\mathcal{FL}(H, K)$ into $\mathcal{L}(K)$ is continuous in the norm topology.*

PROOF. Let $S \in \mathcal{FL}(H, K)$. Then $|S| = (S^*S)^{1/2}$ is regular (invertible) in $\mathcal{L}(H)$, and $V_S = S \cdot |S|^{-1}$ is a partial isometry of H into K satisfying $R_S = V_S V_S^*$. Hence R_S depends continuously on S.

COROLLARY. *For each $S \in \mathcal{FL}(H, K)$ there is a neighborhood $\mathcal{N}$ of S such that for all $T \in \mathcal{N}$ there is a unitary element U of $\mathcal{L}(K)$ satisfying $R_S = U^* R_T U$.*

PROOF. It follows from Proposition 2 that one can choose $\mathcal{N}$ so small that

$$\|R_T - R_S\| < 1 \quad \text{for all } T \in \mathcal{N}. \tag{1.14}$$

It follows then from Riesz-Sz.-Nagy [22, §105] that there are partial isometries $V, \tilde{V}$ of K satisfying

$$(1.15)\qquad R_T = VV^*,\ \ R_S = V^*V,\ \ 1 - R_T = \tilde{V}\tilde{V}^*,\ \ 1 - R_S = \tilde{V}^*\tilde{V}.$$

Then $U = V + \tilde{V}$ satisfies the conditions of the corollary.

2. **On compact and Fredholm operators relative to a von Neumann algebra.** Let H be a complex Hilbert space. The commutant $\mathfrak{M}'$ of a subset $\mathfrak{M}$ of $\mathcal{L}(H)$ is the set of all $T \in \mathcal{L}(H)$ satisfying $ST = TS$ for all $S \in \mathfrak{M}$. An involutive subalgebra $\mathfrak{M}$ of $\mathcal{L}(H)$ is called von Neumann if $\mathfrak{M} = \mathfrak{M}''$. A von Neumann algebra $\mathfrak{M}$ is called a factor if its center consists of the scalar operators of H only.

In the following let $\mathfrak{M}$ be a von Neumann algebra of continuous linear operators of H. Let $P(\mathfrak{M})$ denote the complete lattice of projections of $\mathfrak{M}$ with the usual order relation

$$(2.1)\qquad E \leqq F \Leftrightarrow EF = E$$

where $E, F \in P(\mathfrak{M})$. The relations $\sim$ and $\prec$ in $P(\mathfrak{M})$ are defined by

$$(2.2)\qquad E \sim F \Leftrightarrow E = V^*V, \qquad F = VV^* \quad \text{for some } V \in \mathfrak{M}$$

and

$$(2.3)\qquad E \prec F \Leftrightarrow E \sim G \leqq F \quad \text{for some } G \in P(\mathfrak{M}).$$

Call $E \in P(\mathfrak{M})$ finite if $F \leqq E$ and $E \sim F$ imply $E = F$. $P_f(\mathfrak{M})$ denotes the lattice of finite projections of $\mathfrak{M}$. For the basic properties of $P(\mathfrak{M})$ and $P_f(\mathfrak{M})$ we refer to Dixmier [12].

Let $[E]$ be the $\sim$-equivalence class of $E \in P(\mathfrak{M})$. Let $\mathcal{J}$ be the free abelian group generated by the equivalence classes of finite projections of $\mathfrak{M}$. Let $\mathcal{R}$ be the subgroup of $\mathcal{J}$ generated by all elements of the form $[E + F] - [E] - [F]$ with $EF = 0$ and E, F in $P_f(\mathfrak{M})$. The quotient group $I(\mathfrak{M}) = \mathcal{J}/\mathcal{R}$ is called the index group of $\mathfrak{M}$. Let

$$(2.4)\qquad \mathrm{Dim}\colon P_f(\mathfrak{M}) \to I(\mathfrak{M})$$

be the canonical map. Let $I^+(\mathfrak{M})$ be the subsemigroup of $I(\mathfrak{M})$ generated by the elements $\mathrm{Dim}\, E$, $E \in P_f(\mathfrak{M})$. For α, β in $I(\mathfrak{M})$ define $\alpha \geqq \beta$ if $\alpha - \beta$ is in $I^+(\mathfrak{M})$. With that order relation $I(\mathfrak{M})$ becomes a lattice group, and one has

$$(2.5)\qquad \mathrm{Dim}\, E \geqq \mathrm{Dim}\, F \Leftrightarrow E \succ F.$$

For an alternative description of the index group see Breuer [8] and [9, Appendix].

Let $T \in \mathfrak{M}$. Call T finite if its range projection R_T is finite. Let $\mathfrak{m}_0$ denote the set of all finite elements of $\mathfrak{M}$. The norm closure of $\mathfrak{m}_0$, *notation*: $\mathfrak{m}$, is a two-sided *-ideal of $\mathfrak{M}$. Its elements are called *compact* (relative to $\mathfrak{M}$).

To define Fredholm elements of $\mathfrak{M}$ we first generalize the concept of a closed subspace of H. Let K be a linear subspace of H and the projection of H onto the norm closure of K be denoted by P_K. Call K essentially closed (or closed relative to $\mathfrak{M}$), if there is a nondecreasing sequence

$$E_1 \leqq E_2 \leqq E_3 \leqq \cdots \tag{2.6}$$

in $P(\mathfrak{M})$ satisfying the following three conditions

(i) $E_n(H) \subseteq K$ for all $n = 1, 2, 3, \cdots$,

(ii) $P_K = \sup\{E_n/n = 1, 2, \cdots\}$,

(iii) $P_K - E_1$ is finite.

Call $T \in \mathfrak{M}$ a Fredholm element of $\mathfrak{M}$ if the null projections N_T and N_{T^*} are finite and if $T(H)$ is essentially closed.

PROPOSITION 3. *Suppose $\mathfrak{M}$ is properly infinite. Then $T \in \mathfrak{M}$ is Fredholm iff T is regular (invertible) modulo $\mathfrak{m}$.*

PROOF. See Breuer [9, Theorem 1].

Let $\mathfrak{F}(\mathfrak{M})$ denote the set of Fredholm elements of $\mathfrak{M}$. Proposition 3 implies that $\mathfrak{F}(\mathfrak{M})$ is an open subset of $\mathfrak{M}$ (with respect to the norm topology) and that $\mathfrak{F}(\mathfrak{M})$ is an involutive monoid (i.e., $1 \in \mathfrak{F}(\mathfrak{M})$ and S, T in $\mathfrak{F}(\mathfrak{M})$ imply S^* and ST in $\mathfrak{F}(\mathfrak{M})$). The index map

$$\text{Index: } \mathfrak{F}(\mathfrak{M}) \to I(\mathfrak{M}) \tag{2.7}$$

is defined by

$$\text{Index } T = \text{Dim } N_T - \text{Dim } N_{T^*}. \tag{2.8}$$

The following additional notation will be used. $G\mathfrak{M}$, resp. $\mathfrak{U}\mathfrak{M}$, is the group of regular, resp. unitary, elements of $\mathfrak{M}$. If $E, F \in P(\mathfrak{M})$, then

$$\mathcal{J}_{\mathfrak{M}}(E, F) = \{V \in \mathfrak{M} \mid E = V^*V, F = VV^*\}; \tag{2.9}$$

$\mathcal{J}_{\mathfrak{M}}$ denotes the set of all partial isometries of $\mathfrak{M}$. All subsets of $\mathfrak{M}$ are equipped with the norm topology.

3. **Some remarks on tensor products of C^*-algebras.** Let H, K be complex Hilbert spaces with positive Hermitian forms $\langle\, ,\rangle_H$ and $\langle\, ,\rangle_K$. The algebraic tensor product $H \otimes K$ over $\mathbf{C}$ is a prehilbert space with respect to the form

$$(3.1) \qquad \langle\ ,\ \rangle = \langle\ ,\ \rangle_H \otimes \langle\ ,\ \rangle_K.$$

The completion of $H \otimes K$ in the norm associated to $\langle\ ,\ \rangle$ is a Hilbert space denoted by $H \hat{\otimes} K$.

Let $S \in \mathcal{L}(H)$, $T \in \mathcal{L}(K)$. Define

$$(3.2) \qquad S \otimes T : H \otimes K \to H \otimes K$$

by linearity and

$$(3.3) \qquad (S \otimes T)(u \otimes v) = (Su) \otimes (Tv).$$

Then

$$(3.4) \qquad \|S \otimes T\| = \|S\| \cdot \|T\|.$$

Hence $S \otimes T$ is continuous. The unique continuous linear extension of $S \otimes T$ to $H \hat{\otimes} K$ is an element of $\mathcal{L}(H \hat{\otimes} K)$ still denoted by $S \otimes T$.

Let $\mathfrak{M}$, $\mathfrak{N}$ be abstract C^*-algebras. A norm $\|\ \|_\alpha$ defined on the algebraic tensor product $\mathfrak{M} \otimes \mathfrak{N}$ is admissible if the completion of $\mathfrak{M} \otimes \mathfrak{N}$ in $\|\ \|_\alpha$ is a C^*-algebra. Let

$$(3.5) \qquad \rho : \mathfrak{M} \to \mathcal{L}(H_\rho), \qquad \sigma : \mathfrak{N} \to \mathcal{L}(H_\sigma)$$

be representations (*-homomorphisms). Let $H_{\rho\otimes\sigma} = H_\rho \hat{\otimes} H_\sigma$. Then

$$(3.6) \qquad \rho \otimes \sigma : \mathfrak{M} \otimes \mathfrak{N} \to \mathcal{L}(H_{\rho\otimes\sigma})$$

is defined by

$$(3.7) \qquad (\rho \otimes \sigma)\left(\sum_{i=1}^{n} x_i \otimes y_i \right) = \sum_{i=1}^{n} (\rho x_i) \otimes (\sigma y_i).$$

For $z \in \mathfrak{M} \otimes \mathfrak{N}$ define

$$(3.8) \qquad \|z\|_* = \sup\{\|(\rho \otimes \sigma)z\| \mid \rho, \sigma \text{ representations of } \mathfrak{M}, \mathfrak{N}\}.$$

Then $\|\ \|_*$ is an admissible norm of $\mathfrak{M} \otimes \mathfrak{N}$.

Let $\mathfrak{M}$, $\mathfrak{N}$ be C^*-subalgebras of $\mathcal{L}(H)$, $\mathcal{L}(K)$. Their operator tensor product $\mathfrak{M} \otimes_{\text{op}} \mathfrak{N}$ is the linear subspace of $\mathcal{L}(H \hat{\otimes} K)$ generated by all elements $S \otimes T$, $S \in \mathfrak{M}$, $T \in \mathfrak{N}$. It is quite obvious that there is a canonical isomorphism between the operator and algebraic tensor product.

$$(3.9) \qquad \mathfrak{M} \otimes \mathfrak{N} \cong \mathfrak{M} \otimes_{\text{op}} \mathfrak{N}.$$

Via this isomorphism and admissible norm $\|\ \|_*$ of $\mathfrak{M} \otimes \mathfrak{N}$ coincides with the operator norm $\| \ \|$ of $\mathfrak{M} \otimes_{\mathrm{op}} \mathfrak{N}$ (Wulfson [26]). The completion of $\mathfrak{M} \otimes \mathfrak{N}$ in $\| \ \|_*$ is denoted by $\mathfrak{M} \hat{\otimes} \mathfrak{N}$.

Proposition 4. *Let $\mathfrak{M}$, $\mathfrak{N}$ be C^*-algebras.*

(i) *If $\| \ \|_\alpha$ is an admissible norm of $\mathfrak{M} \otimes \mathfrak{N}$, then $\|x\|_* \leqq \|x\|_\alpha$ for all $x \in \mathfrak{M} \otimes \mathfrak{N}$.*

(ii) *If $\mathfrak{M}$ or $\mathfrak{N}$ is postliminal, then $\| \ \|_*$ is the only admissible norm of $\mathfrak{M} \otimes \mathfrak{N}$.*

This proposition is proved in Takesaki [25].

Corollary 1. *If $\mathfrak{i}$ is an ideal of $\mathfrak{M} \hat{\otimes} \mathfrak{N}$ satisfying $(\mathfrak{M} \otimes \mathfrak{N}) \cap \mathfrak{i} = 0$, then $\mathfrak{i} = 0$.*

Proof. For $x \in \mathfrak{M} \otimes \mathfrak{N}$ define $\|x\|_\alpha = \inf \|x + \mathfrak{i}\|_*$. Then $\| \ \|_\alpha$ is an admissible norm of $\mathfrak{M} \otimes \mathfrak{N}$. One has $\|x\|_\alpha \leqq \|x\|_*$ by definition and $\|x\|_\alpha \geqq \|x\|_*$ by (i) of Proposition 4. Hence $(\mathfrak{M} \hat{\otimes} \mathfrak{N})/\mathfrak{i}$ is isomorphic to $\mathfrak{M} \hat{\otimes} \mathfrak{N}$ and consequently $\mathfrak{i} = 0$.

Corollary 2. *Let $\mathfrak{a}$ be an ideal of $\mathfrak{M}$ and $\mathfrak{M}/\mathfrak{a}$ be postliminal. Then there is a canonical isomorphism*

$$\mathfrak{M} \hat{\otimes} \mathfrak{N} / \mathfrak{a} \hat{\otimes} \mathfrak{N} \cong (\mathfrak{M}/\mathfrak{a}) \hat{\otimes} \mathfrak{N}. \tag{3.10}$$

Proof. There is a commutative diagram

$$\begin{array}{ccc}
\mathfrak{M} \otimes \mathfrak{N} & \xrightarrow{\quad\psi\quad} & \\
\downarrow \varphi & & \searrow \\
(\mathfrak{M} \otimes \mathfrak{N})/\mathfrak{a} \otimes \mathfrak{N} & \xrightarrow{\quad\kappa\quad} & (\mathfrak{M}/\mathfrak{a}) \otimes \mathfrak{N}
\end{array}$$

where φ, ψ are canonically defined and κ is uniquely determined by the commutativity of the diagram. κ is an isomorphism. For $x \in \mathfrak{M} \otimes \mathfrak{N}$ define

$$\|\kappa\varphi x\|_\alpha = \inf \|x + \text{kernel } \varphi\|_*. \tag{3.11}$$

Then $\| \ \|_\alpha$ is an admissible norm of $\mathfrak{M}/\mathfrak{a} \otimes \mathfrak{N}$. Since $\mathfrak{M}/\mathfrak{a}$ is postliminal, (ii) of Proposition 4 implies $\|\kappa\varphi x\|_\alpha = \|\psi x\|_*$. Hence κ extends uniquely to an isomorphism (3.10).

Remark. If $\mathfrak{a}, \mathfrak{b}$ are ideals of $\mathfrak{M}$, $\mathfrak{N}$ and $\mathfrak{M}/\mathfrak{a}$ or $\mathfrak{M}/\mathfrak{b}$ is postliminal, then

$$\mathfrak{M} \hat{\otimes} \mathfrak{N} / \mathfrak{i} \cong (\mathfrak{M}/\mathfrak{a}) \hat{\otimes} (\mathfrak{M}/\mathfrak{b}) \tag{3.12}$$

where $\mathfrak{i}$ is the closed ideal of $\mathfrak{M} \hat{\otimes} \mathfrak{N}$ generated by $\mathfrak{M} \otimes \mathfrak{b} + \mathfrak{a} \otimes \mathfrak{N}$.

COROLLARY 3. *Let $\mathfrak{M}$ be commutative with unit element. Let M be the maximal ideal space of $\mathfrak{M}$ with the Gelfand topology. Let $\mathcal{C}(M, \mathfrak{N})$ be the C^*-algebra of continuous maps of M into $\mathfrak{N}$ with the usual norm*

$$\|f\| = \sup\{\|f(p)\| \mid p \in M\}. \tag{3.13}$$

Then

$$\mathfrak{M} \hat{\otimes} \mathfrak{N} \cong \mathcal{C}(M, \mathfrak{N}) \tag{3.14}$$

canonically.

Since $\mathfrak{M}$ is postliminal, this follows from part (ii) of Proposition 4. A direct proof is given in Takesaki [**25**].

