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AN APPLICATION OF LINEAR PROGRAMMING TO RATIONAL APPROXIMATION

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This is a report on a program for rational Chebyshev approximation of functions of either one or several variables. In addition, this program can be used to get good results for two types of simultaneous rational approximation. The algorithm used is a version of the differential-correction algorithm introduced by Cheney and Loeb [2].

Let $T \equiv \{t_1, \cdots, t_N\}$ be a finite set, where $T \subset R$ (the real line) or $T \subset R^k$. Let $f, \phi_1, \cdots, \phi_m, \Psi_1, \cdots, \Psi_n$ be functions defined on T. We then define the set of generalized rational functions

$$R_n^m \equiv \left\{ \frac{P}{Q} = \sum_{i=1}^{m} p_i\phi_i \Big/ \sum_{j=1}^{n} q_j\Psi_j \mid p_i, q_j \in R \text{ for all } i,j;\ Q > 0 \text{ on } T \right\}.$$

Our object is to choose $R \in R_n^m$ to minimize $\|f - R\|_T \equiv \max_{t \in T}|f(t) - R(t)|$.

The differential-correction algorithm for solving this problem is as follows:

(i) Choose any initial approximation $R_0 = P_0/Q_0 \in R_n^m$.

(ii) Having found $R_k = P_k/Q_k$, compute $\Delta_k \equiv \|f - R_k\|_T$, and choose P_{k+1} and Q_{k+1} as a solution of the following minimization problem (which can be solved by linear programming):

$$\text{minimize the expression } \max_{t \in T} \frac{|f(t)Q(t) - P(t)| - \Delta_k Q(t)}{Q_k(t)}.$$

under the restrictions $|q_j| \leqq 1, j = 1, \cdots, n$.

Define $\Delta^* \equiv \inf_{R \in R_n^m} \|f - R\|_T$. We say that $R^* \in R_n^m$ is a best approximation if $\|f - R^*\|_T = \Delta^*$. Barrodale, Powell, and Roberts [1] have proved that if R_k is not a best approximation, then $Q_{k+1} > 0$ on T and $\Delta_{k+1} < \Delta_k$. Furthermore, $\Delta_k \to \Delta^*$, and this convergence is quadratic if $T \subset R$, $N \geqq m + n - 1$, $\phi_i = \Psi_i = t^{i-1}$ for all i, and a non-degenerate best approximation (i.e., one for which either numerator or denominator has greatest allowable degree) exists. The proof of quadratic convergence does not generalize easily to functions of

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more than one variable, although the convergence appears quadratic for the examples we have tried.

Now suppose functions f_1 and f_2 on T are to be approximated simultaneously, in the sense that $R^* \in R_n^m$ is to be chosen to minimize

$$\max(\|f_1 - R^*\|_T, \|f_2 - R^*\|_T) = \inf_{R \in R_n^m} [\max(\|f_1 - R\|_T, \|f_2 - R\|_T)].$$

To solve this problem, let $\bar{T} \equiv \{t_1, \cdots, t_N, t_{N+1}, \cdots, t_{2N}\}$, where $t_{N+1}, \cdots, t_{2N}$ are chosen arbitrarily (but distinct from each other and from $t_1, \cdots, t_N$). It can be shown that the problem is equivalent to minimizing $\|\bar{f} - \sum_{i=1}^m p_i \bar{\phi}_i / \sum_{j=1}^n q_j \bar{\Psi}_j\|_{\bar{T}}$ over p_i, $q_j \in R$, $\sum_{j=1}^n q_j \bar{\Psi}_j > 0$ on $\bar{T}$, where $\bar{f}$, $\bar{\phi}_i$ and $\bar{\Psi}_j$ are given by

	$1 \leqq \alpha \leqq N$	$N + 1 \leqq \alpha \leqq 2N$	
$\bar{f}(t_\alpha) =$	$f_1(t_\alpha)$	$f_2(t_{\alpha-N})$	
$\bar{\phi}(t_\alpha) =$	$\phi_i(t_\alpha)$	$\phi_i(t_{\alpha-N})$	$(i = 1, \cdots, m)$
$\bar{\Psi}_j(t_\alpha) =$	$\Psi_j(t_\alpha)$	$\Psi_j(t_{\alpha-N})$	$(j = 1, \cdots, n)$

The problem of simultaneously approximating functions $f_1, \cdots, f_\ell$ where $\ell > 2$ on T is equivalent to the problem of simultaneously approximating their upper and lower envelopes (see Dunham [**3**]).

The final problem is an example of a modified form of simultaneous approximation, suggested to us by Günter Meinardus, in which the denominators of the approximating rational functions are required to be the same. This type of simultaneous approximation can also be treated by reducing it to a problem of approximating one function.

We close with some examples. The stopping criterion used was $\Delta_k - \Delta_{k+1} < 10^{-8} \Delta_k$, and the initial guess was $P_0/Q_0 \equiv 1/1$. In example 1, the third approximation required 79.5 seconds to execute on a CDC-3600. Observe that the addition of basis functions x^2 and y^2 does not reduce Δ^* very much.

EXAMPLE 1. Approximation of x^y on the 11 by 11 grid formed by uniformly subdividing the rectangle with corners $(1/2, 0)$, $(1, 0)$, $(1/2, 1)$, $(1, 1)$.

Best Approximation	Δ^*	No. iterations
$\dfrac{.639 + 1.00x - .653y + .632xy}{.638 + 1.00x + .954y - .974xy}$	6.13×10^{-4}	9
$\dfrac{.617 + .997x - .618y + .585xy}{.615 + 1.00x + .967y - 1.00xy + .0001x^2 + .006y^2}$	5.81×10^{-4}	10
$\dfrac{.494 + .968x - .499y + .389xy + .022x^2 + .006y^2}{.483 + 1.00x + .881y - .985xy}$	5.55×10^{-4}	10

EXAMPLE 2. Simultaneous approximation (in the sense of Meinardus) of x^y and $x^3 - y^3$ on the 5 by 5 grid formed by uniformly subdividing the same rectangle as above.

$$R_1 = \frac{.772 - .192x - .490y}{1.00 - .478x - .432y} \qquad R_2 = \frac{-.132 + .610x - .468y}{1.00 - .478x - .432y}$$

$$\Delta^* = 1.12 \times 10^{-1} \qquad 9 \text{ iterations}$$

Lee and Roberts [4] consider various algorithms for ordinary rational approximation. Extensive computer comparisons are given.

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