Let $\mathfrak{M}$, $\mathfrak{N}$ be von Neumann algebras of operators of H, K. Then the von Neumann algebra $\mathfrak{M} \hat{\hat{\otimes}} \mathfrak{N}$ of operators of $H \hat{\otimes} K$ is defined as the bicommutant of $\mathfrak{M} \otimes_{\mathrm{op}} \mathfrak{N}$,

$$\mathfrak{M} \hat{\hat{\otimes}} \mathfrak{N} = (\mathfrak{M} \otimes_{\mathrm{op}} \mathfrak{N})''. \tag{3.15}$$

Let (E_i), $i = 1, 2, \cdots$, be a sequence of pairwise orthogonal equivalent projections of the von Neumann algebra $\mathfrak{M}$. Let

$$E = E_1, \qquad F = \sum_{i=1}^{\infty} E_i. \tag{3.16}$$

Let L be a separable complex Hilbert space with orthonormal base (φ_i), $i = 1, 2, \cdots$. Let e_i be the orthogonal projection of L on the subspace $\mathbf{C} \cdot e_i$. Then there is an isomorphism

$$\Phi : F(H) \to E(H) \hat{\otimes} L \tag{3.17}$$

inducing a spatial isomorphism

$$\Phi^{\#} : \mathfrak{M}_F \to \mathfrak{M}_E \hat{\hat{\otimes}} \mathcal{L}(L) \tag{3.18}$$

such that

$$\Phi^{\#}(E_i) = E \otimes e_i, \qquad i = 1, 2, 3, \cdots \tag{3.19}$$

(Dixmier [**12**, I, §2, Proposition 5]). In the following let

$$\mathfrak{M} = \mathfrak{M}_E \hat{\hat{\otimes}} \mathcal{L}(L) \tag{3.20}$$

and E be finite and L be separable and infinite dimensional.

LEMMA 2.

$$\mathfrak{m} \cap (\mathfrak{M}_E \otimes \mathcal{L}(L)) = \mathfrak{M}_E \otimes \mathfrak{C}(L). \tag{3.21}$$

PROOF. The relation

$$\mathfrak{m} \supseteq \mathfrak{M}_E \hat{\otimes} \mathfrak{C}(L) \tag{3.22}$$

is quite obvious. Let $\mathcal{Z}_E$ be the center of $\mathfrak{M}_E$. Then

$$\mathcal{Z} = \mathcal{Z}_E \otimes 1_L \tag{3.23}$$

is the center of $\mathfrak{M}$. One has

$$\mathcal{Z} \cap \mathfrak{m} = \{0\} \tag{3.24}$$

because $\mathfrak{M}$ is properly infinite. Let Q be the set of all irreducible representations

$$\pi : \mathfrak{M}_E \hat{\otimes} \mathcal{L}(L) \to \mathcal{L}(H_\pi) \tag{3.25}$$

with

$$\text{Kernel}\, \pi \geqq \mathfrak{m} \cap (\mathfrak{M}_E \hat{\otimes} \mathcal{L}(L)). \tag{3.26}$$

For $\pi \in Q$ define

$$\lambda_\pi = \pi \mid \mathfrak{M}_E \otimes 1_L, \qquad \mu_\pi = \pi | E \otimes \mathcal{L}(L). \tag{3.27}$$

Then

$$\text{Kernel}\, \lambda_\pi \subseteq \text{Kernel}\, \pi. \tag{3.28}$$

One has

$$\bigcap_{\pi \in Q} \text{Kernel}\, \pi = \mathfrak{m} \cap (\mathfrak{M} \hat{\otimes} \mathcal{L}(L)) \tag{3.29}$$

(Dixmier [12, 2.9.7]). The relations (3.24), (3.28) and (3.29) imply

$$\mathcal{Z} \cap \bigcap_{\pi \in Q} \text{Kernel}\, \lambda_\pi = \{0\}. \tag{3.30}$$

Since $\mathfrak{M}_E \otimes 1_L$ is a finite von Neumann algebra, (3.30) implies

$$\bigcap_{\pi \in Q} \text{Kernel}\, \lambda_\pi = \{0\} \tag{3.31}$$

Dixmier [12, III, §5, Proposition 2]). Let

$$S = \sum_{i=1}^{n} T_i \otimes T_i' \in \mathfrak{m} \cap \mathfrak{M}_E \otimes \mathcal{L}(L). \tag{3.32}$$

Then

$$\sum \lambda_\pi(T_i) \cdot \mu_\pi(T_i') = 0 \quad \text{for all } \pi \in Q. \tag{3.33}$$

Observe that

$$\mu_\pi(T_i') \in \lambda_\pi(\mathfrak{M}_E \otimes 1_L)' \tag{3.34}$$

and that the bicommutant of $\lambda_\pi(\mathfrak{M}_E \otimes 1_L)$ is a factor. Therefore, using a result of Murray and von Neumann [**21**, Theorem III] (see also Dixmier [**12**, I, §2, exercise 6a]) there is a matrix $(a_{ij})_{i,j=1,\cdots,n}$ of complex numbers such that

$$(3.35)\qquad \sum a_{ij}T_i \in \text{Kernel } \lambda_\pi, \qquad T_i' - \sum a_{ij}T_j' \in \text{Kernel } \mu_\pi.$$

Observe that

$$(3.36)\qquad \text{Kernel } \mu_\pi = E \otimes \mathfrak{C}(L).$$

The relations (3.35) and (3.36) imply

$$(3.37)\qquad S \in \text{Kernel } \lambda_\pi \otimes \mathcal{L}(L) + \mathfrak{M}_E \otimes \mathfrak{C}(L).$$

Since (3.37) holds for all $\pi \in Q$, (3.31) implies

$$(3.38)\qquad S \in \mathfrak{M}_E \otimes \mathfrak{C}(L)$$

concluding the proof of the lemma.

PROPOSITION 5. *Let* $\mathfrak{B}$ *be a postliminal* C^**-subalgebra of* $\mathcal{L}(L)$. *Suppose that* $\mathfrak{C}(L) \subseteq \mathfrak{B}$. *Then*

$$(3.39)\qquad \mathfrak{m} \cap (\mathfrak{M}_E \hat{\otimes} \mathfrak{B}) = \mathfrak{M}_E \hat{\otimes} \mathfrak{C}(L).$$

PROOF. Let φ be the canonical map of $\mathfrak{M}_E \hat{\otimes} \mathfrak{B}$ onto $(\mathfrak{M}_E \hat{\otimes} \mathfrak{B})/\mathfrak{M}_E \hat{\otimes} \mathfrak{C}(L)$. Let κ be the canonical isomorphism of $(\mathfrak{M}_E \hat{\otimes} \mathfrak{B})/\mathfrak{M}_E \hat{\otimes} \mathfrak{C}(L)$ onto $\mathfrak{M}_E \hat{\otimes} (\mathfrak{B}/\mathfrak{C}(L))$ according to Corollary 2 of Proposition 4. Let $\mathfrak{i}$ be the image of $\mathfrak{m} \cap \mathfrak{M}_E \hat{\otimes} \mathfrak{B}$ under $\kappa \circ \varphi$. Lemma 2 implies

$$(3.40)\qquad \mathfrak{i} \cap (\mathfrak{M}_E \otimes \mathfrak{B}/\mathfrak{C}(L)) = \{0\}.$$

Hence Corollary 1 of Proposition 4 implies $\mathfrak{i} = 0$. Hence (3.39).

Problem. Does

$$(3.41)\qquad \mathfrak{m} \cap (\mathfrak{M}_E \hat{\otimes} \mathcal{L}(L)) = \mathfrak{M}_E \hat{\otimes} \mathfrak{C}(L)$$

hold, too?

4. **Remarks on Banach space and** C^***-algebra bundles.** For the basic facts on Banach space bundles, i.e. vector bundles with Banach spaces as fibres one is referred to Lang [**18**]. Let X be a topological space. For any Banach space bundle Ξ over X let P_Ξ be the projection of Ξ and $\Xi_x = P_\Xi^{-1}(x)$. Let Ξ_1, Ξ_2 be Banach space bundles over X. In this section we only consider morphisms

$$(4.1)\qquad h : \Xi_1 \to \Xi_2$$

that induce the identity map on the base space, i.e.,

$$(4.2)\qquad P_{\Xi_1} = P_{\Xi_2} \cdot h.$$

Let Γ be the section functor which associates to each Banach space bundle Ξ over X the $\mathcal{C}(X, \mathbf{C})$-module $\Gamma(\Xi)$ of (continuous) sections of Ξ and to each morphism (4.1) the module homomorphism

$$(4.3)\qquad \Gamma(h) : \Gamma(\Xi_1) \to \Gamma(\Xi_2)$$

defined by

$$(4.4)\qquad (\Gamma(h)T)_x = h_x T_x \quad \text{for all } x \in X \text{ and all } T \in \Gamma(\Xi_1).$$

Observe that

$$(4.5)\qquad \text{Kernel } \Gamma(h) = \{T \in \Gamma(\Xi_1) \mid T_x \in \text{Kernel } h_x \text{ for all } x \in X\}.$$

PROPOSITION 6. *Let X be paracompact. Let*

$$(4.6)\qquad 0 \to \Xi' \xrightarrow{h} \Xi \xrightarrow{h''} \Xi'' \to 0$$

be an exact sequence of Banach space bundles over X. Then

$$(4.7)\qquad 0 \to \Gamma(\Xi') \xrightarrow{\Gamma(h')} \Gamma(\Xi) \xrightarrow{\Gamma(h'')} \Gamma(\Xi'') \to 0$$

is exact.

PROOF. It is obvious that $\Gamma(h')$ is injective and that the image of $\Gamma(h')$ is equal to the kernel of $\Gamma(h'')$. The nontrivial part is to show that $\Gamma(h'')$ is surjective. For this we need a continuous selection theorem and the open mapping theorem to verify lower semicontinuity. Let $(U_i, \Phi_i, E_i)_{i\in I}$, $(U_i, \Phi_i'', E_i'')_{i\in I}$ be atlases of Ξ, Ξ''. Let $T \in \Gamma(\Xi'')$. For each $x \in U_i$ the set $\Phi_{i,x}(h'')^{-1}T_x$ is a closed affine subspace of the Banach space E_i. Let W be open in E_i. Since h'' is surjective, $(\Phi_{i,x}'' h'' \Phi_{i,x}^{-1})W$ is open in E_i'' by the open mapping theorem. Suppose that $\Phi_{i,x}'' T_x \in (\Phi_{i,x}'' h'' \Phi_{i,x}^{-1})W$ for some $x \in U_i$. Since $y \to \Phi_{i,y}'' T_y$ is a continuous map of U_i into E_i'', it follows that $\Phi_{i,y}'' T_y$ is contained in $(\Phi_{i,x}'' h'' \Phi_{i,x}^{-1})W$ for all y in some neighborhood of x. Hence $x \to \Phi_{i,x}(h'')^{-1}(T_x)$ is a lower semicontinuous map of U_i into the closed affine subspaces of E_i. Since the closed affine subspaces of E_i are convex, it follows from a continuous selection theorem (Michael [19]) that there is a continuous map S_i of U_i into E_i satisfying $h''\Phi_{i,x}^{-1}(S_i(x)) = T_x$ for all $x \in U_i$. Let $(\lambda_i)_{i\in I}$ be a partition of unity subordinate to the cover $(U_i)_{i\in I}$. Define $S \in \Gamma(\Xi)$ by

$$(4.8)\qquad S_x = \sum \lambda_i(x) \Phi_{i,x}^{-1} S_i(x)$$

where $\lambda_i(x)\Phi_{i,x}^{-1}S_i(x) = 0$ if $x \notin U_i$. Then

(4.9) $$\Gamma(h'')S = T$$

which shows that $\Gamma(h'')$ is surjective.

In the following we also use the notion of a C^*-algebra bundle. Let Ξ be a Banach space bundle over X. Assume

(C^*B1) Each Ξ_x has been given the structure of a C^*-algebra. Let $(U_i, \Phi_i, \mathfrak{R}_i)_{i\in I}$ be an atlas of Ξ satisfying the following condition.

(C^*B2) All $\mathfrak{R}_i$ are C^*-algebras. For each $x \in U_i$ the map $\Phi_{i,x}$ of Ξ_x onto $\mathfrak{R}_i$ is a C^*-algebra isomorphism.

We say that an atlas $(U_i, \Phi_i, \mathfrak{R}_i)_{i\in I}$ of Ξ satisfying (C^*B2) is a C^*-algebra atlas of Ξ. Two such atlases are equivalent if their union is again a C^*-algebra atlas. The equivalence class of a C^*-algebra atlas of the Banach space bundle Ξ is said to define the structure of a C^*-algebra bundle (which is still denoted by Ξ).

In the following we assume that X is compact. Let Ξ be a C^*-algebra bundle over X. For each $T \in \Gamma(\Xi)$ define

(4.10) $$\|T\| = \sup\{\|T_x\|_x \mid x \in X\},$$

where $\| \ \|_x$ denotes the norm of Ξ_x. With respect to this norm and the obvious structure of an involutive complex algebra $\Gamma(\Xi)$ is a C^*-algebra. Let Y be another compact space. Let $\mathcal{C}(Y, \Xi_x)$ be the C^*-algebra of continuous maps of Y into Ξ_x. Then

(4.11) $$\mathcal{C}\,.\,(Y, \Xi) = \dot{\bigcup_{x\in X}} \mathcal{C}(Y, \Xi_x),$$

where $\dot{\bigcup}$ denotes disjoint union, can naturally be equipped with the structure of a C^*-algebra bundle. (Every atlas of Ξ gives rise to an atlas of $\mathcal{C}\,.\,(Y, \Xi)$.)

LEMMA 3. *There is a natural isomorphism of the C^*-algebra $\Gamma\mathcal{C}\cdot(Y, \Xi)$ onto the C^*-algebra of all continuous maps*

(4.12) $$f : X \times Y \to \Xi$$

satisfying

(4.13) $$f(x, y) \in \Xi_x.$$

PROOF. Since X, Y are compact, there is a natural homeomorphism

(4.14) $$\mathcal{C}(X \times Y, \Xi) \cong \mathcal{C}(X, \mathcal{C}(Y, \Xi))$$

(Bourbaki, *Topologie générale,* Chapter X, §5, Theorem 3). It is easy to see that this homeomorphism induces the C^*-algebra isomorphism described in Lemma 3.

CHAPTER II. VECTOR BUNDLES RELATIVE TO $\mathfrak{M}$.

In this chapter $\mathfrak{M}$ denotes always a properly infinite and semifinite von Neumann algebra of operators of a complex Hilbert space H.

1. **Definition of $\mathfrak{M}$-vector bundles and their morphisms.** Let ξ and X be topological spaces, and p_ξ be a continuous map of ξ onto X. Assume

(VB1) For each $x \in X$, the fibre $\xi_x = p_\xi^{-1}(x)$ has been given the structure of a Hilbert space.

Let $\{U_i\}_{i\in I}$ be an open cover of X, let $\{E_i\}_{i\in I}$ be a family in $P\mathfrak{M}$ and let $\{\varphi_i\}_{i\in I}$ be a family of maps

$$\varphi_i : p_\xi^{-1}(U_i) \to U_i \times E_i(H). \tag{1.1}$$

Denote by q_i the projection of $U_i \times E_i(H)$ onto U_i. Suppose that the following conditions hold.

(VB2) Each φ_i is a homeomorphism satisfying $p_\xi = q_i \circ \varphi_i$ and inducing an isometric isomorphism $\varphi_{i,x}$ of ξ_x onto $E_i(H)$ for each $x \in U_i$.

(VB3) For each $x \in U_i \cap U_j$ define $g_{ij}(x) \in \mathcal{L}(H)$ by

$$g_{ij}(x)(v) = (\varphi_{i,x} \circ \varphi_{j,x}^{-1})(E_j(v)) \quad \text{for all } v \in H. \tag{1.2}$$

Then $x \to g_{ij}(x)$ is a continuous map of $U_i \cap U_j$ into $\mathfrak{M}$,

$$g_{ij} : U_i \cap U_j \to \mathfrak{M}. \tag{1.3}$$

It follows that the range of g_{ij} is contained in $\mathcal{J}_{\mathfrak{M}}(E_j, E_i)$. We say that the family $(U_i, E_i, \varphi_i)_{i\in I}$ satisfying these conditions is an $\mathfrak{M}$-atlas of ξ and that each of its members is a chart. Two $\mathfrak{M}$-atlases are equivalent if their union is an $\mathfrak{M}$-atlas. The equivalence class of an $\mathfrak{M}$-atlas of ξ is an $\mathfrak{M}$-vector bundle with ξ as its total space, p_ξ its projection and X as its base space. Such $\mathfrak{M}$-vector bundles are usually denoted by their total space ξ. An $\mathfrak{M}$-vector bundle is said to be of *finite type* if it admits an atlas with finitely many charts. If the $\mathfrak{M}$-vector bundle ξ admits an atlas $(U_i, E_i, \varphi_i)_{i\in I}$ such that all E_i are equivalent then ξ is said to be of *constant fibre dimension.*

Let ξ be an $\mathfrak{M}$-vector bundle over X.

LEMMA 1. *If X is compact, then ξ is of finite type. If ξ is of finite type then there exists an $\mathfrak{M}$-atlas $(U_i, E_i, \varphi_i)_{i=1,\cdots,n}$ such that $E_iE_j = 0$ for $i \neq j$.*

PROOF. Use Dixmier [12, Chapter III, §8, Corollary 2 of Theorem 1].

REMARK. Using Dixmier's corollary one can prove a similar lemma under the weaker hypothesis that ξ is of countable type, i.e., that ξ admits an atlas with countably many charts. But Lemma 1 is all that we need in the following.

LEMMA 2. *If X is connected, then ξ is of constant fibre dimension. If ξ is of constant fibre dimension, then there is an atlas of ξ of the form $(U_i, E, \varphi_i)_{i\in I}$. In that case E is called the projection of this atlas. The equivalence class of E is uniquely determined by ξ.*

The proof is obvious.

Let ξ, ξ' be $\mathfrak{M}$-vector bundles over X, X'. A pair of maps (T, f): $\xi \times X \to \xi' \times X'$ is a morphism if the following two conditions hold.

(Mor 1) The relation $p_{\xi'} \circ T = f \circ p_\xi$ holds and T induces a partial isometry T_x of ξ_x into ξ'_{fx} for each $x \in X$.

(Mor 2) Let $(U_i, E_i, \varphi_i)_{i\in I}$ and $(U_j', E_j', \varphi_j')_{j\in J}$ be $\mathfrak{M}$-atlases of ξ, resp. ξ'.

For each $x \in U_i \cap f^{-1}(U_j')$ define $T_{ij,x} \in \mathcal{L}(H)$ by

$$(1.4) \qquad T_{ij,x}(v) = (\varphi'_{j,x} \circ T_x \circ \varphi_{i,x}^{-1})(E_i(v)) \quad \text{for all } v \in H.$$

Then $x \to T_{ij,x}$ is a continuous map of $U_i \cap f^{-1}(U_j')$ into $\mathfrak{M}$. It follows that T_{ij} maps $U_i \cap f^{-1}(U_j')$ into $\mathcal{J}_{\mathfrak{M}}(E_i, E_j')$.

PROPOSITION 1. *Let $\mathfrak{M}$ be countably decomposable. Let X be a topological space. Let ξ be an $\mathfrak{M}$-vector bundle over X with an atlas $(U_i, E_i, \varphi_i)_{i\in I}$ such that $E_i \sim 1$ for all $i \in I$. Then ξ is $\mathfrak{M}$-isomorphic to the trivial $\mathfrak{M}$-vector bundle $X \times H$ over X. Any two $\mathfrak{M}$-isomorphisms of ξ onto $X \times H$ are homotopic.*

PROOF. It follows from $E_i \sim 1$ and Lemma 2 that there is also an $\mathfrak{M}$-atlas whose transition functions take their values in the unitary group $\mathfrak{U}\mathfrak{M}$ of $\mathfrak{M}$. Since $\mathfrak{M}$ is countably decomposable, $\mathfrak{U}\mathfrak{M}$ is contractible in its norm topology (Breuer [10]). It follows from Dold [13] that the principal bundle (with group $\mathfrak{U}\mathfrak{M}$) associated to ξ admits a cross section. Hence ξ is $\mathfrak{M}$-equivalent to the product bundle $X \times H$ (Steenrod [23, Part I, §8]). Let V, $\tilde{V}$ be two $\mathfrak{M}$-isomorphisms of ξ onto $X \times H$. Then $\tilde{V} \circ V^*$ is an $\mathfrak{M}$-automorphism of $X \times H$, i.e., a continuous map of X into $\mathfrak{U}\mathfrak{M}$. Hence there is a homotopy $W_t : X \to \mathfrak{U}\mathfrak{M}$, $0 \leqq t \leqq 1$, with $W_0 = 1$, $W_1 = \tilde{V} \circ V^*$. Then $V_t = W_t \circ V$ is a homotopy between V and $\tilde{V}$.

2. **The Hom-functor.** Let ξ, η be $\mathfrak{M}$-vector bundles over X. Let $(U_i, \varphi_i, E_i)_{i\in I}$, $(U_i, \psi_i, F_i)_{i\in I}$ be $\mathfrak{M}$-atlases of ξ, resp. η, with the same open cover $(U_i)_{i\in I}$. Let $x \in U_i$. Define

$$(2.1) \qquad \mathrm{Hom}(\xi_x, \eta_x) = \{\psi_{i,x}^{-1} T \varphi_{i,x} \mid T \in \mathfrak{M}\}.$$

This definition is independent of the given atlases. $\mathrm{Hom}(\xi_x, \eta_x)$ is a linear subspace of $\mathcal{L}(\xi_x, \eta_x)$. For $T_x \in \mathrm{Hom}(\xi_x, \eta_x)$ define $\Phi_{i,x} T_x \in F_i \mathfrak{M} E_i$ by

$$(2.2) \qquad (\Phi_{i,x} T_x)(v) = (\psi_{i,x} T_x \varphi_{i,x}^{-1})(E_i v) \quad \text{for all } v \in H.$$

Then

$$(2.3) \qquad \Phi_{i,x} : \mathrm{Hom}(\xi_x, \eta_x) \to F_i \mathfrak{M} E_i$$

is a spatial isomorphism (induced by $\varphi_{i,x}, \psi_{i,x}$). It follows that $\mathrm{Hom}(\xi_x, \eta_x)$ is a weakly closed subspace of $\mathcal{L}(\xi_x, \eta_x)$. In particular $\mathrm{Hom}(\xi_x, \eta_x)$ is a Banach space. Define

$$(2.4) \qquad \mathrm{Hom}(\xi, \eta) = \dot{\bigcup_{x \in X}} \mathrm{Hom}(\xi_x, \eta_x).$$

Let

$$(2.5) \qquad p_{\mathrm{Hom}(\xi,\eta)} : \mathrm{Hom}(\xi, \eta) \to X$$

be the canonical projection. Define

$$(2.6) \qquad \Phi_i : p_{\mathrm{Hom}(\xi,\eta)}^{-1}(U_i) \to U_i \times F_i \mathfrak{M} E_i$$

to be the unique map whose restriction to $\mathrm{Hom}(\xi_x, \eta_x)$ is $\Phi_{i,x}$, $x \in U_i$. Then $(U_i, \Phi_i, F_i\mathfrak{M}E_i)_{i \in I}$ is an atlas of $\mathrm{Hom}(\xi, \eta)$ which defines the structure of a Banach space bundle on $\mathrm{Hom}(\xi, \eta)$ with $F_i\mathfrak{M}E_i$ as fibres. We call $(U_i, \varphi_i, F_i\mathfrak{M}E_i)_{i \in I}$ the *spatial atlas* of $\mathrm{Hom}(\xi, \eta)$ induced by $(U_i, \varphi_i, E_i)_{i \in I}$ and $(U_i, \psi_i, F_i)_{i \in I}$. The class of spatial atlases of $\mathrm{Hom}(\xi, \eta)$ induced by the $\mathfrak{M}$-atlases of ξ and η is said to define the structure of the Hom-bundle $\mathrm{Hom}(\xi, \eta)$.

Let ξ', η' be another pair of $\mathfrak{M}$-vector bundles over X. Let

$$(2.7) \qquad V : \xi \to \xi', \qquad W : \eta \to \eta'$$

be morphisms (as defined in §1). Define

$$(2.8) \qquad (V, W)_x^{\#} : \mathrm{Hom}(\xi_x, \eta_x) \to \mathrm{Hom}(\xi_x', \eta_x')$$

by

$$(2.9) \qquad (V, W)_x^{\#} T_x = W_x T_x V_x^{*} \quad \text{for all } T_x \in \mathrm{Hom}(\xi_x, \eta_x).$$

Define

$$(2.10) \qquad (V, W)^{\#} : \mathrm{Hom}(\xi, \eta) \to \mathrm{Hom}(\xi', \eta')$$

to be the map whose restriction to $\mathrm{Hom}(\xi_x, \eta_x)$ is $(V, W)_x^{\#}$. The maps $(V, W)^{\#}$ induced by pairs V, W of morphisns are called the morphisms of the Hom-bundles of pairs of $\mathfrak{M}$-vector bundles.

We are mainly interested in the case $\xi = \eta$. Then we write

$$\text{end}\,\xi = \text{Hom}(\xi, \xi). \tag{2.11}$$

It is clear that the above considerations can be repeated by choosing $\xi = \eta$ and in addition $(U_i, \varphi_i, E_i) = (U_i, \psi_i, F_i)$ and $V = W$. We thus can define spatial atlases of end ξ, the structure of the endomorphism bundle end ξ and morphisms

$$V^{\#} = (V, V)^{\#} : \text{end}\,\xi \to \text{end}\,\xi' \tag{2.12}$$

of endomorphism bundles induced by morphisms $V : \xi \to \xi'$. The fibre end ξ_x of end ξ at $x \in X$ is a von Neumann algebra which is spatially isomorphic to a reduced algebra of $\mathfrak{M}$. In particular, end ξ is always a C^*-algebra bundle.

In addition to the general hypotheses of this chapter let $\mathfrak{M}$ in the following also be countably decomposable. Let ξ be an $\mathfrak{M}$-vector bundle over X with an atlas $(U_i, \varphi_i, E)_{i \in I}$ and finite dimensional fibre, $E \in P_f(\mathfrak{M})$. Let $c(E)$ be the central cover of E, i.e.,

$$c(E) = \inf\{F \mid F \geqq E \text{ and } F \in P(\mathcal{Z})\} \tag{2.13}$$

where $\mathcal{Z} = \mathfrak{M} \cap \mathfrak{M}'$ is the center of $\mathfrak{M}$. Then there is an infinite sequence $(E_j)_{j=1,2,3,\dots}$ satisfying $E = E_1 \sim E_j$, $E_jE_k = 0$ for all j and $k \neq j$ and $c(E) = \sum_{j=1}^{\infty} E_j$. Therefore, according to §4 of Chapter I, there is a separable infinite dimensional Hilbert space L and an isomorphism

$$E(H) \hat{\otimes} L \cong c(E)(H) \tag{2.14}$$

inducing an isomorphism

$$\mathfrak{M}_E \hat{\hat{\otimes}} \mathcal{L}(L) \cong \mathfrak{M}_{c(E)}. \tag{2.15}$$

Let $\xi \hat{\otimes} L$ be the disjoint union of all $\xi_x \hat{\otimes} L$. Then $(U_i, \varphi_i \otimes 1_L, c(E))_{i \in I}$ is an $\mathfrak{M}$-atlas of $\xi \hat{\otimes} L$ defining the structure of an $\mathfrak{M}$-vector bundle on $\xi \hat{\otimes} L$.

Proposition 2. *end*$(\xi \hat{\otimes} L)$ *is spatially isomorphic to the trivial bundle* $X \times \mathfrak{M}_{c(E)}$. *Any two spatial isomorphisms of* end$(\xi \hat{\otimes} L)$ *onto* $X \times \mathfrak{M}_{c(E)}$ *are homotopic.*

Proof. Since $c(E)$ is properly infinite, it follows from Proposition 1 that there is an $\mathfrak{M}$-isomorphism V of $\xi \hat{\otimes} L$ onto the trivial bundle $X \times c(E)(H)$. Then $V^{\#}$ is an isomorphism of end ξ onto $X \times \mathfrak{M}_{c(E)}$. If V_t, $0 \leqq t \leqq 1$, is a homotopy of V, then $V_t^{\#}$, $0 \leqq t \leqq 1$, is a homotopy of $V^{\#}$.

3. **Finite $\mathfrak{M}$-vector bundles and classifying spaces.** Let ξ be an $\mathfrak{M}$-vector bundle over X with an atlas $(U_i, E_i, \varphi_i)_{i \in I}$. If all projections E_i are finite relative to $\mathfrak{M}$ then ξ is said to be finite relative to $\mathfrak{M}$ (or briefly: finite). In that case define the fibre demension by

$$\text{Dim } \xi_x = \text{Dim } E_i \in I(\mathfrak{M}) \quad \text{for } x \in U_i, \tag{3.1}$$

where $I(\mathfrak{M})$ is the index group of $\mathfrak{M}$ as defined in §2 of Chapter I. The definition of Dim ξ_x is independent of the given atlas. The function $x \to \text{Dim } \xi_x$ of X into $I(\mathfrak{M})$ is locally constant.

LEMMA 3. *Let X be paracompact. Let ξ be an $\mathfrak{M}$-vector bundle of finite type over X. Then there is a projection E of $\mathfrak{M}$ and an injective morphism of ξ into the trivial $\mathfrak{M}$-vector bundle $X \times E(H)$. If ξ is finite, then E can be chosen to be finite.*

PROOF. Let $(U_i, E_i, \varphi_i)_{i=1,\cdots,n}$ be an atlas of ξ satisfying $E_iE_j = 0$ for $i \neq j$. Let $E = \sum_{i=1}^n E_i$ and let $\lambda_i : X \to [0, 1]$ be continuous functions satisfying

(1) support $\lambda_i \subset U_i$,

(2) $\sum_{i=1}^n \lambda_i = 1$.

For each $x \in X$ and $v_x \in \xi_x$ define

$$T_x v_x = \sum \sqrt{\lambda_i(x)}\varphi_i(v_x) \tag{3.2}$$

where $\sqrt{\lambda_i(x)}\varphi_i(v_x) = 0$ if $x \notin U_i$. Then T_x is an isometry of ξ_x into $E(H)$. For each $x \in U_i$ define

$$T_{i,x}(v) = (T_x \circ \varphi_{i,x}^{-1})(E_i(v)) \quad \text{for all } v \in H, \tag{3.3}$$

Then $T_{i,x} \in \mathfrak{M}$. Obviously $x \to T_{i,x}$ is a continuous map of U_i into $\mathfrak{M}$. Thus the map

$$T : \xi \to X \times E(H) \tag{3.4}$$

defined by $T(v_x) = (x, T_x v_x)$ for $v_x \in \xi_x$ is an injective morphism. If all E_i, $i = 1, \cdots, n$, are finite, then their supremum E is known to be finite (Dixmier [12, III, §2, Proposition 5]).

Let E be a finite projection of $\mathfrak{M}$. The equivalence class

$$\mathcal{M}_E = \{F \in P\mathfrak{M} \mid F \sim E\} \tag{3.5}$$

of E equipped with the norm topology is called the Grassmannian of E. Equip

$$\mathcal{B}_E = \{(F, v) \in \mathcal{M}_E \times H \mid Fv = v\} \tag{3.6}$$

with the topology induced by $\mathcal{M}_E \times H$ and let

$$P : \mathcal{B}_E \to \mathcal{M}_E \tag{3.7}$$

be the canonical projection onto $\mathcal{M}_E$. For each $F \in \mathcal{M}_E$ define

$$\mathcal{N}_F = \{F' \in \mathcal{M}_E \mid \|F - F'\| < 1\}. \tag{3.8}$$

Let

$$FF' = V_{F,F'}|FF'| \tag{3.9}$$

be the polar decomposition. Define

$$\Phi_F : P^{-1}(\mathcal{N}_F) \to \mathcal{N}_F \times F(H) \tag{3.10}$$

by

$$\Phi_F(F', v) = (F', V_{F,F'}(v)). \tag{3.11}$$

Observe that $F' \in \mathcal{N}_F$ implies

$$F = V_{F,F'}V^*_{F,F'}, \quad F' = V^*_{F,F'}V_{F,F'} \tag{3.12}$$

(Riesz-Sz.-Nagy [22, §105]) and that $F' \to V_{F,F'}$ is a continuous map of $\mathcal{N}_F$ into $\mathfrak{M}$. Moreover, for $F' \in \mathcal{N}_E \cap \mathcal{N}_F$ and $v \in E(H)$

$$(\Phi_F \circ \Phi_E^{-1})(F', v) = (F', V_{E,F'}V^*_{E,F'}(v)). \tag{3.13}$$

Hence the family $(\mathcal{N}_F, F, \Phi_F)_{F\in\mathcal{M}_E}$ is an $\mathfrak{M}$-atlas of $\mathcal{B}_E$. The equivalence class of this atlas is called the *Grassmann vector bundle* of E. If $E \sim F$, then the Grassmann vector bundles of E and F are equal.

PROPOSITION 3. *Let X be paracompact. Let ξ be a finite $\mathfrak{M}$-vector bundle of finite type over X. Suppose that the fibre dimension of ξ is constant and equal to* Dim F *for some finite $F \in P\mathfrak{M}$. Then there is a continuous map $f : X \to \mathcal{M}_F$ such that ξ is $\mathfrak{M}$-isomorphic to the induced bundle $f^*(\mathcal{B}_F)$.*

PROOF. Use all the notation of the proof of Lemma 3 and define $f(x) = R_{T_x}$ (range projection of T_x). One has $R_{T_x} = R_{T_{i,x}}$ and $T_{i,x} \in \mathfrak{M}$ for all $x \in U_i$ which implies $f(x) \in \mathcal{M}_F$ for $x \in U_i$. Proposition 2 of Chapter I and the continuity of $x \to T_{i,x}$ on U_i imply that f is continuous on U_i. Since $(U_i)_{i=1,\cdots,n}$ is an open cover of X, f is a continuous map of X into $\mathcal{M}_F$. The pair (T, f) can canonically be considered as a map $\xi \times X \to \mathcal{B}_F \times \mathcal{M}_F$. To show that ξ is $\mathfrak{M}$-isomorphic to $f^*(\mathcal{B}_F)$ it suffices to show that (T, f) is an injective morphism. The injectivity and axiom (Mor 1) are trivial. To verify axiom (Mor 2) consider the atlas $(U_i, E_i, \varphi_i)_{i=1,\cdots,n}$ of ξ used in the proof of Lemma 3 and the atlas $(\mathcal{N}_E, E, \Phi_E)_{E\in\mathcal{M}_F}$ of $\mathcal{B}_F$ defined above. Let $x \in U_i$ and $y \in U_i \cap f^{-1}(\mathcal{N}_{f(x)})$. Then

$$(3.14) \qquad (\Phi_{f(x)} \circ T \circ \varphi_{i,y}^{-1})(v) = (fy, V_{f(x),f(y)}T_{i,y}(v)) \quad \text{for } v \in E_i(H).$$

Thus $y \to V_{f(x),f(y)} \circ T_{i,y}$ is a continuous map of $U_i \cap f^{-1}(\mathcal{N}_{fx})$ into $\mathfrak{M}$ which implies (Mor 2).

PROPOSITION 4. *Let X be compact, F a finite projection of $\mathfrak{M}$, $f_t : X \to \mathcal{M}_F$ $(0 \leqq t \leqq 1)$ a homotopy. Then the induced bundles $f_0^*(\mathcal{B}_F)$ and $f_1^*(\mathcal{B}_F)$ are $\mathfrak{M}$-isomorphic $\mathfrak{M}$-vector bundles over X.*

PROOF. Let $t_0 \in [0, 1]$. Since X is compact, there is a $\delta > 0$ such that for all $x \in X$ and all $t \in X$ and all $t \in [t_0 - \delta, t_0 + \delta] \cap [0, 1]$ the relation

$$(3.15) \qquad \|f_{t_0}(x) - f_t(x)\| < 1$$

holds. Let

$$(3.16) \qquad f_t(x)f_{t_0}(x) = V_{t,t_0}(x) \cdot |f_t(x)f_{t_0}(x)|$$

be the polar decomposition. It follows from (3.15) and the continuity of the polar decomposition that $(V_{t,t_0}(x))_{x\in X}$ is a continuous family of partial isometries in $\mathfrak{M}$ satisfying

$$(3.17) \qquad V_{t,t_0}(x)V^*_{t,t_0}(x) = f_t(x),\ V^*_{t,t_0}(x)V_{t,t_0}(x) = f_{t_0}(x)$$

for all $x \in X$. Hence this family induces an $\mathfrak{M}$-isomorphism

$$(3.18) \qquad V_{t,t_0} : f^*_{t_0}(\mathcal{B}_F) \to f_t^*(\mathcal{B}_F).$$

The connectedness of $[0, 1]$ then implies that $f_0^*(\mathcal{B}_F)$ is $\mathfrak{M}$-isomorphic to $f_1^*(\mathcal{B}_F)$.

COROLLARY 1. *Let X be compact, Y paracompact, $f_t : X \to Y$ $(0 \leqq t \leqq 1)$ a homotopy and η a finite $\mathfrak{M}$-vector bundle of finite type over Y. Then $f_0^*(\eta)$ is $\mathfrak{M}$-isomorphic to $f_1^*(\eta)$.*

PROOF. Without loss of generality we can assume that the fibre dimension of η is constant. Then it follows from Proposition 3 that there is a finite projection $F \in \mathfrak{M}$ and a continuous map $g : Y \to \mathcal{M}_F$ such that $\eta \cong g^*(\mathcal{B}_F)$. Define the homotopy $h_t : X \to \mathcal{M}_F$ by $h_t = g \circ f_t$. Thus Proposition 4 implies

$$(3.19) \quad f_0^*(\eta) \cong f_0^*g^*(\mathcal{B}_F) = h_0^*(\mathcal{B}_F) \cong h_1^*(\mathcal{B}_F) = f_1^*g^*(\mathcal{B}_F) \cong f_1^*(\eta).$$

Corollary 2. *Every $\mathfrak{M}$-vector bundle over the one-sphere S^1 is $\mathfrak{M}$-isomorphic to a trivial $\mathfrak{M}$-vector bundle.*

Proof. For each $E \in P\mathfrak{M}$ the Grassmannian $\mathscr{M}_E$ is simply connected (Breuer [**10**]). Hence the corollary follows from Propositions 3 and 4.

Proposition 5. *Let $\mathfrak{M}$ be countably decomposable. Let X be a topological space. Let $E \in P\mathfrak{M}$ be finite and f, g be continuous maps of X into $\mathscr{M}_E$. If $f^*\mathscr{B}_E$ and $g^*\mathscr{B}_E$ are $\mathfrak{M}$-isomorphic, then f and g are homotopic.*

Proof. Since $f^*\mathscr{B}_E \cong g^*\mathscr{B}_E$, there is a continuous map $x \to V_x$ of X into $\mathfrak{M}$ such that

$$f(x) = V_x{}^*V_x, \qquad g(x) = V_xV_x{}^*. \tag{3.20}$$

Define the maps $\tilde{f}$, $\tilde{g}$ of X into $\mathscr{M}_{1-E}$ by $\tilde{f}(x) = 1 - f(x)$, $\tilde{g}(x) = 1 - g(x)$. Since E is finite, $1 - E$ is equivalent to 1. Proposition 1 of §1 implies that there are $\mathfrak{M}$-isomorphisms

$$\Phi : \tilde{f}^*\mathscr{B}_{1-E} \to X \times H, \qquad \Psi : \tilde{g}^*\mathscr{B}_{1-F} \to X \times H. \tag{3.21}$$

Define

$$T : X \to \mathfrak{U}\mathfrak{M} \tag{3.22}$$

by

$$T(x) = V_x + \Psi_x{}^{-1} \circ \Phi_x. \tag{3.23}$$

Then T is continuous and satisfies

$$g(x) = T(x)f(x)T^*(x). \tag{3.24}$$

(It is well known that two equivalent finite projections of $\mathfrak{M}$ are unitarily equivalent (Dixmier [**12**, III, §2, Proposition 6]). Formula (3.23) is a generalization of that proposition to continuous families of finite projections of $\mathfrak{M}$.) Since $\mathfrak{U}\mathfrak{M}$ is contractible (Breuer [**10**]), there is a homotopy

$$T_t : X \to \mathfrak{U}\mathfrak{M}, \qquad 0 \leqq t \leqq 1, \tag{3.25}$$

satisfying $T_0 = 1$ (constant map of X on the unit element) and $T_1 = T$. Then

$$f_t(x) = T_t(x)f(x)T_t{}^*(x), \qquad 0 \leqq t \leqq 1, x \in X, \tag{3.26}$$

defines a homotopy f_t, $0 \leqq t \leqq 1$, between f and g.

4. Direct sums, orthogonal complements, definition of $K_{\mathfrak{M}}(X)$.

LEMMA 4. *Let ξ, η be $\mathfrak{M}$-vector bundles over X. Let $(U_i, E_i, \varphi_i)_{i \in I}$ and $(U_j', E_j', \varphi_j')_{j \in J}$ be $\mathfrak{M}$-atlases of ξ; let $(U_i, F_i, \psi_i)_{i \in I}$ and $(U_j', F_j', \psi_j')_{j \in J}$ be $\mathfrak{M}$-atlases of η. Suppose that*

$$(4.1) \qquad E_iF_i = 0 \quad \text{for all } i \in I, \qquad E_j'F_j' = 0 \quad \text{for all } j \in J.$$

Then $(U_i, E_i + F_i, \varphi_i + \psi_i)_{i \in I}$, $(U_j', E_j' + F_j', \varphi_j' + \psi_j')_{j \in J}$ are $\mathfrak{M}$-equivalent atlases of

$$(4.2) \qquad \xi \oplus \eta = \bigcup_{x \in X} \{x\} \times (\xi_x \oplus \eta_x).$$

The proof is obvious.

Since atlases of the $\mathfrak{M}$-vector bundles ξ, η satisfying the conditions of Lemma 4 always exist, the direct sum $\xi \oplus \eta$ can canonically be equipped with the structure of an $\mathfrak{M}$-vector bundle. This structure will simply be denoted by $\xi \oplus \eta$.

LEMMA 5. *Let E be a finite projection of $\mathfrak{M}$. Let f be a continuous map of X into $\mathcal{M}_E$ and let $\xi = f^*(\mathcal{B}_E)$ be the induced bundle. Let η be an $\mathfrak{M}$-vector subbundle of ξ. Let*

$$(4.3) \qquad (\xi \ominus \eta)_x = \xi_x \ominus \eta_x$$

be the orthogonal complement of ξ_x in η_x. Then

$$(4.4) \qquad \xi \ominus \eta = \bigcup_{x \in X} \{x\} \times (\xi \ominus \eta)_x$$

can canonically be equipped with the structure of an $\mathfrak{M}$-vector bundle over X satisfying

$$(4.5) \qquad \xi \cong \eta \oplus (\xi \ominus \eta)$$

where $\cong$ means $\mathfrak{M}$-isomorphic.

PROOF. Without loss of generality we can assume that the fibre dimension of η is constant and equal to Dim F for some $F \leqq E$. Since we have $\eta \subseteq \xi$ and $\xi \subseteq X \times H$ we also have $\eta \subseteq X \times H$ and this inclusion is a morphism. It follows that the projection $f'(x)$ of H onto η_x is in $\mathcal{M}_F$ and that $f' : X \to \mathcal{M}_F$ is continuous. Define the continuous map $f'' : X \to \mathcal{M}_{E-F}$ by $f''(x) = f(x) - f'(x)$. Then the fibre of $f''^*(\mathcal{B}_{E-F})$ at x is equal to $(\xi \ominus \eta)_x$. Thus $\xi \ominus \eta$ can be given the $\mathfrak{M}$-vector bundle structure of $f'^*(\mathcal{B}_{E-F})$. The relation (4.5) is trivial.

LEMMA 6 (UNIQUENESS OF $\mathfrak{M}$-VECTOR SUBBUNDLES). *Let ξ, η be $\mathfrak{M}$-vector bundles over X. Let ξ', η' be $\mathfrak{M}$-vector subbundles of ξ, η. Let T be an $\mathfrak{M}$-isomorphism of ξ onto η which induces a bijection of*

ξ' onto η'. Then the restriction T' of T to ξ' is an $\mathfrak{M}$-isomorphism of ξ' onto η'.

PROOF. This is quite trivial and therefore omitted (see N. Bourbaki, *Théorie des ensembles,* Chapitre 4, §2, CST 8 and CST 12).

PROPOSITION 6. *Let X be paracompact. Let ξ be a finite $\mathfrak{M}$-vector bundle of finite type over X. Let η be an $\mathfrak{M}$-vector subbundle of ξ. Then $\xi \ominus \eta$ admits one and only one structure of an $\mathfrak{M}$-vector bundle which makes it an $\mathfrak{M}$-vector subbundle of ξ (via the natural inclusion). If we equip $\xi \ominus \eta$ with this structure, then ξ is $\mathfrak{M}$-isomorphic to the direct sum $\eta \oplus (\xi \ominus \eta)$.*

PROOF. The existence of an $\mathfrak{M}$-vector bundle structure on $\xi \ominus \eta$ which makes it an $\mathfrak{M}$-vector subbundle satisfying $\xi \cong \eta \oplus (\xi \ominus \eta)$ follows from Proposition 3 and Lemma 5. The uniqueness follows from Lemma 6.

PROPOSITION 7. *Let X be paracompact. Let ξ be a finite $\mathfrak{M}$-vector bundle of finite type over X. Then there are a finite $\mathfrak{M}$-vector bundle η over X and a finite projection E of $\mathfrak{M}$ such that $\xi \oplus \eta$ is $\mathfrak{M}$-isomorphic to the trivial bundle $X \times E(H)$.*

PROOF. This is an easy consequence of Lemma 3 and Proposition 6.

It is easy to see that the direct sum $\oplus$ of $\mathfrak{M}$-vector bundles has the following properties, where $\cong$ means $\mathfrak{M}$-isomorphic.

(i) $\xi \oplus (\eta \oplus \xi) \cong (\xi \oplus \eta) \oplus \xi$,
(ii) $\xi \oplus \eta \cong \eta \oplus \xi$,
(iii) $\xi \oplus 0 \cong \xi$,
(iv) $\xi \cong \eta$ and $\xi' \cong \eta'$ implies $\xi \oplus \xi' \cong \eta \oplus \eta'$,
(v) ξ and η $\mathfrak{M}$-infinite implies $\xi \oplus \eta$ $\mathfrak{M}$finite.

It follows that $\oplus$ induces the structure of a commutative monoid on the set of isomorphism classes of $\mathfrak{M}$-finite vector bundles over X. Denote thiș monoid by $\mathrm{Vect}_{\mathfrak{M}}(X)$. Observe that $\mathrm{Vect}_{\mathfrak{M}}$ is a contravariant functor of the category of topological spaces and continuous maps in the category of commutative monoids.

DEFINITION 1. Let X be compact. $K_{\mathfrak{M}}(X)$ denotes the Grothendieck group of $\mathrm{Vect}_{\mathfrak{M}}(X)$. Let ξ be a finite $\mathfrak{M}$-vector bundle over X. $[\xi]_{\mathfrak{M}}$ denotes the class of ξ in $K_{\mathfrak{M}}(X)$.

Let $E \in P\mathfrak{M}$ be finite. The class of the trivial $\mathfrak{M}$-vector bundle $X \times E(H)$ is uniquely determined by $\mathrm{Dim}\, E \in I(\mathfrak{M})$. The map $\mathrm{Dim}\, E \to [X \times E(H)]_{\mathfrak{M}}$ of $I^+(\mathfrak{M})$ into $K_{\mathfrak{M}}(X)$ extends to an injective isomorphism $I(\mathfrak{M}) \subseteq K_{\mathfrak{M}}(X)$. Therefore the class of $X \times E(H)$ in $K_{\mathfrak{M}}(X)$ will usually be denoted by $\mathrm{Dim}\, E$.

Observe that $K_{\mathfrak{M}}$ is a contravariant functor of the category of compact spaces and continuous maps in the category of commutative groups. Let X, Y be compact. Let f, g be homotopic maps of X into Y. Proposition 4 implies that $K_{\mathfrak{M}}(f) = K_{\mathfrak{M}}(g)$. If X is contractible then $K_{\mathfrak{M}}(X) = I(\mathfrak{M})$.

Let x_0 be a point of X and i: $\{x_0\} \to X$ be the inclusion. Then $K_{\mathfrak{M}}(i)$ is a homomorphism of $K_{\mathfrak{M}}(X)$ onto $I(\mathfrak{M})$ inducing the identity isomorphism on $I(\mathfrak{M}) \subseteq K_{\mathfrak{M}}(X)$. It follows that

$$K_{\mathfrak{M}}(X) = \operatorname{kernel}(K_{\mathfrak{M}}(i)) \oplus I(\mathfrak{M}). \tag{4.6}$$

5. **Clutching data of $\mathfrak{M}$-vector bundles over $S^2 \times X$.** In this section $\mathfrak{M}$ is also assumed to be countably decomposable. Let X be a compact space. Let $S^2 = \mathbf{C} \cup \{\infty\}$ be the Riemann sphere, and one point compactification of $\mathbf{C}$. Let

$$D_0 = \{z \in S^2 \mid |z| \leqq 1\}, \qquad D_\infty = \{z \in S^2 \mid |z| \geqq 1\}. \tag{5.1}$$

Then $S^2 = D_0 \cup D_\infty$ and $S^1 = D_0 \cap D_\infty$.

Proposition 8. *Let ξ_0, resp. ξ_∞, be finite $\mathfrak{M}$-vector bundles over $D_0 \times X$, resp. $D_\infty \times X$. Let*

$$\varphi : \xi_0|S^1 \times X \to \xi_\infty|S^1 \times X \tag{5.2}$$

be an $\mathfrak{M}$-isomorphism. Then there are an $\mathfrak{M}$-vector bundle ξ over $S^2 \times X$ and $\mathfrak{M}$-isomorphisms

$$U_0 : \xi|D_0 \times X \to \xi_0, \qquad U_\infty : \xi|D_\infty \times X \to \xi_\infty \tag{5.3}$$

such that

$$\varphi = U_\infty \circ U^{-1} \quad (\textit{restricted to } \xi_0|S^1 \times X). \tag{5.4}$$

Moreover, ξ is unique up to isomorphism.

Proof. Without loss of generality we can assume that the fibre dimensions of ξ_0 and ξ_∞ are constant. Choose a (necessarily finite) projection $E \in \mathfrak{M}$ such that $\operatorname{Dim} E$ is the common fibre dimension of ξ_0 and ξ_∞. According to Proposition 3 there are continuous maps

$$f_0 : D_0 \times X \to \mathcal{M}_E, \quad f_\infty : D_\infty \times X \to \mathcal{M}_E \tag{5.5}$$

and $\mathfrak{M}$-isomorphisms

$$V_0 : \xi_0 \to f_0^*(\mathcal{B}_E), \qquad V_\infty : \xi_\infty \to f_\infty^*(\mathcal{B}_E). \tag{5.6}$$

Then

$$\psi = V_\infty \varphi V_0^{-1} : f_0^*(\mathcal{B}_E)|S^1 \times X \to f_\infty^*(\mathcal{B}_E)|S^1 \times X \tag{5.7}$$

is an $\mathfrak{M}$-isomorphism. Define

$$\tilde{f}_0 : S^1 \times X \to \mathcal{M}_{1-E}, \qquad \tilde{f}_\infty : S^1 \times X \to \mathcal{M}_{1-E} \tag{5.8}$$

by

$$\tilde{f}_0(z, x) = 1 - f_0(z, x), \qquad \tilde{f}_\infty(z, x) = 1 - f_\infty(z, x). \tag{5.9}$$

Since $1 - E$ is properly infinite, there is an $\mathfrak{M}$-isomorphsim

$$\tilde{\psi} : \tilde{f}_0^*(\mathcal{B}_{1-E}) \to \tilde{f}_\infty^*(\mathcal{B}_{1-E}). \tag{5.10}$$

Both ψ and $\tilde{\psi}$ can canonically be viewed as continuous maps of $S^1 \times X$ into the space of partial isometries of $\mathfrak{M}$ (equipped with the norm topology) satisfying

$$\begin{aligned} f_0(z, x) &= \psi^*(z, x)\psi(z, x), & f_\infty(z, x) &= \psi(z, x)\psi^*(z, x), \\ \tilde{f}_0(z, x) &= \tilde{\psi}^*(z, x)\tilde{\psi}(z, x), & \tilde{f}_\infty(z, x) &= \tilde{\psi}(z, x)\tilde{\psi}^*(z, x) \end{aligned} \tag{5.11}$$

for all $(z, x) \in S^1 \times X$. Define

$$\bar{T} : S^1 \times X \to \mathfrak{U}\mathfrak{M} \tag{5.12}$$

by

$$\bar{T}(z, x) = \psi(z, x) + \tilde{\psi}(z, x). \tag{5.13}$$

Then $\bar{T}$ induces the isomorphism ψ and we have

$$f_\infty(z, x) = \bar{T}(z, x)f_0(z, x)\bar{T}^*(z, x) \tag{5.14}$$

for all $(z, x) \in S^1 \times X$. Using the contractibility of $\mathfrak{U}\mathfrak{M}$ (Breuer [10]) we can define a homotopy

$$\bar{T}_t : S^1 \times X \to \mathfrak{U}\mathfrak{M}, \qquad 0 \leqq t \leqq 1, \tag{5.15}$$

satisfying

$$T_0 = 1, \qquad \bar{T}_1 = \bar{T}. \tag{5.16}$$

define the extension

$$T : D_0 \times X \to \mathfrak{U}\mathfrak{M} \tag{5.17}$$

of $\bar{T}$ by

$$T(z, x) = \begin{cases} \bar{T}_{|z|}(\exp(i \cdot \arg z), x) & \text{for } 0 < |z| \leqq 1, \\ 1 & \text{for } z = 0. \end{cases} \tag{5.18}$$

Define

$$f : S^2 \times X \to \mathfrak{M}_E \tag{5.19}$$

by

$$(5.20)\qquad f(z,x) = \begin{cases} T(z,x)f_0(z,x)T^*(z,x) & \text{for } (z,x) \in D_0 \times X, \\ f_\infty(z,x) & \text{for } (z,x) \in D_\infty \times X. \end{cases}$$

It follows from (5.14) that f is well defined and continuous. Define

$$(5.21)\qquad \xi = f^*(\mathcal{B}_E).$$

Then T induces an $\mathfrak{M}$-isomorphism

$$(5.22)\qquad W_0 : \xi \mid D_0 \times X \to f_0{}^*(\mathcal{B}_E).$$

Let

$$(5.23)\qquad W_\infty : \xi \mid D_\infty \times X \to f_\infty{}^*(\mathcal{B}_E)$$

be the identity isomorphism. Then

$$(5.24)\qquad \psi = W_\infty \circ W_0{}^{-1} \quad (\text{restricted to } f_0{}^*(\mathcal{B}_E) \mid S^1 \times X).$$

Define the $\mathfrak{M}$-isomorphisms (5.3) by

$$(5.25)\qquad U_0 = V_0{}^{-1}W_0, \qquad U_\infty = V_\infty{}^{-1} \circ W_\infty.$$

Then (5.4) follows from (5.7) and (5.25).

Suppose that ξ' is another $\mathfrak{M}$-vector bundle over $S^2 \times X$ with $\mathfrak{M}$-isomorphisms

$$(5.26)\qquad U_0{}' : \xi' \mid D_0 \times X \to \xi_0, \qquad U_\infty{}' : \xi' \mid D_\infty \times X \to \xi_\infty$$

satisfying

$$(5.27)\qquad \varphi = U_\infty{}' \circ (U_0{}')^{-1} \quad (\text{restricted to } \xi_0 \mid S^1 \times X).$$

Then the $\mathfrak{M}$-isomorphisms

$$(5.28)\qquad \begin{aligned} U_0{}^{-1}U_0{}' &: \xi'|D_0 \times X \to \xi|D_0 \times X, \\ U_\infty{}^{-1}U_\infty{}' &: \xi'|D_\infty \times X \to \xi|D_\infty \times X, \end{aligned}$$

coincide on $\xi'|S^1 \times X$ and consequently give rise to an $\mathfrak{M}$-isomorphism of ξ' onto ξ.

Definition 1. *The bundle ξ of Proposition* 8 *is denoted by* $\xi_0 \cup_\varphi \xi_\infty$.

Proposition 9. *The $\mathfrak{M}$-isomorphism class of $\xi_0 \cup_\varphi \xi_\infty$ depends on the homotopy class of the $\mathfrak{M}$-isomorphism φ only.*

Proposition 10. *Let π_0 resp. π_∞, be the natural projection of $D_0 \times X$, resp. $D_\infty \times X$, on X. Let ζ be a finite $\mathfrak{M}$-vector bundle over $S^2 \times X$. Then there are a finite $\mathfrak{M}$-vector bundle ξ over X and an $\mathfrak{M}$-automorphism*

(5.29) $$\varphi : \pi_0{}^*(\xi)|S^1 \times X \to \pi_\infty{}^*(\xi)|S^1 \times X$$

such that the following hold:

(i) *the restriction of* φ *to* $\pi_0{}^*(\xi) \mid \{1\} \times X$ *is homotopic to the identity automorphism,*

(ii) ξ *is* $\mathfrak{M}$*-isomorphic to* $\pi_0{}^*(\xi) \cup_\varphi \pi_\infty{}^*(\xi)$,

(iii) *the homotopy class of* φ *is uniquely determined by* (i) *and* (*ii*).

The proofs of Propositions 9 and 10 are similar to the proofs of the corresponding propositions on complex finite dimensional vector bundles in Husemoller [15, 9(7.6) and 10(2.3)]. One has to replace Husemoller's Proposition 9(7.1) by the above Proposition 8.

DEFINITION 2. The $\mathfrak{M}$-vector bundle $\pi_0{}^*(\xi) \cup_\varphi \pi_\infty{}^*(\xi)$ of Proposition 10 is denoted by $[\xi, \varphi]$. The $\mathfrak{M}$-automorphism φ is called a clutching function of ξ.

PROPOSITION 11. *The clutching functions of the* $\mathfrak{M}$*-vector bundle* ξ *over* X *are in natural* 1-1 *correspondence with the unitary elements of the* C^**-algebra* $\Gamma\, \mathcal{C}.\,(S^1, \text{ end } \xi)$. *Moreover, the homotopies of clutching functions of* ξ *correspond to the continuous paths of the unitary group of* $\Gamma\, \mathcal{C}.\,(S^1, \text{end } \xi)$.

PROOF. The maps (5.29) can canonically be viewed as maps

(5.30) $$\varphi : S^1 \times X \to \text{end } \xi$$

satisfying

(5.31) $$\varphi(z, x) \in \text{end } \xi_x \quad \text{for all } (z, x) \in S^1 \times X.$$

The first part then follows from Lemma 3 of Chapter I. The second part is proved similarly using in addition the canonical C^*-algebra isomorphism

(5.32) $$\Gamma \mathcal{C}.\,([0, 1], \mathcal{C}.\,(S^1, \text{end } \xi)) \cong \Gamma \mathcal{C}.\,([0, 1] \times S^1, \text{end } \xi).$$

REMARK. The methods of this section can also be used to obtain clutching data of $\mathfrak{M}$-vector bundles over CW-triads. However, this more general construction has been omitted, since it is not used in the following.

CHAPTER III. THE INDEX OF A COMPACT FAMILY OF $\mathfrak{M}$-FREDHOLM OPERATORS

In this chapter $\mathfrak{M}$ is a semifinite and properly infinite von Neumann algebra of operators of a complex Hilbert space H. X is a compact space. If $E \in P\mathfrak{M}$, then $\Theta_{E,X}$ denotes the trivial $\mathfrak{M}$-vector bundle $X \times E(H)$. The projection $1 - E$ is denoted by $E^\perp$.

1. Definition of the index of a map $X \to \mathfrak{F}(\mathfrak{M})$. Let

$$T : X \to \mathfrak{F}(\mathfrak{M}) \tag{1.1}$$

be a continuous map. Call a projection E of $\mathfrak{M}$ a choice for T if the following hold:

(i) $E^{\perp}$ is finite,
(ii) the range of $T_x E$ is closed for all $x \in X$,
(iii) $\inf(N_{T_x}, E) = 0$ for all $x \in X$.

LEMMA 1. *For each continuous map T of the compact space X into $\mathfrak{F}(\mathfrak{M})$ there is a choice $E \in P(\mathfrak{M})$.*

PROOF. Lemma 1 of Chapter I and the definition of $\mathfrak{F}(\mathfrak{M})$ imply that the following holds: For each $x \in X$ there is a projection $E_x \in \mathfrak{M}$ and a neighborhood U_x of x satisfying

(i′) $E_x{}^{\perp}$ is finite,
(ii′) the range of $T_y E_x$ is closed for all $y \in U_x$,
(iii′) $\inf(N_{T_y}, E_x) = 0$ for all $y \in U$.

Let $U_{x_1}, \cdots, U_{x_n}$ be a finite subcover of $(U_x)_{x \in X}$. Then (i′)–(iii′) imply that

$$E = \inf(E_{x_1}, \cdots E_{x_n}) \tag{1.2}$$

is a choice for T.

Let E be a choice for the continuous map T of X into $\mathfrak{F}(\mathfrak{M})$. We want to define an $\mathfrak{M}$-vector bundle $\rho_{TE}^{\perp}$ over X whose fibre over $x \in X$ is the orthogonal complement of the range of TE, i.e.,

$$(\rho_{TE}^{\perp})_x = H \ominus T_x E(H) = R_{T_x E}^{\perp}(H). \tag{1.3}$$

Any bundle over X is well determined if its portion over each connected component of X is known. Therefore we can, without loss of generality, assume that X is connected. Proposition 2 of Chapter I implies that the map

$$r_{TE}^{\perp} : X \to P\mathfrak{M} \tag{1.4}$$

defined by

$$r_{TE}^{\perp}(c) = R_{T_x E}^{\perp} \tag{1.5}$$

is continuous. Observe that $R_{T_x E}^{\perp}$ is finite for all $x \in X$. Since the Grassmannian $\mathcal{M}_G$ of a finite $G \in P\mathfrak{M}$ is the connected component of G in $P\mathfrak{M}$ (Breuer [10]) and since X is connected, there is a finite $F \in P\mathfrak{M}$ such that the range of $r_{TE}^{\perp}$ is contained in $\mathcal{M}_F$. Define

$$\rho_{TE}^{\perp} = (r_{TE}^{\perp})^*(\mathcal{B}_F). \tag{1.6}$$

In view of (1.5), $\rho_{TE}^{\perp}$ satisfies (1.3).

LEMMA 2. *Let E', E be choices of T such that $E' \geqq E$. Then*

$$\rho_{TE}^{\perp} \cong \rho_{TE'}^{\perp} \oplus \Theta_{X,E'-E} \tag{1.7}$$

and

$$\rho_{TE}^{\perp} \oplus \Theta_{X,E'^{\perp}} \cong \rho_{TE'}^{\perp} \oplus \Theta_{X,E_{\perp}} . \tag{1.8}$$

PROOF. (1.7) implies (1.8) so it suffices to prove (1.7). $E' \geqq E$ implies that $\rho_{TE'}^{\perp}$ is an $\mathfrak{M}$-vector subbundle of $\rho_{TE}^{\perp}$. Proposition 6 of Chapter II implies

$$\rho_{TE}^{\perp} \cong \rho_{TE'}^{\perp} \oplus (\rho_{TE}^{\perp} \ominus \rho_{TE'}^{\perp}). \tag{1.9}$$

Let $T_x(E' - E) = V_x|T_x(E' - E)|$ be the polar decomposition. Then the continuous family $(V_x)_{x \in X}$ of partial isometries of $\mathfrak{M}$ induces an isomorphism

$$\Theta_{X,E'-E} \cong \rho_{TE}^{\perp} \ominus \rho_{TE'}^{\perp} . \tag{1.10}$$

(1.10) and (1.9) imply (1.7).

LEMMA 3. *Let E, E' be choices of T. Then*

$$\operatorname{Dim} E^{\perp} - [\rho_{TE}^{\perp}]_{\mathfrak{M}} = \operatorname{Dim} E'^{\perp} - [\rho_{TE'}^{\perp}]_{\mathfrak{M}} . \tag{1.11}$$

PROOF. This follows from (1.8) and the fact that $E'' = \inf(E', E)$ is a choice.

DEFINITION 1. If $T : X \to \mathfrak{F}(\mathfrak{M})$ is continuous and E a choice of T, then

$$\operatorname{Index} T = \dim E^{\perp} - [\rho_{TE}^{\perp}]_{\mathfrak{M}} . \tag{1.12}$$

In view of Lemma 3 this definition of the index of T is independent of the choice of E.

2. Homotopy invariance and additivity of the index.

PROPOSITION 1. *Let X be compact and $T_t : X \to \mathfrak{F}(\mathfrak{M})$, $0 \leqq t \leqq 1$, be a homotopy. Then*

$$\operatorname{Index} T_0 = \operatorname{Index} T_1. \tag{2.1}$$

PROOF. Without loss of generality we assume that X is connected. Define $T : X \times [0, 1] \to \mathfrak{F}(\mathfrak{M})$ by $T(x, t) = T_t x$. Since $X \times [0, 1]$ is compact, there is a choice E of T. Then E is also a choice of each T_t, $0 \leqq t \leqq 1$. Define

$$r_{TE}^{\perp} : X \times [0, 1] \to P\mathfrak{M}, \qquad r_{T_t E}^{\perp} : X \to P\mathfrak{M} \tag{2.2}$$

by $r_{TE}^{\perp}(x, t) = R_{T(x,t)E}^{\perp}$, $r_{T_t E}^{\perp}(x) = R_{T_t(x)E}^{\perp}$. Since $X \times [0, 1]$ is connected, the range of $r_{TE}^{\perp}$ is contained in the connected component of a finite projection $F \in P\mathfrak{M}$ which is the Grassmannian $\mathcal{M}_F$. Obviously

$$r_{TE}^{\perp}(x, t) = r_{T_t E}^{\perp}(x). \tag{2.3}$$

Hence $r_{T_t E}^{\perp} : X \to \mathcal{M}_F$, $0 \leqq t \leqq 1$, is a homotopy. It follows from Proposition 4 of Chapter II that

$$\rho_{T_0 E}^{\perp} = (r_{T_0 E}^{\perp})^* \mathcal{B}_F \cong (r_{T_1 E}^{\perp})^* \mathcal{B}_F = \rho_{T_1 E}^{\perp}. \tag{2.4}$$

Hence

$$\begin{aligned} \text{Index } T_0 &= \text{Dim } E^{\perp} - [\rho_{T_0 E}^{\perp}]_{\mathfrak{M}} \\ &= \text{Dim } E^{\perp} - [\rho_{T_1 E}^{\perp}]_{\mathfrak{M}} = \text{Index } T_1. \end{aligned} \tag{2.5}$$

Lemma 3. *Let $\mathfrak{M}^+$ be the space of positive Hermitian elements of $\mathfrak{M}$. Then $\mathfrak{F}(\mathfrak{M}) \cap \mathfrak{M}^+$ is contractible.*

Proof. A deformation of the identity map of $\mathfrak{F}(\mathfrak{M}) \cap \mathfrak{M}^+$ onto itself into the constant map of $\mathfrak{F}(\mathfrak{M}) \cap \mathfrak{M}^+$ on $1 \in \mathfrak{F}(\mathfrak{M}) \cap \mathfrak{M}^+$ is given by $f_t(T) = t \cdot 1 + (1 - t)T$ for $t \in [0, 1]$ and $T \in \mathfrak{F}(\mathfrak{M}) \cap \mathfrak{M}^+$.

Proposition 2. *Let X be compact. Let S, T be continuous maps of X into $\mathfrak{F}(\mathfrak{M})$. Then S^* and TS are continuous maps of X into $\mathfrak{F}(\mathfrak{M})$ satisfying*

$$\text{Index } S^* = - \text{Index } S \tag{2.6}$$

and

$$\text{Index } TS = \text{Index } T + \text{Index } S. \tag{2.7}$$

Proof. The first part follows from the fact that $\mathfrak{F}(\mathfrak{M})$ is a monoid closed under involution (Chapter I, §2). Let $S = V|S|$ be the polar decomposition. Then $|S|$ maps X continuously into $\mathfrak{F}\mathfrak{M} \cap \mathfrak{M}^+$. Proposition 1 and Lemma 3 imply

$$\text{Index } S = \text{Index } V = - \text{Index } V^* = - \text{Index } S^*. \tag{2.8}$$

Let E be a choice of T. Then TE is homotopic to T and Proposition 1 implies

$$\text{Index } TS = \text{Index } (TE)(ES). \tag{2.9}$$

Let $TE = U|TE|$ be the polar decomposition. Proposition 1 and

Lemma 3 imply

$$\text{(2.10)} \qquad \text{Index}\,(TE)(ES) = \text{Index}\,U(ES).$$

Let F be a choice of ES, then F is also a choice of $U(ES)$ because E is a choice of U. Observe that

$$\text{(2.11)} \quad H \ominus (U_x ES_x F)(H) = (H \ominus U_x E(H)) + (U_x[E(H) \ominus ES_x F(H)]).$$

Hence

$$\text{(2.12)} \qquad \rho^{\perp}_{UESF} \cong \rho^{\perp}_{UE} \oplus (\rho^{\perp}_{ESF} \ominus \Theta_{X,E^{\perp}})$$

and consequently

$$\text{(2.13)} \quad \begin{aligned} \text{Index}\,(UES) &= \text{Dim}\,F^{\perp} - [\rho^{\perp}_{ESF}]_{\mathfrak{M}} + \text{Dim}\,E^{\perp} - [\rho^{\perp}_{UE}]_{\mathfrak{M}} \\ &= \text{Index}\,(ES) + \text{Index}\,U. \end{aligned}$$

Since ES, U are homotopic to S, resp. T, (via straight lines), (2.13) and Proposition 1 imply

$$\text{(2.14)} \qquad \text{Index}\,(UES) = \text{Index}\,T + \text{Index}\,S.$$

The equations (2.9), (2.10) and (2.14) imply (2.7).

3. **Isomorphism between $[X, \mathfrak{F}\mathfrak{M}]$ and $K_{\mathfrak{M}}(X)$.** Let $\mathcal{C}(X, \mathfrak{F}\mathfrak{M})$ be the topological monoid of continuous maps of X into $\mathfrak{F}\mathfrak{M}$ with the topology of uniform convergence. Let $[X, \mathfrak{F}\mathfrak{M}]$ be the monoid of homotopy classes of continuous maps of X into $\mathfrak{F}\mathfrak{M}$. If $S \in \mathcal{C}(X, \mathfrak{F}\mathfrak{M})$, then $[S]$ denotes the homotopy class of S. Lemma 3 implies that $[S^*]$ is a two-sided inverse of $[S]$. Hence $[X, \mathfrak{F}\mathfrak{M}]$ is a group. The results of §2 can be reformulated by saying that there is a group homomorphism

$$\text{(3.1)} \qquad \text{index} : [X, \mathfrak{F}\mathfrak{M}] \to K_{\mathfrak{M}}(X)$$

such that the diagram

$$\text{(3.2)} \qquad \begin{array}{ccc} \mathcal{C}(X, \mathfrak{F}\mathfrak{M}) & & \\ \big\downarrow [\;] & \overset{\text{Index}}{\searrow} & \\ [X, \mathfrak{F}\mathfrak{M}] & \xrightarrow{\text{index}} & K_{\mathfrak{M}}(X) \end{array}$$

is commutative.

Theorem 1. *For any compact space X the map index is an isomorphism of $[X, \mathfrak{F}\mathfrak{M}]$ onto $K_{\mathfrak{M}}(X)$.*

Proof. *Injectivity.* Let $T \in \mathcal{C}(X, \mathfrak{F}\mathfrak{M})$ have index zero. Then we have

(3.3) $$\operatorname{Dim} E^{\perp} = [\rho_{TE}^{\perp}]_{\mathfrak{M}} .$$

In terms of $\mathfrak{M}$-vector bundles this means that there is a finite $\mathfrak{M}$-vector bundle η over X such that

(3.4) $$\Theta_{X,E^{\perp}} \oplus \eta \cong \rho_{TE}^{\perp} \oplus \eta.$$

Proposition 7 of Chapter II and (3.4) imply that there is a finite projection $F' \in \mathfrak{M}$ such that

(3.5) $$\Theta_{X,E^{\perp}} \oplus \Theta_{X,F'} \cong \rho_{TE}^{\perp} \oplus \Theta_{X,F'} .$$

Because of $E \sim 1$ we can choose $F' \leqq E$. Then $F = E - F'$ is still a choice of T. Lemma 2 of §1 (relation (1.7)) implies that there is an $\mathfrak{M}$-isomorphism

(3.6) $$V : \Theta_{X,F^{\perp}} \rightarrow \rho_{TF}^{\perp}.$$

Hence $x \rightarrow V_x + T_x F$ is a continuous map of X into the group $G\mathfrak{M}$ of regular elements of $\mathfrak{M}$. This map is homotopic within $\mathfrak{F}\mathfrak{M}$ to the given map $x \rightarrow T_x$ (by the straight line $tV + TF$, $0 \leqq t \leqq 1$, since all V_x are of finite rank). On the other hand it is also homotopic within $G\mathfrak{M}$ to the constant map $x \rightarrow 1 \in G\mathfrak{M}$ because $G\mathfrak{M}$ is contractible (Breuer [**10**]).

Surjectivity. Let ξ be an $\mathfrak{M}$-finite vector bundle over X. Since the index is additive and $K_{\mathfrak{M}}(X)$ is generated by the elements of the form $[\xi]$, it suffices to show that there is a map $T : X \rightarrow \mathfrak{F}\mathfrak{M}$ such that

(3.7) $$\operatorname{Index} T = [\xi]_{\mathfrak{M}} .$$

In view of Lemma 3 of Chapter II we can also assume that ξ is an $\mathfrak{M}$-subbundle of $\Theta_{X,1} = X \times H$. Proposition 1 of Chapter II implies that there is an isomorphism

(3.8) $$V : \Theta_{X,1} \rightarrow \Theta_{X,1} \ominus \xi.$$

Then $x \rightarrow V_x$ is a continuous map of X into $\mathfrak{F}\mathfrak{M}$. Define $T = V^*$. Using Proposition 2 and the fact that the unit element 1 of $\mathfrak{M}$ is a choice of V we get

(3.9) $$\operatorname{Index} T = -\operatorname{Index} V = -(\operatorname{Dim} 0 - [\xi]) = [\xi]_{\mathfrak{M}} .$$

Corollary 1. *The index map induces an isomorphism*

(3.10) $$\pi_0 \mathfrak{F}(\mathfrak{M}) \cong I(\mathfrak{M}).$$

Proof. In Theorem 1 choose for X a one point space $\{p\}$ and observe that $K_{\mathfrak{M}}(\{p\}) = I(\mathfrak{M})$.

Corollary 2. *The fundamental group of* $\mathfrak{F}(\mathfrak{M})$ *is trivial,*

$$\pi_1 \mathfrak{F}(\mathfrak{M}) = \{0\}. \tag{3.11}$$

Proof. In Theorem 1 choose $X = S^1$ and apply Corollary 2 of Proposition 4 of Chapter II.

Chapter IV. The Periodicity Theorem for $K_{\mathfrak{M}}$.

In this chapter $\mathfrak{M}$ is a countably decomposable semifinite and properly infinite von Neumann algebra of operators of a complex Hilbert space H.

1. **Some elementary properties of the $K_{\mathfrak{M}}$-functor.** In this section we state some lemmas on $K_{\mathfrak{M}}$ whose proofs are elementary and do not require the periodicity theorem. The proofs will only be indicated. In Chapter II, §4, $K_{\mathfrak{M}}$ has been defined as a contravariant functor from the category of *compact* spaces and continuous maps into the category of abelian groups and homomorphisms. We define the reduced $K_{\mathfrak{M}}$-functor by extending $K_{\mathfrak{M}}$ to the locally compact spaces as follows.

Definition 1. *Let* X be *locally compact* and $\dot{X} = X \cup \{\infty\}$ be its one point compactification. Let i_∞ be the inclusion map of the point ∞ into X. Define

$$K_{\mathfrak{M}}(X) = \text{kernel}\,[K_{\mathfrak{M}}(i_\infty) : K_{\mathfrak{M}}(\dot{X}) \to I(\mathfrak{M})]\,. \tag{1.1}$$

It is easy to see that this definition extends $K_{\mathfrak{M}}$ to a contravariant functor from the category of locally compact spaces and proper maps into the category of abelian groups and homomorphisms. One always has

$$K_{\mathfrak{M}}(\dot{X}) \cong K_{\mathfrak{M}}(X) \oplus I(\mathfrak{M}). \tag{1.2}$$

Thus $K_{\mathfrak{M}}(X)$ is the part of $K_{\mathfrak{M}}(\dot{X})$ depending on the topology of $\dot{X}$. The other part $I(\mathfrak{M})$ depends on the von Neumann algebra only. If $X = \mathbf{R}^n$, then $\dot{X}$ is the n-sphere S^n. (1.2) specializes to

$$K_{\mathfrak{M}}(S^n) \cong K_{\mathfrak{M}}(\mathbf{R}^n) \oplus I(\mathfrak{M}). \tag{1.3}$$

If $\mathfrak{M} = \mathcal{L}(H)$, then we use the more common notation

$$K = K_{\mathcal{L}(H)}, \qquad \text{Vect} = \text{Vect}_{\mathcal{L}(H)}, \tag{1.4}$$

Let X be a paracompact space. Let a be a complex finite dimensional vector bundle and ξ be a finite $\mathfrak{M}$-vector bundle over X. Without loss of generality we assume in the following construction that the fibre dimensions of a and ξ are constant and equal to $n \in Z^+$, resp.

$\operatorname{Dim} E \in I(\mathfrak{M})^+$. Choose an atlas $(U_j, \varphi_j, \mathbf{C}^n)_{j\in J}$ of a whose transition functions map into the unitary group $U(n)$ of $\mathbf{C}^n$ (such a reduction of the structure group is possible because X is paracompact; any two such reductions are $U(n)$-equivalent (see Steenrod [23, Part I, 12.9 and 12.13])). Choose an $\mathfrak{M}$-atlas $(U_j, \psi_j, E)_{j\in J}$ of ξ (whose transition functions map by definition into the unitary group $\mathfrak{U}\mathfrak{M}_E$ of $\mathfrak{M}_E$). Let F be a projection of $\mathfrak{M}$ such that

$$(1.5) \qquad \operatorname{Dim} F = n \cdot \operatorname{Dim} E \quad \text{and} \quad F \geqq E.$$

This is possible because $\mathfrak{M}$ is properly infinite. Choose an isomorphism

$$(1.6) \qquad \gamma : \mathbf{C}^n \otimes E(H) \to F(H)$$

that induces a von Neumann algebra isomorphism

$$(1.7) \qquad \gamma^{\#} : \mathcal{L}(\mathbf{C}^n) \otimes \mathfrak{M}_E \to \mathfrak{M}_F.$$

Then $(U_j, \gamma^{\#} \circ (\varphi_j \otimes \psi), F)_{j\in J}$ is an $\mathfrak{M}$-atlas of the tensor product $a \otimes \xi$ of the vector bundles a, ξ. Its equivalence class depends on the vector bundle structure of a and the $\mathfrak{M}$-vector bundle structure of ξ only. Thus $a \otimes \xi$ can canonically be equipped with the structure of an $\mathfrak{M}$-vector bundle. This construction can also be made if the fibre dimensions of a, ξ are not constant. One always has

$$(1.8) \qquad \operatorname{Dim}(a \otimes \xi)_x = \operatorname{Dim} a_x \cdot \operatorname{Dim} \xi_x \quad \text{for all } x \in X.$$

Let $a, b, \cdots$ be complex finite dimensional vector bundles over X; let $\xi, \eta, \cdots$ be finite $\mathfrak{M}$-vector bundles over X. Let $\cong$, resp. $\cong_{\mathfrak{M}}$, denote isomorphic, resp. $\mathfrak{M}$-isomorphic. Then we have

(i) $a \cong b$ and $\xi \cong_{\mathfrak{M}} \eta$ imply $a \otimes \xi \cong_{\mathfrak{M}} b \otimes \eta$,

(ii) $a \otimes (\xi + \eta) \cong_{\mathfrak{M}} (a \otimes \xi) \oplus (a \otimes \eta)$,

(iii) $(a \oplus b) \otimes \xi \cong_{\mathfrak{M}} (a \otimes \xi) \oplus (b \otimes \xi)$,

(iv) $(a \otimes b) \otimes \xi \cong_{\mathfrak{M}} a \otimes (b \otimes \xi)$.

In the following we assume that X is locally compact. Let

$$(1.9) \qquad [\ ] : \operatorname{Vect}(\dot{X}) \to K(\dot{X}), \qquad [\ ]_{\mathfrak{M}} : \operatorname{Vect}_{\mathfrak{M}}(\dot{X}) \to K_{\mathfrak{M}}(\dot{X})$$

be the canonical homomorphisms. Define

$$(1.10) \qquad \bar{\delta} : \operatorname{Vect}(\dot{X}) \times \operatorname{Vect}_{\mathfrak{M}}(\dot{X}) \to K_{\mathfrak{M}}(\dot{X})$$

by

$$(1.11) \qquad \bar{\delta}(a, \xi) = [a \otimes \xi]_{\mathfrak{M}}.$$

In the following $a, b, \cdots$, resp. $\xi, \eta, \cdots$, also denote isomorphism classes of vector bundles, resp. $\mathfrak{M}$-vector bundles.

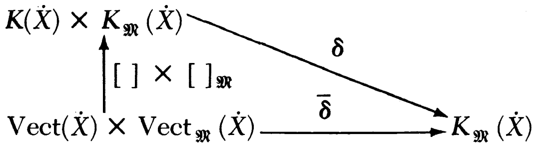
Lemma 1. *There is a unique map*

$$\delta : K(\dot X) \times K_{\mathfrak{M}}(\dot X) \to K_{\mathfrak{M}}(\dot X) \tag{1.12}$$

that defines the structure of a $K(\dot X)$*-module on* $K_{\mathfrak{M}}(\dot X)$ *and satisfies*

$$\delta([a], [\xi]_{\mathfrak{M}}) = [a \otimes \xi]_{\mathfrak{M}}. \tag{1.13}$$

Condition (1.13) *can also conveniently be expressed by saying that the diagram*



is commutative.

One proves Lemma 1 by using the above properties of $\otimes$, the commutativity of the ring $K(\dot X)$ and the universal properties of the ring $K(\dot X)$ (with respect to the semiring $\mathrm{Vect}(\dot X)$) and of the group $K_{\mathfrak{M}}(\dot X)$ (with respect to the monoid $\mathrm{Vect}_{\mathfrak{M}}(\dot X)$). This is very similar to the proof that $K(\dot X)$ is a ring given in Milnor [20]. In the present paper the details are omitted.

In the following we write

$$\delta([a], [\xi]_{\mathfrak{M}}) = [a] \cdot [\xi]_{\mathfrak{M}} \tag{1.15}$$

as is more usual in the theory of modules.

Lemma 2. $K_{\mathfrak{M}}(X)$ *is a submodule of* $K_{\mathfrak{M}}(\dot X)$.

Proof. Note that $K(i_\infty)$, resp. $K_{\mathfrak{M}}(i_\infty)$, associates to $[a] \in K(\dot X)$, resp. $[\xi]_{\mathfrak{M}} \in K_{\mathfrak{M}}(\dot X)$, the dimension of the fibre of a, resp. ξ, at ∞. Similarly as in K-theory one shows

$$K_{\mathfrak{M}}(\dot X) = \{[\xi]_{\mathfrak{M}} - \mathrm{Dim}\,\xi_\infty \mid \xi \in \mathrm{Vect}_{\mathfrak{M}}(\dot X)\}. \tag{1.16}$$

(This also follows from the surjectivity of the index map (Theorem 1 of Chapter II).) Using the distributive laws and (1.8) one easily verifies

$$K_{\mathfrak{M}}(i_\infty)([a] - [b])([\xi]_{\mathfrak{M}} - \mathrm{Dim}\,\xi_\infty) = 0. \tag{1.17}$$

Hence $K_{\mathfrak{M}}(X)$ is a $K(\dot X)$-module.

Lemmas 1 and 2 generalize the fact that $K(\dot X)$ is a commutative ring and $K(X)$ an ideal of $K(\dot X)$. Some other properties of the K-functor generalize verbally to the $K_{\mathfrak{M}}$-functor. In particular one can generalize the exact cohomology sequence of Atiyah [1, Proposition 2.4.4]. A formal consequence of it is the following

LEMMA 3. *Let X, Y be locally compact. Then there is a natural exact sequence*

(1.18) $0 \to K_{\mathfrak{M}}(X \times Y) \to K_{\mathfrak{M}}(\dot{X} \times \dot{Y}) \to K_{\mathfrak{M}}(\dot{X}) \oplus K_{\mathfrak{M}}(\dot{Y})$.

Using this lemma one can easily prove the following generalization of (1.3).

LEMMA 4. *Let X be locally compact. Then*

(1.19) $$K_{\mathfrak{M}}(S^n \times X) \cong K_{\mathfrak{M}}(R^n \times X) \oplus K_{\mathfrak{M}}(X).$$

Finally we want to generalize the external multiplication. Let X, Y be locally compact. Let

(1.20) $$P_{\dot{X}} : \dot{X} \times \dot{Y} \to \dot{X}, \qquad P_{\dot{Y}} : \dot{X} \times \dot{Y} \to \dot{Y}$$

be the natural projections. Then a Z-linear map

(1.21) $$\lambda : K(\dot{X}) \otimes_Z K_{\mathfrak{M}}(\dot{Y}) \to K_{\mathfrak{M}}(\dot{Y} \times \dot{X})$$

is defined by the relation

(1.22) $$\lambda([a] \otimes [\xi]_{\mathfrak{M}}) = (K(P_{\dot{X}})[a]) \cdot (K_{\mathfrak{M}}(P_{\dot{Y}})[\xi]_{\mathfrak{M}})$$

for all $a \in \text{Vect}(X)$ and $\xi \in \text{Vect}_{\mathfrak{M}}(X)$. It follows from Lemma 3 that λ induces a map

(1.23) $$\lambda : K(X) \otimes_Z K_{\mathfrak{M}}(Y) \to K_{\mathfrak{M}}(X \times Y).$$

The image of $[a] \otimes [\xi]_{\mathfrak{M}} \in K(\dot{X}) \otimes K_{\mathfrak{M}}(\dot{Y})$ under λ is denoted by $[a] \cdot [\xi]_{\mathfrak{M}}$. In a similar way one can define a Z-linear map

(1.24) $$\lambda' : K_{\mathfrak{M}}(\dot{Y}) \otimes_Z K(\dot{X}) \to K_{\mathfrak{M}}(\dot{Y} \times \dot{X})$$

that induces a map

(1.25) $$\lambda' : K_{\mathfrak{M}}(Y) \otimes_Z K(X) \to K_{\mathfrak{M}}(Y \times X).$$

The image of $[\xi]_{\mathfrak{M}} \otimes [a] \in K_{\mathfrak{M}}(\dot{Y}) \times K(\dot{X})$ under λ' is denoted by $[\xi]_{\mathfrak{M}} \cdot [a]$. Observe that we consider $[a] \cdot [\xi]_{\mathfrak{M}}$ and $[\xi]_{\mathfrak{M}} \cdot [a]$ as elements of different $K(\dot{X})$-modules. If we define

(1.26) $$i : \dot{X} \times \dot{Y} \to \dot{Y} \times \dot{X}$$

by $i(x, y) = (y, x)$, then one obviously has

(1.27) $$K_{\mathfrak{M}}(i)([\xi]_{\mathfrak{M}} \cdot [a]) = [a] \cdot [\xi]_{\mathfrak{M}}.$$

2. On Fredholm sections of endomorphism bundles. Let F be a projection of $\mathfrak{M}$. The inclusion map of the reduced algebra $\mathfrak{M}_F = F\mathfrak{M}F$ into $\mathfrak{M}$ does not induce a homomorphism of the group of unitary (or regular) elements of $\mathfrak{M}_F$ into the group of unitary (or regular) elements of $\mathfrak{M}$, unless $F = 1$, nor does the inclusion induce a map of $\mathfrak{F}(\mathfrak{M}_F)$ into $\mathfrak{F}(\mathfrak{M})$, unless $F^\perp$ is finite. When dealing with these multiplicative structures the appropriate map ι_F of $\mathfrak{M}_F$ into $\mathfrak{M}$ is given by

$$\iota_F(T) = T + F^\perp. \tag{2.1}$$

It is obvious that ι_F induces an injective homomorphism of $\mathfrak{U}(\mathfrak{M}_F)$, $G(\mathfrak{M}_F)$, resp. $\mathfrak{F}(\mathfrak{M}_F)$, into $\mathfrak{U}(\mathfrak{M})$, $G(\mathfrak{M})$, resp. $\mathfrak{F}(\mathfrak{M})$.

Let X be a compact space. Let ξ be a finite $\mathfrak{M}$-vector bundle over X with

$$\operatorname{Dim} \xi_x = \operatorname{Dim} E \tag{2.2}$$

for all $x \in X$. Let L be a separable infinite dimensional complex Hilbert space. Choose a trivialization

$$V : \xi \hat{\otimes} L \to X \times c(E)(H). \tag{2.3}$$

A section

$$T : X \to \operatorname{end}(\xi \hat{\otimes} L) \tag{2.4}$$

is called a *Fredholm section* if

$$V_x^{\#} T_x = V_x T_x V_x^* \in \mathfrak{F}(\mathfrak{M}_{c(E)}) \tag{2.5}$$

for all $x \in X$. This definition is independent of the choice of V because $\mathfrak{F}(\mathfrak{M}_{c(E)})$ is invariant under inner automorphisms of $\mathfrak{M}_{c(E)}$.

We want to describe certain subalgebras of the C^*-algebra $\Gamma \operatorname{end}(\xi \hat{\otimes} L)$ and their Fredholm sections.

First observe that $\operatorname{end}(\xi_x \otimes L)$ and $\operatorname{end} \xi_x \hat{\hat{\otimes}} \mathcal{L}(L)$ are both isomorphic to $\mathfrak{M}_{c(E)}$. It is easy to see that the canonical homomorphism

$$\operatorname{end} \xi \hat{\hat{\otimes}} \mathcal{L}(L) \to \operatorname{end}(\xi \hat{\otimes} L) \tag{2.6}$$

is an isomorphism. Let $\mathfrak{b}$ be a closed $*$-subalgebra of $\mathcal{L}(L)$. Define

$$\operatorname{end} \xi \hat{\otimes} \mathfrak{b} = \bigcup_{x \in X} (\operatorname{end} \xi_x \hat{\otimes} \mathfrak{b}). \tag{2.7}$$

The tensor product of a spatial atlas of $\operatorname{end} \xi$ (see §2 of Chapter II) with the trivial atlas of the trivial C^*-algebra bundle $X \times \mathfrak{b}$ is an atlas of $\operatorname{end} \xi \hat{\otimes} \mathfrak{b}$ which gives $\operatorname{end} \xi \hat{\otimes} \mathfrak{b}$ the structure of a C^*-algebra subbundle of the C^*-algebra bundle $\operatorname{end} \xi \hat{\hat{\otimes}} \mathcal{L}(L)$.

It follows that $\Gamma(\text{end}\,\xi \hat{\otimes} \mathfrak{b})$ is a C^*-subalgebra of the C^*-algebra $\Gamma(\text{end}\,\xi \hat{\hat{\otimes}} \mathcal{L}(L))$.

Let $\mathfrak{b}$ be a postliminal C^*-subalgebra of $\mathcal{L}(L)$ containing the ideal $\mathfrak{C}(L)$ of compact operators of L. Let $\bar{\mathfrak{b}} = \mathfrak{b}/\mathfrak{C}(L)$ be the quotient C^*-algebra and

$$(2.7) \qquad p : \mathfrak{b} \to \bar{\mathfrak{b}}$$

be the canonical projection. Let $\mathfrak{m}_x$ be the ideal of compact elements of end $\xi_x \hat{\hat{\otimes}} \mathcal{L}(L)$. Then Proposition 5 of Chapter I says that

$$(2.8) \qquad \mathfrak{m}_x \cap \text{end}\,\xi_x \hat{\otimes} \mathfrak{b} = \text{end}\,\xi_x \hat{\otimes} \mathfrak{C}(L).$$

Let

$$(2.9) \qquad \pi_{\xi,x} : \text{end}\,\xi_x \hat{\otimes} \mathfrak{b} \to \text{end}\,\xi_x \hat{\otimes} \bar{\mathfrak{b}}$$

be the canoncial map (tensor product of the identity map of end ξ_x with p_x). The collection of all maps $\pi_{\xi,x}$, $x \in X$, gives rise to a C^*-algebra bundle morphism

$$(2.10) \qquad \pi_\xi : \text{end}\,\xi \hat{\otimes} \mathfrak{b} \to \text{end}\,\xi \hat{\otimes} \bar{\mathfrak{b}}.$$

Applying the section functor we obtain a C^*-algebra homomorphism

$$(2.11) \qquad \Gamma(\pi_\xi) : \Gamma(\text{end}\,\xi \hat{\otimes} \mathfrak{b}) \to \Gamma(\text{end}\,\xi \hat{\otimes} \bar{\mathfrak{b}}).$$

Proposition 1. *The homomorphism $\Gamma(\pi_\xi)$ is surjective. The element T of $\Gamma(\text{end}\,\xi \hat{\otimes} \mathfrak{b})$ is a Fredholm section if and only if $\Gamma(\pi_\xi)(T)$ is a regular element of $\Gamma(\text{end}\,\xi \hat{\otimes} \bar{\mathfrak{b}})$.*

Proof. The first statement follows immediately from Proposition 6 of Chapter I. The second statement follows easily from (2.8) and Proposition 3 of Chapter I.

In the following we assume in addition to the above that $\bar{\mathfrak{b}}$ is commutative and that $\mathfrak{b}$ contains the identity operator of L. Let $M_{\bar{\mathfrak{b}}}$ be the maximal ideal space of $\bar{\mathfrak{b}}$ equipped with the Gelfand topology. Then there is a canonical C^*-algebra isomorphism

$$(2.12) \qquad \mu_{\xi,x} : \text{end}\,\xi \hat{\otimes} \bar{\mathfrak{b}} \to \mathcal{C}(M_{\bar{\mathfrak{b}}}, \text{end}\,\xi_x)$$

for all $x \in X$ (Chapter I, Corollary 3 of Proposition 4). The collection of all these maps gives rise to a C^*-algebra bundle isomorphism

$$(2.13) \qquad \mu_\xi : \text{end}\,\xi \hat{\otimes} \bar{\mathfrak{b}} \to \mathcal{C}.(M_{\bar{\mathfrak{b}}}, \text{end}\,\xi)$$

(see Chapter I, §4). Define the σ-symbol of end $\xi \hat{\otimes} \mathfrak{b}$ by

$$(2.14) \qquad \sigma_\xi = \mu_\xi \circ \pi_\xi.$$

Obviously

$$\Gamma(\sigma_\xi) = \Gamma(\mu_\xi) \circ \Gamma(\pi_\xi). \tag{2.15}$$

Proposition 1 can be reformulated in terms of the σ-symbol as follows.

COROLLARY 1. *$\Gamma(\sigma_\xi)$ is a C^*-algebra homomorphism of* $\Gamma(\text{end}\, \xi \hat{\otimes} \mathfrak{b})$ *onto* $\Gamma \mathcal{C}.(M_{\bar{\mathfrak{b}}}, \text{end}\, \xi)$. *The section T of* end $\xi \hat{\otimes} \mathfrak{b}$ *is a Fredholm section iff* $(\Gamma(\sigma_\xi)T(x, m))$ *is a regular element of* end ξ_x *for all* $(x, m) \in X \times M_{\bar{\mathfrak{b}}}$.

Examples of algebras $\mathfrak{b}$ satisfying the above assumptions arise from the theory of singular integral operators. Because of this one can view such algebras $\mathfrak{b}$ as abstract algebras of singular integral operators. For the proof of the periodicity theorem we need a very special and well-known algebra of singular integral operators which is defined in the following.

Let $L^2(S^1)$ be the Hilbert space of complex Lebesgue square integrable functions of the 1-sphere $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$. For $f \in \mathcal{C}(S^1, C)$ define $M_f \in \mathcal{L}(L^2(S^1))$ as usual by

$$M_f(g) = f \cdot g \quad \text{for all } g \in L^2(S^1). \tag{2.16}$$

Let $f_n(z) = z^n/2\pi$, $n \in \mathbf{Z}$. Then $(f_n)_{n\in \mathbf{Z}}$ is a complete o.n.s. of $L^2(S^1)$. Let L be the closure of the span of $(f_n)_{n\in \mathbf{Z}^+}$. Let Q be the projection of $L^2(S^1)$ onto L. Define

$$W : \mathcal{C}(S^1, \mathbf{C}) \to \mathcal{L}(L) \tag{2.17}$$

by $W_f(g) = QM_f g$ for all $g \in L$. Then W is a linear isometry of $\mathcal{C}(S^1, \mathbf{C})$ into $\mathcal{L}(L)$, but not an algebra homomorphism. The commutators

$$[W_f, W_g] = W_f W_g - W_g W_f, \qquad f, g \in \mathcal{C}(S^1, \mathbf{C}), \tag{2.18}$$

are always compact. One has

$$\mathfrak{C}(L) \cap \text{Range } W = \{0\}. \tag{2.19}$$

Let $\mathfrak{a}$ be the *-subalgebra of $\mathcal{L}(L)$ generated by $\mathfrak{C}(L)$ and Range W. Then

$$\mathfrak{a} = \mathfrak{C}(L) + \text{Range } W. \tag{2.20}$$

Let $\bar{\mathfrak{a}} = \mathfrak{a} / \mathfrak{C}(L)$. The canonical map W of $\mathcal{C}(S^1, \mathbf{C})$ into $\mathfrak{a}$ composed with the projection p of $\mathfrak{a}$ onto $\bar{\mathfrak{a}}$ is a C^*-algebra isomorphism

$$p \circ W : \mathcal{C}(S^1, \mathbf{C}) \cong \bar{\mathfrak{a}}. \tag{2.21}$$

It follows that S^1 is the maximal ideal space of $\bar{\mathfrak{A}}$ and that $\mu = (p \circ W)^{-1}$ is the Gelfand isomorphism. Observe that the map

$$e = W \circ \mu \circ p \tag{2.22}$$

is idempotent. Its kernel is $\mathfrak{C}(L)$ and its range is Range(W). Hence the algebraic direct sum (2.20) is also topologically direct. Hence $\mathfrak{A}$ is closed!

Proposition 2. *W_f is Fredholm iff f is regular. If W_f is Fredholm, then the index of W_f is the negative winding number of f,*

$$\text{Index } W_f = -\omega(f). \tag{2.23}$$

Proof. The first part is trivial. The map ω which associates to each regular $f \in \mathcal{C}(S^1, C)$ its winding number $\omega(f)$ induces an isomorphism of $\pi_0 G\mathcal{C}(S^1, C)$ onto Z. Hence there is a $k \in Z$ such that

$$\text{Index } W_f = k\omega(f)$$

for all $f \in G\mathcal{C}(S^1, C)$. Choosing for f the identity map of S^1, i.e. $f(z) = z$, one sees that $k = -1$.

3. **The periodicity theorem.** Let X be a locally compact space and $\dot{X} = X \cup \{\infty\}$ be its one point compactification. Using the index isomorphism

$$\text{index} : [\dot{X}, \mathfrak{F}\mathfrak{M}] \to K_{\mathfrak{M}}(\dot{X}) \tag{3.1}$$

of Chapter III and the results of §1 — §2 of this chapter we will construct a homomorphism

$$\alpha : K_{\mathfrak{M}}(\mathbf{R}^2 \times X) \to K_{\mathfrak{M}}(X). \tag{3.2}$$

This will be the analogue of the corresponding construction in K-theory given by Atiyah [3].

The elements of $\text{Vect}_{\mathfrak{M}}(S^2 \times \dot{X})$ are by Proposition 10 of Chapter II of the form $[\xi, \varphi]$, where ξ is a finite $\mathfrak{M}$-vector bundle over $\dot{X}$ and φ is a clutching function of ξ. We can consider φ as a unitary element of the C^*-algebra $\Gamma\mathcal{C}.(S^1, \text{end } \xi)$ (see Proposition 11 of Chapter II). Let $\mathfrak{A}$ be the algebra of singular integral operators defined in §2. Let

$$\sigma : \text{end } \xi \hat{\otimes} \mathfrak{A} \to \mathcal{C}.(S^1, \text{end } \xi) \tag{3.3}$$

be the σ-symbol of the C^*-algebra bundle end $\xi \hat{\otimes} \mathfrak{A}$. Then

$$\Gamma(\sigma) : \Gamma(\text{end } \xi \hat{\otimes} \mathfrak{A}) \to \Gamma\mathcal{C}.(S^1, \text{end } \xi) \tag{3.4}$$

is a surjective C^*-algebra homomorphism (Corollary 1 of Proposition 1). Let

$$\gamma_\xi : \Gamma\mathcal{C}.(S^1, \text{end}\,\xi) \to \Gamma(\text{end}\,\xi \,\hat{\otimes}\, \mathfrak{a}) \tag{3.5}$$

be a global continuous section of $\Gamma(\sigma)$. Such sections exist according to Bartle-Graves [**6**], and any two such sections are homotopic (via a straight line because the kernel of $\Gamma(\sigma)$ is a linear space).

In the following we assume first that $\dot{X}$ is connected. Then the fibre dimension of ξ is constant. Choose a projection E of $\mathfrak{M}$ such that Dim E is the fibre dimension of ξ. Let $F = c(E)$ be the central cover of E. Let L be a separable infinite dimensional complex Hilbert space. Let

$$V : \xi \,\hat{\otimes}\, L \to \dot{X} \times F(H) \tag{3.6}$$

be an $\mathfrak{M}$-isomorphism (Chapter II, Proposition 1). Observe that any two such trivializations of $\xi \,\hat{\otimes}\, L$ are homotopic.

The trivialization V of $\xi \,\hat{\otimes}\, L$ induces a trivialization

$$V^{\#} : \text{end}\,\xi \,\hat{\hat{\otimes}}\, \mathcal{L}(L) \to \dot{X} \times \mathfrak{M}_F \tag{3.7}$$

(see Chapter II, Proposition 2). Applying the section functor Γ one arrives at a C^*-algebra isomorphism

$$\Gamma(V^{\#}) : \Gamma(\text{end}\,\xi \,\hat{\hat{\otimes}}\, \mathcal{L}(L)) \to \mathcal{C}(\dot{X}, \mathfrak{M}_F). \tag{3.8}$$

Let

$$\iota_F : \mathfrak{M}_F \to \mathfrak{M} \tag{3.9}$$

be the map defined by (2.1).

Since φ is a unitary element of $\Gamma\mathcal{C}.(S^1, \text{end}\,\xi)$ it follows from the corollary of Proposition 1 that $(\iota_F \circ \Gamma(V^{\#}) \circ \gamma_\xi)\varphi$ is an element of $\mathcal{C}(\dot{X}, \mathfrak{F}\mathfrak{M})$. The homotopy class of the map

$$(\iota_F \circ \Gamma(V^{\#}) \circ \gamma_\xi)\varphi : \dot{X} \to \mathfrak{F}\mathfrak{M} \tag{3.10}$$

depends on the homotopy class of φ only. Hence it depends on the element $[\xi, \varphi]$ of $\text{Vect}_{\mathfrak{M}}(S^2 \times \dot{X})$ only. We denote the homotopy class of (3.10) by $\Delta_{[\xi,\varphi]}$.

If $\dot{X}$ is not connected, then the restriction of ξ to each connected component of $\dot{X}$ and φ give rise to a continuous map of that component into $\mathfrak{F}\mathfrak{M}$ whose homotopy class again depends on $[\xi, \varphi]$ only. Thus $[\xi, \varphi]$ also gives rise to a homotopy class of continuous maps of X into $\mathfrak{F}\mathfrak{M}$ which is denoted by $\Delta_{[\xi,\varphi]}$.

Define

(3.11) $$\Delta : \mathrm{Vect}_{\mathfrak{M}}(S^2 \times \dot{X}) \to [\dot{X}, \mathfrak{F}\mathfrak{M}]$$

by $[\xi, \varphi] \to \Delta_{[\xi, \varphi]}$.

PROPOSITION 3. *Δ is a monoid homomorphism.*

PROOF. Choose $\mathfrak{M}$-embeddings

(3.12) $$\xi \subseteq \dot{X} \times E(H), \qquad \eta \subseteq \dot{X} \times F(H)$$

with

(3.13) $$EF = 0, \qquad E \sim F, \qquad E + F = 1.$$

ξ, resp. η, are also $\mathfrak{M}_E$-, resp. $\mathfrak{M}_F$-, vector bundles over $\dot{X}$. Applying the above definition of Δ to $\xi, \varphi, \mathfrak{M}_E$, resp. $\eta, \psi, \mathfrak{M}_F$, we get homotopy classes

(3.14) $$\Delta^E_{[\xi, \varphi]} \in [\dot{X}, \mathfrak{F}\mathfrak{M}_E], \qquad \Delta^F_{[\eta, \psi]} \in [\dot{X}, \mathfrak{F}\mathfrak{M}_F].$$

Let

(3.15) $$h : \dot{X} \to \mathfrak{F}\mathfrak{M}_E, \qquad k : \dot{X} \to \mathfrak{F}\mathfrak{M}_F$$

be maps whose homotopy classes are $\Delta^E_{[\xi, \varphi]}$, resp. $\Delta^F_{[\eta, \psi]}$. Then $h + F$, resp. $E + k$, represents $\Delta_{[\xi, \varphi]}$, resp. $\Delta_{[\eta, \psi]}$. Hence $(h + F)(E + k)$ represents $\Delta_{[\xi, \varphi]} + \Delta_{[\eta, \psi]}$ (Chapter III, Proposition 2). On the other hand $h + k$ represents $\Delta_{[\xi \oplus \eta, \varphi \oplus \psi]}$. But $h + k = (h + F)(E + k)$. Hence

(3.16) $$\Delta_{[\xi \oplus \eta, \varphi \oplus \psi]} = \Delta_{[\xi, \varphi]} + \Delta_{[\eta, \psi]}.$$

One has a canonical $\mathfrak{M}$-isomorphism

(3.17) $$[\xi \oplus \eta, \varphi \oplus \psi] \cong [\xi, \varphi] \oplus [\eta, \psi].$$

The last two relations imply Proposition 3.

Composing Δ with the index map we obtain a monoid homomorphism

(3.18) $$\operatorname{index} \Delta : \mathrm{Vect}_{\mathfrak{M}}(S^2 \times \dot{X}) \to K_{\mathfrak{M}}(\dot{X}).$$

Since $K_{\mathfrak{M}}(S^2 \times \dot{X})$ is universal with respect to $\mathrm{Vect}_{\mathfrak{M}}(S^2 \times X)$ there is a unique group homomorphism

(3.19) $$\dot{\alpha} : K_{\mathfrak{M}}(S^2 \times \dot{X}) \to K_{\mathfrak{M}}(\dot{X})$$

satisfying

(3.20) $$\dot{\alpha}([\xi, \varphi]_{\mathfrak{M}}) = \operatorname{index}(\Delta_{[\xi, \varphi]})$$

for all $[\xi, \varphi]_{\mathfrak{M}} \in K_{\mathfrak{M}}(S^2 \times \dot{X})$.

Lemma 5. *The restriction of $\dot{\alpha}$ to the subgroup $K_{\mathfrak{M}}(\mathbf{R}^2 \times X)$ of $K_{\mathfrak{M}}(S^2 \times \dot{X})$ is a group homomorphism*

$$\alpha_X : K_{\mathfrak{M}}(\mathbf{R}^2 \times X) \to K_{\mathfrak{M}}(X). \tag{3.21}$$

If Y is another locally compact space, then we have commutative diagrams

$$(\mathrm{D}_1) \qquad \begin{array}{ccc} K(\mathbf{R}^2 \times Y) \otimes K_{\mathfrak{M}}(X) & \longrightarrow & K_{\mathfrak{M}}(\mathbf{R}^2 \times Y \times X) \\ \Big\downarrow \alpha_X \otimes 1 & & \Big\downarrow \alpha_{X \times Y} \\ K(Y) \otimes K_{\mathfrak{M}}(X) & \longrightarrow & K_{\mathfrak{M}}(Y \times X) \end{array}$$

and

$$(\mathrm{D}_2) \qquad \begin{array}{ccc} K_{\mathfrak{M}}(\mathbf{R}^2 \times X) \otimes K(Y) & \longrightarrow & K_{\mathfrak{M}}(\mathbf{R}^2 \times X \times Y) \\ \Big\downarrow \alpha_X \otimes 1 & & \Big\downarrow \alpha_{X \times Y} \\ K_{\mathfrak{M}}(X) \otimes K(Y) & \longrightarrow & K_{\mathfrak{M}}(X \times Y) \end{array}$$

where the horizontal maps are defined by external multiplication.

This lemma is a simple consequence of the lemmas of §1.

Let $\varphi_n(z) = z^n$ for all complex numbers z. Let ξ be the trivial complex line bundle over the one point space $\{x\}$. Then $[\xi, \varphi_n]$ is a complex line bundle over S^2 denoted by ξ_n. Define the Bott class b in $K(S^2)$ by

$$b = [\zeta_{-1}] - [\zeta_0]. \tag{3.22}$$

It is obvious that b is contained in the subgroup $K(\mathbf{R}^2)$ of $K(S^2)$. The definition of α_X in Lemma 5 gives rise to a map

$$\alpha_{\{x\}} : K(\mathbf{R}^2) \to \mathbf{Z}. \tag{3.23}$$

Lemma 6. *$\alpha_{\{x\}}$ is an isomorphism satisfying*

$$\alpha_{\{x\}}([\zeta_n]) = -n, \qquad n \in \mathbf{Z}, \tag{3.24}$$

and consequently

$$\alpha_{\{x\}}(b) = 1. \tag{3.25}$$

Proof. This is an obvious consequence of the definition of $\alpha_{\{x\}}$ and Proposition 2.

Returning to the general case we define

$$\beta_X : K_{\mathfrak{M}}(X) \to K_{\mathfrak{M}}(\mathbf{R}^2 \times X) \tag{3.26}$$

by taking the external product of any $[\xi]_{\mathfrak{M}} - [\eta]_{\mathfrak{M}} \in K_{\mathfrak{M}}(X)$ with b, (3.27) $$\beta_X([\xi]_{\mathfrak{M}} - [\eta]_{\mathfrak{M}}) = b \cdot ([\xi]_{\mathfrak{M}} - [\eta]_{\mathfrak{M}}).$$

PERIODICITY THEOREM. *For any locally compact space X the maps* α_X, β_X *are inverse to each other. Thus we have an isomorphism*

(3.28) $$K_{\mathfrak{M}}(X) \cong K_{\mathfrak{M}}(\mathbf{R}^2 \times X).$$

PROOF. Substituting in (D_1) of Lemma 5 the space Y by the one point space $\{x\}$ one obtains a commutative diagram

(D_1') $$\begin{array}{ccc} K(\mathbf{R}^2) \otimes K_{\mathfrak{M}}(X) & \longrightarrow & K_{\mathfrak{M}}(\mathbf{R}^2 \times X) \\ \downarrow \alpha_{\{x\}} \otimes 1 & & \downarrow \alpha_X \\ Z \otimes K_{\mathfrak{M}}(X) & \longrightarrow & K_{\mathfrak{M}}(X) \end{array}$$

Together with Lemma 6 this implies

(3.29) $$\alpha_X \beta_X([\xi]_{\mathfrak{M}}) = \alpha_{\{x\}}(b) \cdot [\xi]_{\mathfrak{M}} = [\xi]_{\mathfrak{M}}$$

for all $\xi \in \mathrm{Vect}_{\mathfrak{M}}(X)$. Hence α_X is a left inverse of β_X. Substituting Y by $\mathbf{R}^2$ in (D_2) of Lemma 5 one obtains a commutative diagram

(D_2') $$\begin{array}{ccc} K_{\mathfrak{M}}(\mathbf{R}^2 \times X) \otimes K(\mathbf{R}^2) & \longrightarrow & K_{\mathfrak{M}}(\mathbf{R}^2 \times X \times \mathbf{R}^2) \\ \downarrow & & \downarrow \\ K_{\mathfrak{M}}(X) \otimes K(\mathbf{R}^2) & \longrightarrow & K_{\mathfrak{M}}(X \times \mathbf{R}^2) \end{array}$$

Hence

(3.30) $$\alpha_{X \times \mathbf{R}^2}(ub) = (\alpha_X u)b \quad \text{for all } u \in K_{\mathfrak{M}}(X \times \mathbf{R}^2).$$

Define

(3.31) $$j : \mathbf{R}^2 \times X \times \mathbf{R}^2 \to \mathbf{R}^2 \times X \times \mathbf{R}^2$$

by

(3.32) $$j(r, x, s) = (s, x, r).$$

It is easy to see that j is homotopic within the homeomorphisms of $\mathbf{R}^2 \times X \times \mathbf{R}^2$ to the identity map of $\mathbf{R}^2 \times X \times \mathbf{R}^2$. Hence

(3.33) $$K_{\mathfrak{M}}(j) : K_{\mathfrak{M}}(\mathbf{R}^2 \times X \times \mathbf{R}^2) \to K_{\mathfrak{M}}(\mathbf{R}^2 \times X \times \mathbf{R}^2)$$

is the identity map. Define

(3.34) $$i : X \times \mathbf{R}^2 \to \mathbf{R}^2 \times X$$

by

(3.35) $$i(x, r) = (r, x).$$

The maps i, j satisfy the following obvious relations

$$K_{\mathfrak{M}}(j)(u \cdot b) = b \cdot K_{\mathfrak{M}}(i)(u) \quad \text{for all } u \in K_{\mathfrak{M}}(\mathbf{R}^2 \times X) \tag{3.36}$$

and

$$K_{\mathfrak{M}}(i)(v \cdot b) = b \cdot v \quad \text{for all } v \in K_{\mathfrak{M}}(X). \tag{3.37}$$

Using (3.36) and the already proved fact that $\alpha_{X \times \mathbf{R}^2}$ is a left inverse of $\beta_{X \times \mathbf{R}^2}$ one obtains for every $u \in K_{\mathfrak{M}}(\mathbf{R}^2 \times X)$

$$\begin{aligned} \alpha_{X \times \mathbf{R}^2}(u \cdot b) &= \alpha_{X \times \mathbf{R}^2} K_{\mathfrak{M}}(j)(u \cdot b) \\ &= \alpha_{X \times \mathbf{R}^2}(b \cdot K_{\mathfrak{M}}(i)u) = K_{\mathfrak{M}}(i)(u). \end{aligned} \tag{3.38}$$

Together with (3.30) this implies

$$K_{\mathfrak{M}}(i)(u) = (\alpha_X u) \cdot b. \tag{3.39}$$

The relations (3.37) and (3.39) imply

$$\begin{aligned} \beta_X \alpha_X(u) = b\alpha_X(u) &= K_{\mathfrak{M}}(i)(\alpha_X(u) \cdot b) \\ &= K_{\mathfrak{M}}(i) K_{\mathfrak{M}}(i)(u) = u. \end{aligned} \tag{3.40}$$

Hence α_X is a right inverse of β_X. This concludes the proof of the Periodicity Theorem.

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